

Chapter 6

Applications of Integration

6.1 Velocity and Net Change

6.1.1 The position of an object is the coordinate of the object on the line at a given time, often denoted $s(t)$. The displacement over an interval $[a, b]$ is $s(b) - s(a)$, the difference of the object's ending position and beginning position. It can be written as $\int_a^b v(t) dt$ where $v(t)$ is the object's velocity at time t . The distance traveled by the object is $\int_a^b |v(t)| dt$, the sum of the distance traveled along the line to the right and the distance traveled along the line to the left over the given time interval.

6.1.2 If velocity is positive, then $|v(t)| = v(t)$, so the distance and displacement are equal.

6.1.3 The displacement is given by $\int_a^b v(t) dt$, because this quantity is equal to $s(b) - s(a)$.

6.1.4 The net change of a quantity is given by $\int_a^b f'(t) dt$, if $f'(t)$ is the rate of change of the quantity.

6.1.5 The value of Q at time t will be given by $Q(t) = Q(0) + \int_0^t Q'(x) dx$.

6.1.6 If $Q'(t)$ is the growth rate of a population Q at time t , then $\int_a^b Q'(t) dt = Q(b) - Q(a)$, the net change of the population over the time period $[a, b]$.

6.1.7

a. The velocity is positive for $0 \leq t < 1$ and for $3 < t < 5$, so the object is moving in the positive direction on those intervals.

b. The displacement is $\int_0^3 v(t) dt = 12 - 16 = -4$.

c. The distance traveled is $\int_1^5 |v(t)| dt = 16 + 10 = 26$.

d. The displacement is $\int_0^5 v(t) dt = 12 - 16 + 10 = 6$.

e. After five hours the object's position is 6 miles from the original position in the positive direction.

6.1.8

- a. The velocity is negative for $0 < t < 2$ and for $4 < t < 6$, so the object is moving in the negative direction on those intervals.
- b. The displacement is $\int_2^6 v(t) dt = 14 - 10 = 4$.
- c. The distance traveled is $\int_0^6 |v(t)| dt = 20 + 14 + 10 = 44$.
- d. The displacement is $\int_0^8 v(t) dt = -20 + 14 - 10 + 6 = -10$.
- e. After eight minutes the object's position is 10 meters from the original position in the negative direction.

6.1.9

- a. The displacement is the net area, which is $\frac{1}{2} \cdot 2 \cdot 2 - \frac{1}{2} \cdot \frac{4}{3} \cdot 1 + \frac{1}{2} \cdot \frac{5}{3} \cdot 2 = 3$.
- b. The distance traveled is $\frac{1}{2} \cdot 2 \cdot 2 + \frac{1}{2} \cdot \frac{4}{3} \cdot 1 + \frac{1}{2} \cdot \frac{5}{3} \cdot 2 = \frac{13}{3}$.
- c. $s(5) = s(0) + \int_0^5 v(t) dt = 0 + 3 = 3$.
- d. $s(t) = \begin{cases} \int_0^t (-x + 2) dx & \text{if } 0 \leq t \leq 3, \\ \frac{3}{2} + \int_3^t (3x - 10) dx & \text{if } 3 < t \leq 4, \\ 2 + \int_4^t (-2x + 10) dx & \text{if } 4 < t \leq 5 \end{cases} = \begin{cases} -\frac{t^2}{2} + 2t & \text{if } 0 \leq t \leq 3, \\ \frac{3t^2}{2} - 10t + 18 & \text{if } 3 < t \leq 4, \\ -t^2 + 10t - 22 & \text{if } 4 < t \leq 5. \end{cases}$

6.1.10

- a. The displacement is the net area, which is $3 \cdot 3 + \frac{1}{2} \cdot 3 \cdot 2 = 9 + 3 = 12$.
- b. Because $v(t) > 0$ on that interval, the distance traveled is the same as the displacement, so it is 12 also.
- c. $s(5) = s(0) + \int_0^5 v(t) dt = 0 + 12 = 12$.
- d. $s(t) = \begin{cases} 3t & \text{if } 0 \leq t \leq 3, \\ 9 + \int_3^t \left(-\frac{3}{2}x + \frac{15}{2}\right) dx & \text{if } 3 \leq t \leq 5 \end{cases} = \begin{cases} 3t & \text{if } 0 \leq t \leq 3, \\ -\frac{3}{4}t^2 + \frac{15}{2}t - \frac{27}{4} & \text{if } 3 \leq t \leq 5 \end{cases}$.

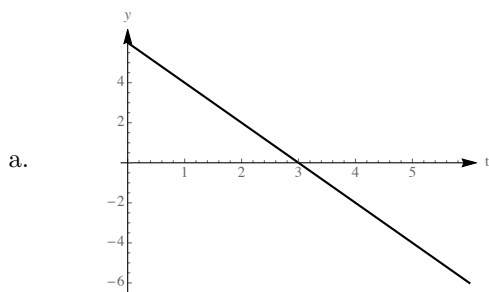
6.1.11

- a. Note that $v(t) > 0$ on $[0, \pi/2]$, so the distance traveled is

$$\int_0^{\pi/2} 3 \sin 2t dt = -\frac{3}{2} \cos 2t \Big|_0^{\pi/2} = -\frac{3}{2}(-1 - 1) = 3.$$

- b. Note that the function is alternately positive then negative (with the same enclosed area) on each interval of length $\pi/2$. The displacement on $[0, \pi]$ is then $3 - 3 = 0$, the displacement on $[0, 3\pi/2]$ is $3 - 3 + 3 = 3$, and the displacement on $[0, 2\pi]$ is $3 - 3 + 3 - 3 = 0$.

- c. The total distance traveled is $3 + 3 + 3 + 3 = 12$.

6.1.12

The velocity is positive (and so the object is moving to the right) on $[0, 3)$ and the velocity is negative (and so the object is moving to the left) on $(3, 6]$.

b. The displacement is $\int_0^6 v(t) dt = \int_0^6 (6 - 2t) dt = (6t - t^2) \Big|_0^6 = 0$ m.

c. The total distance traveled is $\int_0^3 (6 - 2t) dt + \int_3^6 (2t - 6) dt = (6t - t^2) \Big|_0^3 + (t^2 - 6t) \Big|_3^6 = (18 - 9) - (0 - 0) + (36 - 36) - (9 - 18) = 18$ m.

6.1.13

- a. $v(t) = 3t(t - 2)$ which is zero for $t = 0$ and $t = 2$. $v(t) > 0$ on $(2, 3)$ and is negative on $(0, 2)$.

b. The displacement is $\int_0^3 (3t^2 - 6t) dt = (t^3 - 3t^2) \Big|_0^3 = 3^3 - 3(3)^2 = 0$ m.

c. The total distance traveled is $\int_0^2 (6t - 3t^2) dt + \int_2^3 (3t^2 - 6t) dt = (3t^2 - t^3) \Big|_0^2 + (t^3 - 3t^2) \Big|_2^3 = (12 - 8) + (27 - 27) - (8 - 12) = 8$ m.

6.1.14

- a. $v(t) = 4t(t^2 - 6t + 5) = 4t(t - 5)(t - 1)$, which is zero for $t = 0$, $t = 1$, and $t = 5$. $v > 0$ on $(0, 1)$ and is negative on $(1, 5)$.

b. The displacement is $\int_0^5 (4t^3 - 24t^2 + 20t) dt = (t^4 - 8t^3 + 10t^2) \Big|_0^5 = 5^4 - 8(5^3) + 10(5^2) = -125$ m.

c. The distance traveled on the time interval $(0, 1)$ is $\int_0^1 (4t^3 - 24t^2 + 20t) dt = (t^4 - 8t^3 + 10t^2) \Big|_0^1 = 1 - 8 + 10 = 3$ m, and the distance traveled on the time interval $(1, 5)$ is $-\int_1^5 (4t^3 - 24t^2 + 20t) dt = -(t^4 - 8t^3 + 10t^2) \Big|_1^5 = 125 + 3 = 128$ m. Therefore the total distance traveled over the time interval $(0, 5)$ is $3 + 128 = 131$ m.

6.1.15

- a. $v(t) = 3(t^2 - 6t + 8) = 3(t - 4)(t - 2)$, which is zero for $t = 4$ and $t = 2$. $v > 0$ on $(0, 2)$ and $(4, 5)$, while v is negative on $(2, 4)$.

b. The displacement is given by $\int_0^5 (3t^2 - 18t + 24) dt = (t^3 - 9t^2 + 24t) \Big|_0^5 = 5^3 - 9(5^2) + 24(5) = 20$ m.

- c. On the time interval $(0, 2)$ the distance traveled is $\int_0^2 (3t^2 - 18t + 24) dt = (t^3 - 9t^2 + 24t) \Big|_0^2 = 8 - 36 + 48 = 20$ m. On the interval $(2, 4)$, the distance traveled is $-\int_2^4 (3t^2 - 18t + 24) dt = -(t^3 - 9t^2 + 24t) \Big|_2^4 = -(64 - 144 + 96) + (8 - 36 + 48) = 20 - 16 = 4$ m. On the time interval $(4, 5)$, the distance traveled is $\int_4^5 (3t^2 - 18t + 24) dt = (t^3 - 9t^2 + 24t) \Big|_4^5 = 20 - 16 = 4$ m. Therefore the total distance traveled is $20 + 4 + 4 = 28$ m.

6.1.16

- a. The motion is positive for $0 \leq t \leq 4$, because the exponential function is everywhere positive.
- b. The displacement is $\int_0^4 50e^{-2t} dt = (-25e^{-2t}) \Big|_0^4 = 25(1 - e^{-8}) \approx 24.992$ m.
- c. Because the velocity is positive on the given interval, the distance traveled is the same as the displacement, given above.

6.1.17

- a. $s(t) = \int \sin t dt = -\cos t + C$, and because $s(0) = 1$, we must have $C = 2$. Thus, $s(t) = 2 - \cos t$.
- b. $s(t) = s(0) + \int_0^t \sin x dx = 1 + (-\cos x) \Big|_0^t = 2 - \cos t$.

6.1.18

- a. $s(t) = \int (-t^3 + 3t^2 - 2t) dt = -\frac{t^4}{4} + t^3 - t^2 + C$. Because $s(0) = 4$, we have $C = 4$, and thus $s(t) = -\frac{t^4}{4} + t^3 - t^2 + 4$.
- b. $s(t) = s(0) + \int_0^t (-x^3 + 3x^2 - 2x) dx = 4 + \left(-\frac{x^4}{4} + x^3 - x^2\right) \Big|_0^t = -\frac{t^4}{4} + t^3 - t^2 + 4$.

6.1.19

- a. $s(t) = \int (6 - 2t) dt = 6t - t^2 + C$, and because $s(0) = 0$, we must have $C = 0$. Thus, $s(t) = 6t - t^2$.
- b. Also, $s(t) = s(0) + \int_0^t (6 - 2x) dx = (6x - x^2) \Big|_0^t = 6t - t^2$.

6.1.20

- a. $s(t) = \int 3 \sin \pi t dt = -\frac{3}{\pi} \cos \pi t + C$, and because $s(0) = 1$, we must have $C = 1 + \frac{3}{\pi}$. Thus, $s(t) = -\frac{3}{\pi} \cos \pi t + 1 + \frac{3}{\pi}$.
- b. $s(t) = s(0) + \int_0^t 3 \sin \pi x dx = 1 + \left(-\frac{3}{\pi} \cos \pi x\right) \Big|_0^t = -\frac{3}{\pi} \cos \pi t + 1 + \frac{3}{\pi}$.

6.1.21

- a. $s(t) = \int (9 - t^2) dt = 9t - \frac{t^3}{3} + C$, and because $s(0) = -2$, we must have $C = -2$. Thus, $s(t) = 9t - \frac{t^3}{3} - 2$.

$$\text{b. } s(t) = s(0) + \int_0^t (9 - x^2) dx = -2 + \left(9x - \frac{x^3}{3}\right)\bigg|_0^t = 9t - \frac{t^3}{3} - 2.$$

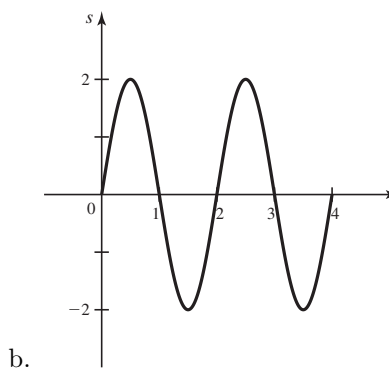
6.1.22

$$\text{a. } s(t) = \int \frac{1}{t+1} dt = \ln(t+1) + C, \text{ and because } s(0) = -4, \text{ we must have } C = -4. \text{ Thus, } s(t) = \ln(t+1) - 4.$$

$$\text{b. } s(t) = s(0) + \int_0^t \frac{1}{1+x} dx = -4 + (\ln(x+1))\bigg|_0^t = \ln(t+1) - 4.$$

6.1.23

$$\text{a. } s(t) = s(0) + \int_0^t 2\pi \cos \pi x dx = 2 \sin \pi x \bigg|_0^t = 2 \sin \pi t.$$



c. The mass reaches its lowest point at $t = 1.5$, $t = 3.5$ and $t = 5.5$.

d. The mass reaches its highest point at $t = .5$, $t = 2.5$, and $t = 4.5$.

6.1.24

$$\text{a. } s(5) = \int_0^5 |400 - 20t| dt = \int_0^5 (400 - 20t) dt = (400t - 10t^2) \bigg|_0^5 = 1750 \text{ m.}$$

$$\text{b. } s(10) = \int_0^{10} |400 - 20t| dt = \int_0^{10} (400 - 20t) dt = (400t - 10t^2) \bigg|_0^{10} = 3000 \text{ m.}$$

$$\text{c. Her velocity is 250 when } 400 - 20t = 250, \text{ or } t = 7.5, \text{ and thus } s(7.5) = \int_0^{7.5} (400 - 20t) dt = (400t - 10t^2) \bigg|_0^{7.5} = 2437.5.$$

6.1.25

$$\text{a. } s(t) = s(0) + \int_0^t 30(16 - x^2) dx = (480x - 10x^3) \bigg|_0^t = 480t - 10t^3 = 10t(48 - t^2).$$

b. Because the velocity is positive, this is given by $s(2) - s(0) = 960 - 80 = 880$ miles.

c. The velocity is 400 when $480 - 30t^2 = 400$, or $t = \sqrt{8/3}$. At this point the plane has traveled $s(\sqrt{8/3}) = 480\sqrt{8/3} - 10\sqrt{8/3}^3 \approx 740.290$ miles.

6.1.26

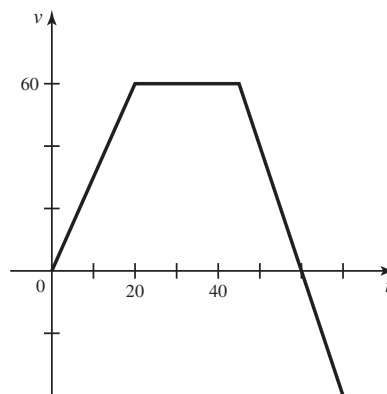
$$\text{a. } s(t) = s(0) + \int_0^t 3 \sin^2(\pi x/2) dx = \frac{3}{2} \int_0^t (1 - \cos(\pi x)) dx = \frac{3}{2} \left(x - \frac{1}{\pi} \sin \pi x\right) \bigg|_0^t = \frac{3}{2} \left(t - \frac{1}{\pi} \sin \pi t\right).$$

b. Because the velocity is positive, this is given by $s(0.25) - s(0) = \frac{3}{2} \left(0.25 - \frac{\sin(\pi/4)}{\pi} \right) = \frac{3}{8} - \frac{3\sqrt{2}}{4\pi} \approx 0.0374$ miles.

c. $s(3) = 4.5$ miles.

6.1.27

a. The velocity has a maximum of 60 for $20 \leq t \leq 45$. The velocity is 0 at $t = 0$ and at $t = 60$.



b. $\int_0^{20} 3t \, dt + \int_{20}^{30} 60 \, dt = 1200$ m.

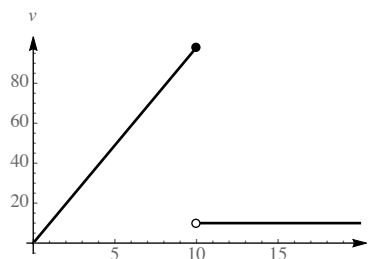
c. $1200 + \int_{30}^{45} 60 \, dt + \int_{45}^{60} (240 - 4t) \, dt = 1200 + 900 + (240t - 2t^2) \Big|_{45}^{60} = 2550$ m.

d. At time $t = 60$ the automobile is at position 2550. In the following 15 seconds, it moves $\int_{60}^{75} (240 - 4t) \, dt = 450$ feet in the opposite direction, so it is at position $2550 - 450 = 2100$.

6.1.28

The velocity is given by

a.
$$v(t) = \begin{cases} 9.8t & \text{if } 0 \leq t \leq 10, \\ 10 & \text{if } t > 10. \end{cases}$$



b. $s(30) = \int_0^{10} 9.8t \, dt + \int_{10}^{30} 10 \, dt = 490 + 200 = 690$ m.

c. We seek t so that $490 + \int_{10}^t 10 \, dx = 3000$, which can be written $10t - 100 = 2510$, so $t = 261$ s.

6.1.29 $v(t) = v(0) + \int_0^t a(x) \, dx = 70 + \int_0^t -32 \, dx = 70 - 32x \Big|_0^t = -32t + 70$.

$s(t) = s(0) + \int_0^t (-32x + 70) \, dx = 10 + (-16x^2 + 70x) \Big|_0^t = -16t^2 + 70t + 10$.

$$\mathbf{6.1.30} \quad v(t) = v(0) + \int_0^t -32 \, dx = 50 + -32x \Big|_0^t = -32t + 50.$$

$$s(t) = s(0) + \int_0^t (-32x + 50) \, dx = 0 + (-16x^2 + 50x) \Big|_0^t = -16t^2 + 50t.$$

$$\mathbf{6.1.31} \quad v(t) = v(0) + \int_0^t -9.8 \, dx = 20 + -9.8x \Big|_0^t = -9.8t + 20.$$

$$s(t) = s(0) + \int_0^t (-9.8x + 20) \, dx = 0 + (-4.9x^2 + 20x) \Big|_0^t = -4.9t^2 + 20t.$$

$$\mathbf{6.1.32} \quad v(t) = v(0) + \int_0^t e^{-x} \, dx = 60 - e^{-x} \Big|_0^t = 60 - (e^{-t} - 1) = 61 - e^{-t}.$$

$$s(t) = s(0) + \int_0^t (61 - e^{-x}) \, dx = 40 + (61x + e^{-x}) \Big|_0^t = 39 + 61t + e^{-t}.$$

$$\mathbf{6.1.33} \quad v(t) = v(0) + \int_0^t -0.01x \, dx = 10 + (-0.005x^2) \Big|_0^t = 10 - 0.005t^2.$$

$$s(t) = s(0) + \int_0^t (10 - .005x^2) \, dx = \left(10x - \frac{1}{600}x^3\right) \Big|_0^t = 10t - \frac{1}{600}t^3.$$

$$\mathbf{6.1.34} \quad v(t) = v(0) + \int_0^t \frac{20}{(x+2)^2} \, dx = 20 - \frac{20}{x+2} \Big|_0^t = 30 - \frac{20}{t+2}.$$

$$s(t) = s(0) + \int_0^t \left(30 - \frac{20}{x+2}\right) \, dx = 10 + (30x - 20 \ln |x+2|) \Big|_0^t = 10 + 20 \ln 2 + 30t - 20 \ln |t+2|.$$

$$\mathbf{6.1.35} \quad v(t) = v(0) + \int_0^t \cos 2x \, dx = 5 + \left(\frac{1}{2} \sin 2x\right) \Big|_0^t = \frac{1}{2} \sin 2t + 5.$$

$$s(t) = s(0) + \int_0^t \left(\frac{1}{2} \sin 2x + 5\right) \, dx = 7 + \left(-\frac{1}{4} \cos 2x + 5x\right) \Big|_0^t = -\frac{1}{4} \cos 2t + 5t + \frac{29}{4}.$$

$$\mathbf{6.1.36} \quad v(t) = v(0) + \int_0^t \frac{2x}{(x^2+1)^2} \, dx. \text{ Let } u = x^2 + 1 \text{ so that } du = 2x \, dx. \text{ Then we have } 0 + \int_1^{t^2+1} \frac{1}{u^2} \, du = \left(-\frac{1}{u}\right) \Big|_1^{t^2+1} = -\frac{1}{t^2+1} + 1.$$

$$s(t) = s(0) + \int_0^t \left(-\frac{1}{x^2+1} + 1\right) \, dx = (-\tan^{-1}(x) + x) \Big|_0^t = -\tan^{-1} t + t.$$

6.1.37

a. The velocity is given by $v(0) + \int_0^t 88 \, dx = 88x \Big|_0^t = 88t$ ft/s.

The position is given by $s(0) + \int_0^t 88x \, dx = 44x^2 \Big|_0^t = 44t^2$ ft.

b. The car travels $s(4) = 44 \cdot 16 = 704$ feet.

c. Because a quarter mile is 1320 feet, we need $44t^2 = 1320$, so $t = \sqrt{30} \approx 5.477$ seconds.

d. We need $44t^2 = 300$, so $t \approx 2.611$ seconds.

e. It reaches that speed when $88t = 178$, or $t = 89/44$ seconds. At that time the racer has traveled $s(89/44) = 44(89/44)^2 = 89^2/44 \approx 180.023$ feet.

6.1.38

a. The velocity is given by $v(0) + \int_0^t -15 \, dx = 60 + (-15x) \Big|_0^t = -15t + 60$.

The position is given by $s(0) + \int_0^t (-15x + 60) \, dx = (-7.5x^2 + 60x) \Big|_0^t = -7.5t^2 + 60t$.

b. The car comes to rest when $v(t) = 0$, which occurs for $t = 4$. At that time $s(4) = 120$ feet.

6.1.39 $v(t) = v(0) + \int_0^t -\frac{1280}{(1+8x)^3} \, dx = 80 + \left(\frac{80}{(1+8x)^2} \right) \Big|_0^t = \frac{80}{(1+8t)^2}$.

$$s(t) = s(0) + \int_0^t \frac{80}{(1+8x)^2} \, dx = \left(-\frac{10}{1+8x} \right) \Big|_0^t = 10 - \frac{10}{1+8t}.$$

Then in the first 0.2 seconds the train travels $s(0.2) - s(0) = 10 - \frac{10}{2.6} = 10 - \frac{50}{13} = \frac{80}{13} \approx 6.154$ miles.

Between time 0.2 and 0.4 the train travels $s(0.4) - s(0.2) = 10 - \frac{10}{4.2} - \left(10 - \frac{10}{2.6} \right) = \frac{50}{13} - \frac{50}{21} = \frac{400}{273} \approx 1.465$ miles.

6.1.40

a. $55 + \int_0^6 (20 - (t/5)) \, dt = 55 + \left(20t - \frac{t^2}{10} \right) \Big|_0^6 = \frac{857}{5} \approx 171.4$.

b. $P(200) = 55 + \int_0^{200} (20 - \frac{t}{5}) \, dt = 55 + \left(20t - \frac{t^2}{10} \right) \Big|_0^{200} = 55$.

6.1.41

a. $P(20) = 250 + \int_0^{20} (30 + 30\sqrt{t}) \, dt = 250 + \left(30t + 20t^{3/2} \right) \Big|_0^{20} = 250 + 600 + 800\sqrt{5} = 850 + 800\sqrt{5} \approx 2639$ people.

b. $P(t) = 250 + \int_0^t (30 + 30\sqrt{x}) \, dx = 250 + \left(30x + 20x^{3/2} \right) \Big|_0^t = 250 + 30t + 20t^{3/2}$ people.

6.1.42

a. $P(15) = 35 + \int_0^{15} (5 + 10 \sin(\pi t/5)) \, dt = 35 + (5t - (50/\pi) \cos(\pi t/5)) \Big|_0^{15} = 35 + (75 + (50/\pi)) - (0 - (50/\pi)) = 110 + 100/\pi \approx 142$ foxes.

$$P(35) = 35 + \int_0^{35} (5 + 10 \sin(\pi t/5)) \, dt = 35 + (5t - (50/\pi) \cos(\pi t/5)) \Big|_0^{35} = 35 + (175 + (50/\pi)) - (0 - (50/\pi)) = 210 + 100/\pi \approx 242$$
 foxes.

b. $P(t) = 35 + \int_0^t (5 + 10 \sin(\pi x/5)) \, dx = 35 + (5x - (50/\pi) \cos(\pi x/5)) \Big|_0^t = 35 + 5t - (50/\pi) \cos(\pi t/5) + (50/\pi)$ foxes.

6.1.43

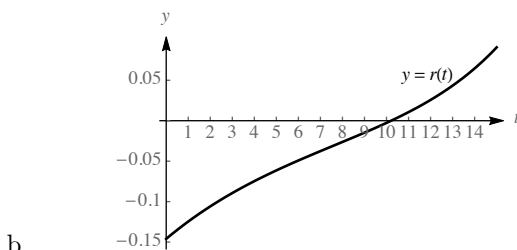
a. $N(20) = 1500 + \int_0^{20} 100e^{-0.25t} \, dt = 1500 + (-400e^{-0.25t}) \Big|_0^{20} = 1500 + (-400e^{-5} + 400) = 1900 - \frac{400}{e^5} \approx 1897$ cells.

$$N(40) = 1500 + \int_0^{40} 100e^{-0.25t} \, dt = 1500 + (-400e^{-0.25t}) \Big|_0^{40} = 1500 + (-400e^{-10} + 400) = 1900 - \frac{400}{e^{10}} \approx 1900$$
 cells.

b. $N(t) = 1500 + \int_0^t 100e^{-0.25x} dx = 1500 + (-400e^{-0.25x}) \Big|_0^t = 1500 + (-400e^{-0.25t} + 400) = -400e^{-0.25t} + 1900$ cells.

6.1.44

a. $r(t) = 0$ when $0.0025e^{0.25t} = 0.1485e^{-0.15t}$, which occurs when $e^{0.40t} = \frac{0.1485}{0.0025} = 59.4$. So $t_0 = \frac{\ln 59.4}{0.40} \approx 10.21$ days.



The tumor decreases in size for the first 10.21 days and then starts to increase in size.

c. $\int_0^{10.21} (0.0025e^{0.25t} - 0.1485e^{-0.15t}) dt = (0.01e^{0.25t} + 0.99e^{-0.15t}) \Big|_0^{10.21} \approx -0.658 \text{ cm}^3$. The tumor decreases in size by 0.658 cm^3 in the first 10.21 days.

6.1.45

a. In the first 35 days the number of barrels produced is $\int_0^{30} 800 dt + \int_{30}^{35} (2600 - 60t) dt = 24000 + 3250 = 27250$.

b. In the first 50 days the number of barrels produced is $27250 + \int_{35}^{40} (2600 - 60t) dt + \int_{40}^{50} 200 dt = 27250 + 1750 + 2000 = 31000$.

c. A constant 200 barrels per day times 20 days yields 4000 barrels.

6.1.46

a. $\int_0^{30} (0.25t^2 + 37.46t + 722.47) dt = \left(\frac{0.25}{3}t^3 + \frac{37.46}{2}t^2 + 722.47t \right) \Big|_0^{30} = 40781.1$.

b. $\int_0^{30} (0.90t^2 - 69.06t + 2053.12) dt = \left(\frac{0.90}{3}t^3 + \frac{69.06}{2}t^2 + 2053.12t \right) \Big|_0^{30} = 38616.6$.

c. $40781.1 + 38616.6 = 79397.7$ millions of cubic feet, or 7.93977×10^{10} cubic feet. There are $5280^3 = 147197952000$ cubic feet in a cubic mile, so the Spokane River contains $0.67 \cdot 147197952000 = 9.86226 \cdot 10^{10}$ cubic feet of water. So the percentage can be calculated by $\frac{7.93977 \times 10^{10}}{9.86226 \cdot 10^{10}} \approx 80.507\%$.

6.1.47

a. $Q(t) = Q(0) + \int_0^t 10^7 e^{-kx} dx = \left(\frac{10^7}{-k} e^{-kx} \right) \Big|_0^t = \frac{10^7(1 - e^{-kt})}{k}$.

b. $\lim_{t \rightarrow \infty} Q(t) = \lim_{t \rightarrow \infty} \frac{10^7(1 - e^{-kt})}{k} = \frac{10^7}{k}$. This represents the total number of barrels extracted if the nation extracts the oil indefinitely where it is assumed that the nation has at least $\frac{10^7}{k}$ barrels of oil in reserve.

c. We seek k so that $\frac{10^7}{k} = 2 \times 10^9$, which gives $k = \frac{1}{200} = 0.005$.

- d. We want T so that $(2 \times 10^7) \int_0^T e^{-0.005t} dt = 2 \times 10^9$, so $(-200e^{-0.005t}) \Big|_0^T = 100$, so $1 - e^{-T/200} = 1/2$, so $T = 200 \ln 2 \approx 138.629$ years.

6.1.48

- a. $\int_0^{60} 3\sqrt{t} dt = (2t^{3/2}) \Big|_0^{60} \approx 929.52$ L.
- b. $Q(t) = \int_0^t 3\sqrt{x} dx = (2x^{3/2}) \Big|_0^t = 2t^{3/2}$ L, where t is measured in minutes.
- c. The tank will be full when $2t^{3/2} = 2000$, or $t = \sqrt[3]{1000^2} = 100$ minutes.

6.1.49

- a. $\int_0^2 20(1 + \cos(\pi t/12)) dt = \left(20t + \frac{240}{\pi} \sin(\pi t/12)\right) \Big|_0^2 = 40 + \frac{240}{\pi} \cdot \frac{1}{2} = 40 + \frac{120}{\pi} \approx 78.197$ m³.
- b. $Q(t) = \int_0^t 20(1 + \cos(\pi x/12)) dx$, which is equal to

$$\left(20x + \frac{240}{\pi} \sin(\pi x/12)\right) \Big|_0^t = 20t + \frac{240}{\pi} \cdot \sin(\pi t/12) \text{ m}^3.$$

- c. The reservoir is full when $20T + \frac{240}{\pi} \sin(\pi T/12) = 2500$, which occurs for $T \approx 122.6$ hours.

6.1.50

- a. $\int_0^1 70(1 + \sin(2\pi t)) dt = \left(70t - \frac{35}{\pi} \cos(2\pi t)\right) \Big|_0^1 = 70 - \frac{35}{\pi} + \frac{35}{\pi} = 70$ mL.
- b.

$$\begin{aligned} \int_0^t 70(1 + \sin(2\pi x)) dx &= \left(70x - \frac{35}{\pi} \cos(2\pi x)\right) \Big|_0^t = 70t - \frac{35}{\pi} \cos(2\pi t) + \frac{35}{\pi} \\ &= \frac{35}{\pi} + 70t - \frac{35}{\pi} \cos(2\pi t) \text{ mL.} \end{aligned}$$

- c.

$$\begin{aligned} \int_t^{t+1} 70(1 + \sin(2\pi x)) dx &= \left(70x - \frac{35}{\pi} \cos(2\pi x)\right) \Big|_t^{t+1} = 70t + 70 - \frac{35}{\pi} \cos(2\pi(t+1)) - (70t - \frac{35}{\pi} \cos(2\pi t)) \\ &= \frac{35}{\pi} + 70t - \frac{35}{\pi} \cos(2\pi t) = 70 \text{ mL.} \end{aligned}$$

6.1.51

- a. $V(t) = V(0) + \int_0^t -\frac{\pi}{2} \sin \frac{\pi x}{2} dx = 6 + \left(\cos \frac{\pi x}{2}\right) \Big|_0^t = 5 + \cos \frac{\pi t}{2}$.
- b. Because $\sin \frac{\pi t}{2}$ is periodic with period 4, the breathing cycle repeats every 4 seconds, so there are $\frac{60}{4} = 15$ breaths per minute.
- c. The lungs are full at $t = 0$, at which time $V(0) = 6$ L, so this is the capacity of the lungs. The tidal volume is the difference between this amount and the amount in the lungs after each exhalation, which is the minimum value of $V(t)$. This minimum occurs at the first positive zero of $V'(t)$, which is at $t=2$. Because $V(2) = 4$, the tidal volume is $6 - 4 = 2$ L.

6.1.52 Note that the general solution for $N(t)$ is

$$\begin{aligned} N(t) &= N(0) + \int_0^t N'(x) dx = N(0) + \int_0^t (A \sin(2\pi x/P) + r) dx \\ &= N(0) + \left(-\frac{PA}{2\pi} \cos(2\pi x/P) + rx \right) \Big|_0^t \\ &= N(0) + rt + \frac{PA}{2\pi} (1 - \cos(2\pi t/P)). \end{aligned}$$

- Using the general solution above, we have $N(t) = 10 + \frac{100}{\pi}(1 - \cos(\pi t/5))$. This is never 0, because it is always 10 or more.
- Using the general solution above, we have that $N(t) = 100 + \frac{100}{\pi}(1 - \cos(\pi t/5))$. The population is never extinct, because it is always 100 or more.
- Using the general solution above, we have $N(t) = 10 + 5t + \frac{250}{\pi}(1 - \cos(\pi t/5))$. Again, for $t \geq 0$ this is always at least 10, so the population never becomes extinct.
- Using the general solution above, we have $N(t) = N(0) - 5t + \frac{250}{\pi}(1 - \cos(\pi t/5))$. Suppose $t = 5k$ for a positive even integer k . Then $\cos(\pi t/5) = \cos(k\pi) = 1$, so $N(t) = N(0) - 25k + 0$, which grows negatively without bound as $k \rightarrow \infty$. Thus, there is no choice of $N(0)$ which will ensure that the population won't become extinct.

6.1.53

- $E = \int_0^{24} (300 - 200 \sin(\pi t/12)) dt = \left(300t + \frac{2400}{\pi} \cos(\pi t/12) \right) \Big|_0^{24} = 7200 \text{ MWh}$. This is equivalent to $7.2 \times 10^6 \cdot 3.6 \times 10^6 = 2.592 \times 10^{13}$ Joules.
- For one day, $\frac{7.2 \times 10^6 \text{ KWh}}{450 \text{ Kwh/kg}} = 16,000 \text{ kg coal needed}$.
For one year, $16000 \text{ kg} \times 365 = 5,840,000 \text{ kg coal needed}$.
- For one day, $\frac{7.2 \times 10^6 \text{ KWh}}{1.6 \times 10^4 \text{ Kwh/g}} = 450 \text{ g U-235 need}$.
For one year, $450 \times 365 = 164,250 \text{ g needed}$.
- $\frac{7.2 \times 10^6 \text{ KWh/day}}{(200 \text{ KW/turbine}) \cdot (24 \text{ hours/day})} = 1500 \text{ turbines}$.

6.1.54

- Photosynthesis occurs during daylight hours. So from approximately 5 AM to 5 PM, plants and trees take in CO_2 to produce oxygen, reducing the amount of CO_2 in the atmosphere.
- $\int_0^{24} N(t) dt \approx 2(0.13) + 2(0.14) + 2(0) + 2(-0.4) + 2(-0.42) + 2(-0.39) + 2(-0.3) + 2(-0.21) + 2(0) + 2(0.24) + 2(0.17) + 2(0.16) = -1.76$.
About 1.76 grams of carbon per square meter are taken out of the coniferous forest during an average July day. (Answers will vary.)

6.1.55

- The additional cost is $\int_{100}^{150} (2000 - .5x) dx = \left(2000x - \frac{x^2}{4} \right) \Big|_{100}^{150} = 96875 \text{ dollars}$.

b. The additional cost is $\int_{500}^{550} (2000 - .5x) dx = \left(2000x - \frac{x^2}{4} \right) \Big|_{500}^{550} = 86875$ dollars.

6.1.56

a. The additional cost is $\int_{100}^{150} (200 - .05x) dx = \left(200x - \frac{x^2}{40} \right) \Big|_{100}^{150} = 9687.5$ dollars.

b. The additional cost is $\int_{500}^{550} (200 - .05x) dx = \left(200x - \frac{x^2}{40} \right) \Big|_{500}^{550} = 8687.5$ dollars.

6.1.57

a. The additional cost is $\int_{100}^{150} (300 + 10x - .01x^2) dx = \left(300x + 5x^2 - \frac{x^3}{300} \right) \Big|_{100}^{150} = 69583.33$ dollars.

b. The additional cost is $\int_{500}^{550} (300 + 10x - .01x^2) dx = \left(300x + 5x^2 - \frac{x^3}{300} \right) \Big|_{500}^{550} = 139583.33$ dollars.

6.1.58

a. The additional cost is $\int_{100}^{150} (3000 - x - .001x^2) dx = \left(3000x - .5x^2 - \frac{x^3}{3000} \right) \Big|_{100}^{150} = 142958.33$ dollars.

b. The additional cost is $\int_{500}^{550} (3000 - x - .001x^2) dx = \left(3000x - .5x^2 - \frac{x^3}{3000} \right) \Big|_{500}^{550} = 109958.33$ dollars.

6.1.59

a. False. This would only be the case if the motion was all in the same direction. If the object changes direction at all, then the distance traveled is greater than the displacement.

b. True. This is because $v(t) = |v(t)|$ in this case.

c. True. This is because $R(t) > 0$ for $0 < t < 10$, but $R(t) < 0$ for $t > 10$.

d. True. The cost of increasing production from A to B is given by $\int_A^B C'(t) dt$, which is geometrically the area under the curve $y = C'(x)$ from A to B . If C' is positive and decreasing, there is more area under the curve from A to $2A$ than from $2A$ to $3A$.

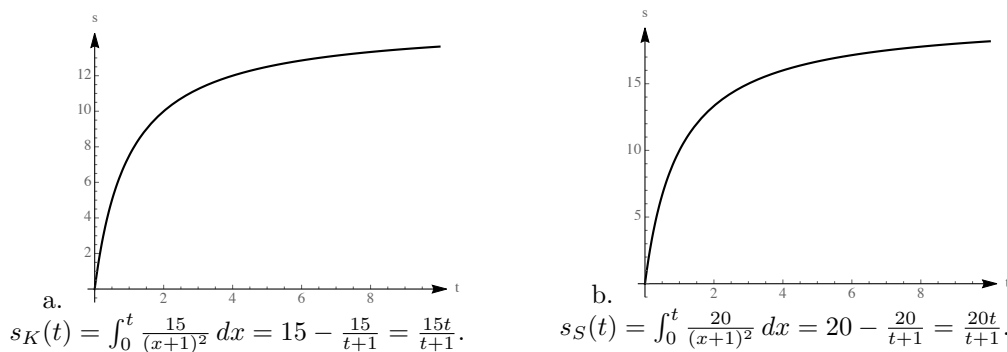
6.1.60 The distance traveled is $\int_0^8 (2t + 6) dt = (t^2 + 6t) \Big|_0^8 = 112$. So the same distance could have been traveled over the given time period at a constant velocity of $\frac{112}{8} = 14$.

6.1.61 The distance traveled is $\int_0^4 \left(1 - \frac{t^2}{16} \right) dt = \left(t - \frac{t^3}{48} \right) \Big|_0^4 = \frac{8}{3}$. So the same distance could have been traveled over the given time period at a constant velocity of $\frac{8/3}{4} = \frac{2}{3}$.

6.1.62 The distance traveled is $\int_0^\pi 2 \sin t dt = (-2 \cos t) \Big|_0^\pi = 4$. So the same distance could have been traveled over the given time period at a constant velocity of $\frac{4}{\pi}$.

6.1.63 The distance traveled is $\int_0^5 t \sqrt{25 - t^2} dt = \frac{1}{2} \int_0^{25} \sqrt{u} du = \left(\frac{1}{3} u^{3/2} \right) \Big|_0^{25} = \frac{125}{3}$. So the same distance could have been traveled over the given time period at a constant velocity of $\frac{125/3}{5} = \frac{25}{3}$.

6.1.64

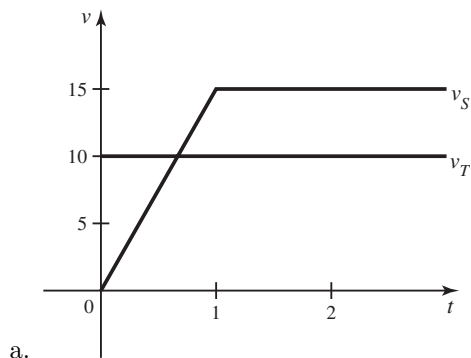


c. They meet when $\frac{15t}{t+1} + \frac{20t}{t+1} = 20$, which occurs for $t = \frac{4}{3}$. (Which represents 1:20 PM.) At this time, Kelly has gone $s_K(4/3) = \frac{20}{7/3} = \frac{60}{7}$ km, and Sandy has gone $s_S(4/3) = \frac{80}{7}$ km.

d. We would need $\frac{At}{t+1} + \frac{Bt}{t+1} = D$ to have a solution. If we solve for t , we obtain $t = \frac{D}{A+B-D}$, so we need $A+B > D$.

e. The maximum distances are A and B respectively, because $\lim_{t \rightarrow \infty} \frac{At}{t+1} = A$ and $\lim_{t \rightarrow \infty} \frac{Bt}{t+1} = B$.

6.1.65



b. After 1 hour, Theo has ridden $1 \cdot 10 = 10$ miles, and Sasha has ridden $\frac{1}{2} \cdot 15 = 7.5$ miles, so Theo has ridden farther.

c. After 2 hours, Theo has ridden $2 \cdot 10 = 20$ miles, and Sasha has ridden $7.5 + 15 \cdot 1 = 22.5$ miles, so Sasha has ridden farther.

d. The times when they arrive at the various mile markers are in the following table:

	10	15	20
Theo	1	$3/2$	2
Sasha	$7/6$	$3/2$	$11/6$

Note that Theo hits the 10 mile marker first, then they are tied as they hit the 15 mile marker, and Sasha hits the 20 mile marker first. The area under v_S is the same as the area under v_T for $t = 1.5$, for $t < 1.5$ the area under v_T is greater, and for $t > 1.5$, the area under v_S is greater.

e. Theo will then hit the 20 mile mark in $18.8/10 = 1.88$ hours. Sasha hits the 20 mile mark at $t = 11/6 \approx 1.833$ hours, so Sasha will win.

f. A head start of 0.2 hours is equivalent for Theo of $10 \cdot 0.2 = 2$ miles. It will take him $18/10 = 1.8$ hours to ride the other 18 miles, while it still takes Sasha about 1.83 hours to cover 20 miles, so Theo will win.

6.1.66

a. $s_A(t) = \int_0^t \frac{4}{x+1} dx = (4 \ln(x+1)) \Big|_0^t = 4 \ln(t+1).$

$$s_B(t) = 2 + \int_0^t \frac{2}{x+1} dx = 2 + (2 \ln(x+1)) \Big|_0^t = 2 + 2 \ln(t+1).$$

- b. This would occur if $4 \ln(t+1) = 2 + 2 \ln(t+1)$, or $\ln(t+1) = 1$, which occurs for $t = e - 1$, which is about 1 hour and 43 minutes.

6.1.67 Let $D(t)$ equal the depth of the snow t hours after it started snowing and let $t = T$ correspond to noon. This means that $D(0) = 0$ and because the snow piles up on the road at a constant rate, $D'(t) = k$ for some positive constant k . It follows that

$$D(t) = D(0) + \int_0^t k dx = kt.$$

Let $v(t)$ equal the velocity of the plow t hours after it started snowing. The plowing rate is inversely proportional to the depth of the snow which means that $v(t) = \frac{C}{kt}$ for some constant C and for $t \geq T$. Because the plow goes twice as far from noon to 1 PM as from 1 PM to 2 PM,

$$\int_T^{T+1} \frac{C}{kt} dt = 2 \int_{T+1}^{T+2} \frac{C}{kt} dt.$$

Evaluating the integrals, we have $\frac{C}{k} \ln \left(\frac{T+1}{T} \right) = \frac{2C}{k} \ln \left(\frac{T+2}{T+1} \right)$, or $\ln \left(\frac{T+1}{T} \right) = \ln \left(\frac{T+2}{T+1} \right)^2$. This implies that $(T+1)^3 = T(T+2)^2$. Expanding both sides and collecting terms gives the quadratic equation $T^2 + T - 1 = 0$, which has a positive root of $T = \frac{-1 + \sqrt{5}}{2} \approx 0.618$ hours ≈ 37 minutes. So it started snowing at approximately 11:23 AM.

6.1.68

- a. $y(t)$ is the position of the projectile, and the derivative of position is velocity, and the derivative of velocity is acceleration. Note that the acceleration force is due to gravity, and only depends on the position y of the projectile.

b. $\frac{1}{2} \frac{d}{dy} (v^2) = \frac{1}{2} \cdot 2v \frac{dv}{dy} = \frac{dy}{dt} \cdot \frac{dv}{dy} = \frac{dv}{dt}.$

c. Because $\frac{dv}{dt} = a(y)$ and $\frac{dv}{dt} = \frac{1}{2} \frac{d}{dy} (v^2)$, we must have $\frac{1}{2} \frac{d}{dy} (v^2) = a(y).$

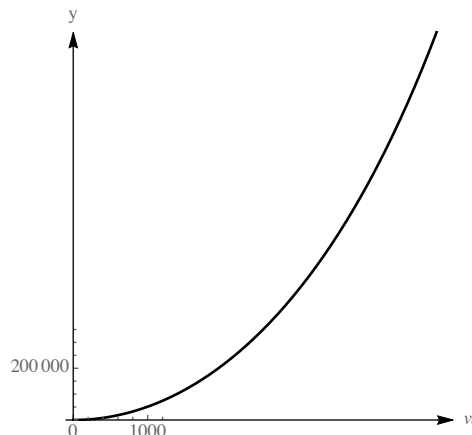
d. $\frac{1}{2} \int \frac{d}{dy} (v^2) dy = \int a(y) dy$, so $\frac{1}{2} (v^2) = \int -\frac{g}{(1+y/R)^2} dy = gR \left(\frac{1}{1+y/R} \right) + D$. Now when $t = 0$, we have $v = v_0$ and $y = 0$, so $\frac{1}{2} v_0^2 = gR + D$, so $D = -gR + \frac{1}{2} v_0^2$, and we can write

$$\frac{1}{2} (v^2 - v_0^2) = gR \left(\frac{1}{1+y/R} - 1 \right).$$

e. When $v = 0$ we have $-\frac{1}{2} v_0^2 \cdot \frac{1}{gR} + 1 = \frac{1}{1+y/R}$, so $\frac{1}{1+y/R} = \frac{2gR - v_0^2}{2gR}$, so $1+y/R = \frac{2gR}{2gR - v_0^2}$, and

$$y = \frac{2gR^2}{2gR - v_0^2} - R = \frac{2gR^2}{2gR - v_0^2} - \frac{2gR^2 - Rv_0^2}{2gR - v_0^2} = \frac{Rv_0^2}{2gR - v_0^2}.$$

- f. $y_{\max}(500) \approx 12,780$ m.
 $y_{\max}(1500) \approx 116,893$ m.
 $y_{\max}(5000) \approx 1,592,990$ m.



- g. The denominator of the expression for $y_{\max}$ is 0 when $2gR = v_0^2$, so when $v_0 = \sqrt{2gR}$. As $v_0 \rightarrow \sqrt{2gR}$, $y_{\max} \rightarrow \infty$.

6.1.69 Using the Fundamental Theorem, we have

$$\int_a^b f'(x) dx = f(b) - f(a) = g(b) - g(a) = \int_a^b g'(x) dx.$$

6.1.70 Let $f(t)$ represent the position of the first runner at time t and let $g(t)$ represent the position of the second runner over the time interval $[a, b]$. We are given that $f(a) = g(a)$ and $f(b) = g(b)$. Then by problem 69, we have

$$\int_a^b f'(t) dt = \int_a^b g'(t) dt,$$

so the displacements of the two runners is the same, even though it might not be the case that $f'(t) = g'(t)$ for every t .

6.1.71 If $f(x)$ is the elevation of trail one at position x and $g(x)$ is the elevation of trail two at position x , then because we are given that $f(a) = g(a)$ and $f(b) = g(b)$, we must have that

$$\int_a^b f'(x) dx = \int_a^b g'(x) dx,$$

so the two trails have the same net change in elevation.

6.1.72 Let $f(x) = 12 \sin(\pi x^2)$ and $g(x) = x^{10}(2 - x)^2$. Note that $f(0) = 0 = g(0)$ and $f(2) = 0 = g(2)$. Thus by problem 69, $\int_0^2 f'(x) dx = \int_0^2 g'(x) dx$, which is equivalent to the statement we are trying to prove.

6.2 Regions Between Curves

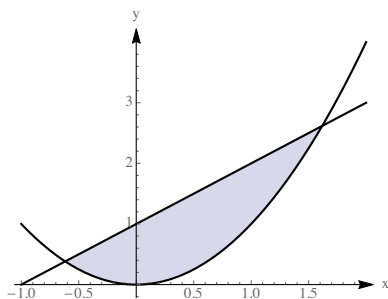
6.2.1 $\int_a^b (f(x) - g(x)) dx + \int_b^c (g(x) - f(x)) dx.$

6.2.2

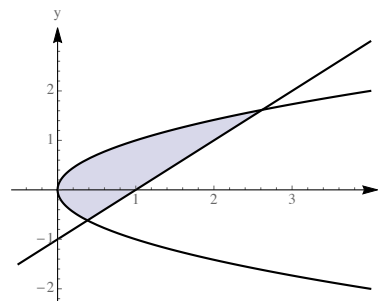
a. $\int_0^4 \left(\sqrt{x} - \frac{1}{2}x \right) dx.$

b. $\int_0^2 (2y - y^2) dy.$

6.2.3



6.2.4



6.2.5

a. The area is given by $\int_0^1 ((2-x) - x) dx = \int_0^1 (2-2x) dx = (2x - x^2) \Big|_0^1 = 1$.

b. The area is that of a triangle with base 2 and height 1, so its area is $\frac{1}{2} \cdot 2 \cdot 1 = 1$.

6.2.6

a. The area is given by $\int_0^1 ((y+1) - 2y) dy = \int_0^1 (1-y) dy = \left(y - \frac{y^2}{2}\right) \Big|_0^1 = \frac{1}{2}$.

b. The area is that of a triangle with base 1 and height 1, so its area is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$.

6.2.7 $\int_0^1 y dy + \int_1^2 (2-y) dy$.

6.2.8 $\int_0^1 \frac{x}{2} dx + \int_1^2 \left(\frac{x}{2} - x + 1\right) dx$.

6.2.9 The curves intersect when $x = x^2 - 2$, or $(x+1)(x-2) = 0$, so at $x = -1$ and $x = 2$. The area is

$$\int_{-1}^2 (x - (x^2 - 2)) dx = \left(2x + \frac{x^2}{2} - \frac{x^3}{3}\right) \Big|_{-1}^2 = 4.5.$$

6.2.10 The curves intersect when $x^2 - 2x = -x^2 + 4x$, or $2x^2 - 6x = 0$. This can be written as $2x(x-3) = 0$, so $x = 0$ and $x = 3$ are the x -values of the points of intersection. The area is given by

$$\int_0^3 (-x^2 + 4x - (x^2 - 2x)) dx = \int_0^3 (-2x^2 + 6x) dx = \left(\frac{-2x^3}{3} + 3x^2\right) \Big|_0^3 = -18 + 27 = 9.$$

6.2.11 The curves intersect when $\frac{6}{x^2+1} = 3x^2$, or $3x^2(x^2+1) = 6$. This can be written as $3x^4 + 3x^2 - 6 = 0$, or $(x^4 + x^2 - 2) = 0$. Factoring gives $(x^2+2)(x^2-1) = (x^2+2)(x-1)(x+1) = 0$, so the curves intersect at $x = -1$ and $x = 1$. Using symmetry, the area is given by

$$2 \int_0^1 \left(\frac{6}{x^2+1} - 3x^2\right) dx = 2 \left(6 \tan^{-1} x - x^3\right) \Big|_0^1 = 2 \left(\frac{6\pi}{4} - 1\right) = 3\pi - 2.$$

6.2.12 The curves intersect where $\sin x = -\cos x$, which is where $\tan x = -1$, or $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$. The area is given by

$$\int_{-\pi/4}^{3\pi/4} (\sin x - (-\cos x)) dx = (-\cos x + \sin x) \Big|_{-\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}.$$

6.2.13 By inspection, the curves intersect at $x = 1$. The area is given by

$$\int_0^1 (3 - x - 2^x) dx = \left(3x - x^2/2 - \frac{2^x}{\ln 2} \right) \Big|_0^1 = 3 - \frac{1}{2} - \frac{2}{\ln 2} - \left(-\frac{1}{\ln 2} \right) = \frac{5}{2} - \frac{1}{\ln 2}.$$

6.2.14 The given curve is 0 when $\sin x = 0$ (because $\cos x - 2$ is never zero), which occurs for $x = 0$ and $x = \pi$. The area is given by

$$\int_0^\pi (0 - (\cos x - 2) \sin x) dx = - \int_0^\pi (\cos x - 2) \sin x dx.$$

Let $u = \cos x - 2$. Then $du = -\sin x dx$, and substituting gives

$$\int_{-1}^{-3} u du = \frac{u^2}{2} \Big|_{-1}^{-3} = \frac{9}{2} - \frac{1}{2} = 4.$$

6.2.15 The curves intersect at $\pi/4$, so the area is given by $\int_0^{\pi/4} \sin x dx + \int_{\pi/4}^{\pi/2} \cos x dx = (-\cos x) \Big|_0^{\pi/4} + (\sin x) \Big|_{\pi/4}^{\pi/2} = (1 - \sqrt{2}/2) + (1 - \sqrt{2}/2) = 2 - \sqrt{2}$.

6.2.16 The curves intersect when $x = x^3$, or $x(x-1)(x+1) = 0$, so at $x = 0$, $x = -1$ and $x = 1$. The area is given by

$$\int_{-1}^0 (x^3 - x) dx + \int_0^1 (x - x^3) dx = 2 \int_0^1 (x - x^3) dx = 2 \left(x^2/2 - x^4/4 \right) \Big|_0^1 = \frac{1}{2}.$$

6.2.17 The curves intersect when $x^2 - 4x = -2x$, or $x^2 - 2x = x(x-2) = 0$, so $x = 0$ and $x = 2$. The area is given by

$$\int_0^2 (-2x - (x^2 - 4x)) dx = \int_0^2 (2x - x^2) dx = \left(x^2 - \frac{x^3}{3} \right) \Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}.$$

6.2.18 The curves intersect when $\frac{\sec^2 x}{4} = 4 \cos^2 x$, which can be written as $\cos^4 x = \frac{1}{16} = \frac{1}{2^4}$. This occurs when $\cos x = \pm \frac{1}{2}$, which occurs at $-\pi/3$ and $\pi/3$. Using symmetry, the area is given by

$$\begin{aligned} 2 \int_0^{\pi/3} (4 \cos^2 x - (1/4) \sec^2 x) dx &= 2 \int_0^{\pi/3} (2(1 + \cos 2x) - (1/4) \sec^2 x) dx \\ &= 2 \left(2x + \sin 2x - (1/4) \tan x \right) \Big|_0^{\pi/3} = \frac{4\pi}{3} + \sqrt{3} - \frac{\sqrt{3}}{2} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}. \end{aligned}$$

6.2.19 The curves intersect when $2y = y^2 - 3$, or $y^2 - 2y - 3 = (y-3)(y+1) = 0$. This occurs for $y = 3$ and $y = -1$. The area is given by

$$\int_{-1}^3 (2y - (y^2 - 3)) dy = \left(y^2 - \frac{y^3}{3} + 3y \right) \Big|_{-1}^3 = (9 - 9 + 9) - (1 + \frac{1}{3} - 3) = \frac{32}{3}.$$

6.2.20 The area is given by

$$\int_0^4 \left(\sqrt{y} - \frac{y}{4} \right) dy = \left(\frac{2y^{3/2}}{3} - \frac{y^2}{8} \right) \Big|_0^4 = \frac{16}{3} - 2 = \frac{10}{3}.$$

6.2.21 Note that the two curves intersect where $y = 2 - y^2$, so for $y^2 + y - 2 = (y+2)(y-1) = 0$. So they intersect for $y = -2$ and $y = 1$. The area is given by

$$\int_{-2}^1 (4 - 2y - 2y^2) dy = \left(4y - y^2 - \frac{2}{3}y^3 \right) \Big|_{-2}^1 = \left(4 - 1 - \frac{2}{3} \right) - \left(-8 - 4 + \frac{16}{3} \right) = 9.$$

6.2.22 The curves $-\sin 2y$ and $\cos y$ intersect where $-\sin 2y = \cos y$, but $-\sin 2y = -2\sin y \cos y$, so we must solve $-2\sin y \cos y = \cos y$. So either $\cos y = 0$ or $\sin y = -\frac{1}{2}$. The first positive value of y satisfying either of these conditions is $y = \frac{\pi}{2}$, where $\cos y = 0$; the first negative value of y is $y = -\frac{\pi}{6}$ where $\sin y = -\frac{1}{2}$. So the area is

$$\begin{aligned} \int_{-\pi/6}^{\pi/2} (\cos y - (-\sin 2y)) dy &= \left(\sin y - \frac{1}{2} \cos 2y \right) \Big|_{-\pi/6}^{\pi/2} \\ &= \sin \frac{\pi}{2} - \frac{1}{2} \cos \pi - \sin \left(-\frac{\pi}{6} \right) + \frac{1}{2} \cos \left(-\frac{\pi}{3} \right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{9}{4}. \end{aligned}$$

6.2.23 The area is given by

$$\int_{-4}^5 \left(8 - y - \frac{(y-2)^2}{3} \right) dy = \left(8y - \frac{y^2}{2} - \frac{(y-2)^3}{9} \right) \Big|_{-4}^5 = 40 - \frac{25}{2} - 3 - (-32 - 8 + 24) = \frac{81}{2}.$$

6.2.24 The curves intersect where $2\sin 2y = 2\cos y$, or $4\sin y \cos y = 2\cos y$. If $\cos y \neq 0$, this reduces to $\sin y = \frac{1}{2}$, which occurs for $y = \frac{\pi}{6}$. If $\cos y = 0$, then we have $y = \pm \frac{\pi}{2}$. The area is therefore given by

$$\begin{aligned} &\int_{-\pi/2}^{\pi/6} (2\cos y - 2\sin 2y) dy + \int_{\pi/6}^{\pi/2} (2\sin 2y - 2\cos y) dy \\ &= (2\sin y + \cos 2y) \Big|_{-\pi/2}^{\pi/6} + (-\cos 2y - 2\sin y) \Big|_{\pi/6}^{\pi/2} \\ &= \left(1 + \frac{1}{2} + 2 + 1 \right) + \left(1 - 2 + \frac{1}{2} + 1 \right) = 5. \end{aligned}$$

6.2.25 The two curves meet at $x = 0$; at that point $y = 1$. So the integration is from $y = 0$ to $y = 1$. Solving for x gives the curves $x = 2y^2 - 2$ and $x = 1 - y^2$. Thus the area of the region is

$$\int_0^1 ((1 - y^2) - (2y^2 - 2)) dy = \int_0^1 (-3y^2 + 3) dy = (-y^3 + 3y) \Big|_0^1 = 2.$$

6.2.26 Note that (by inspection) the curves intersect for $y = 1$ and $y = -1$. Using symmetry, the area is

$$2 \int_0^1 ((2 - y^2) - y) dy = 2 \left(2y - \frac{y^3}{3} - \frac{y^2}{2} \right) \Big|_0^1 = 2 \left(2 - \frac{1}{3} - \frac{1}{2} \right) = 3 - \frac{2}{3} = \frac{7}{3}.$$

6.2.27 The curves intersect when $x^2 - 2 = x$, or $x^2 - x - 2 = (x-2)(x+1) = 0$, or $x = -1$ and $x = 2$. The relevant point for the shaded region is $(-1, -1)$.

The area is

$$\int_{-1}^0 (y - (-\sqrt{y+2})) dy = \int_{-1}^0 (y + \sqrt{y+2}) dy = \left(\frac{y^2}{2} + \frac{2(y+2)^{3/2}}{3} \right) \Big|_{-1}^0 = \frac{2\sqrt{8}}{3} - \left(\frac{1}{2} + \frac{2}{3} \right) = \frac{8\sqrt{2} - 7}{6}.$$

6.2.28 Note that $\sin 2x = 2\sin x \cos x$, so the curves intersect where $2\sin x \cos x = \sin x$, so the curves intersect at $x = 0$ and $x = \pi$, and $x = \pi/3$, because $\cos \pi/3 = \frac{1}{2}$. The area is given by

$$\begin{aligned} &\int_0^{\pi/3} (\sin 2x - \sin x) dx + \int_{\pi/3}^{\pi} (\sin x - \sin 2x) dx \\ &= (-\cos(2x)/2 + \cos x) \Big|_0^{\pi/3} + (-\cos(x) + \cos(2x)/2) \Big|_{\pi/3}^{\pi} \\ &= \left(\frac{1}{4} + \frac{1}{2} - \left(\left(-\frac{1}{2} \right) + 1 \right) \right) + \left(1 + \left(\frac{1}{2} \right) - \left(\left(-\frac{1}{2} \right) + \left(-\frac{1}{4} \right) \right) \right) = \frac{1}{4} + \frac{9}{4} = \frac{5}{2}. \end{aligned}$$

6.2.29 The curves $x = y^2 - 3y + 12$ and $x = -2y^2 - 6y + 30$ intersect where $y^2 - 3y + 12 = -2y^2 - 6y + 30$, or $3y^2 + 3y - 18 = 3(y + 3)(y - 2) = 0$. So the points of intersection are $y = -3$ and $y = 2$. Thus the area is

$$\begin{aligned}\int_{-3}^2 ((-2y^2 - 6y + 30) - (y^2 - 3y + 12)) dy &= \int_{-3}^2 (-3y^2 - 3y + 18) dy \\ &= \left(-y^3 - \frac{3}{2}y^2 + 18y \right) \Big|_{-3}^2 \\ &= -8 - 6 + 36 - 27 + \frac{27}{2} + 54 = \frac{125}{2}.\end{aligned}$$

6.2.30 These two curves intersect where $y^3 - 4y^2 + 3y = y^2 - y$, or $y^3 - 5y^2 + 4y = y(y - 4)(y - 1) = 0$. The three intersection points occur where $y = 0$, $y = 4$, and $y = 1$. Note that from $y = 0$ to $y = 1$ we have $y^2 - y \leq y^3 - 4y^2 + 3y$; from $y = 1$ to $y = 4$ this is reversed. Thus the area between the curves is

$$\begin{aligned}\int_0^1 (y^3 - 4y^2 + 3y - (y^2 - y)) dy + \int_1^4 (y^2 - y - (y^3 - 4y^2 + 3y)) dy \\ &= \int_0^1 (y^3 - 5y^2 + 4y) dy + \int_1^4 (-y^3 + 5y^2 - 4y) dy \\ &= \left(\frac{1}{4}y^4 - \frac{5}{3}y^3 + 2y^2 \right) \Big|_0^1 + \left(-\frac{1}{4}y^4 + \frac{5}{3}y^3 - 2y^2 \right) \Big|_1^4 \\ &= \frac{1}{4} - \frac{5}{3} + 2 - 64 + \frac{320}{3} - 32 + \frac{1}{4} - \frac{5}{3} + 2 = \frac{71}{6}.\end{aligned}$$

6.2.31

a. The area is given by $\int_0^1 (\sqrt{x} - x^3) dx$.

b. The area can also be written $\int_0^1 (\sqrt[3]{y} - y^2) dy$.

6.2.32

a. The area is given by $\int_0^2 (-(x^2 - 4x)) dx + \int_2^4 (-(2x - 8)) dx$.

b. The area can also be written as $\int_{-4}^0 (y/2 + 4 - (2 - \sqrt{y+4})) dy$. Note that in order to solve $y = x^2 - 4x$ for x , we needed to complete the square to obtain $y + 4 = x^2 - 4x + 4 = (x - 2)^2$, so $\sqrt{y+4} = |x - 2|$, and the part of this we need is $x = 2 - \sqrt{y+4}$.

6.2.33 The area is $\int_0^1 (10\sqrt{t} - 7t) dt = \left(\frac{20}{3}t^{3/2} - \frac{7t^2}{2} \right) \Big|_0^1 = \frac{20}{3} - \frac{7}{2} \approx 3.17$ km.

The area under the curve v_1 equals the distance traveled by the first runner and the area under the curve v_2 is the distance traveled by the second runner. So the area between the curves is the difference in the distance covered by the faster runner and the slower runner, which means that the faster runner ran about 3.17 km further than the slower runner.

6.2.34

a. The area of R_1 is $\int_0^1 (3 - x - 2x^2) dx = \left(3x - \frac{x^2}{2} - \frac{2x^3}{3} \right) \Big|_0^1 = 3 - \frac{1}{2} - \frac{2}{3} = \frac{11}{6}$.

b. The area of the R_1 and R_2 together is the area of a triangle with base 3 and height 3 so it is $\frac{9}{2}$.
Therefore, the area of R_2 is $\frac{9}{2} - \frac{11}{6} = \frac{16}{6} = \frac{8}{3}$.

6.2.35

- a. The area of R_1 is given by $\int_0^1 6x(2-x^2)^2 dx$. Let $u = 2-x^2$. Then $du = -2x dx$, and substituting yields

$$\int_2^1 -3u^2 du = (-u^3) \Big|_2^1 = -1 - (-8) = 7.$$

- b. The area of R_1 minus the area of a triangle with base 1 and height 6 is equal to the area of R_2 . Thus, the area of R_2 is $7 - 3 = 4$.

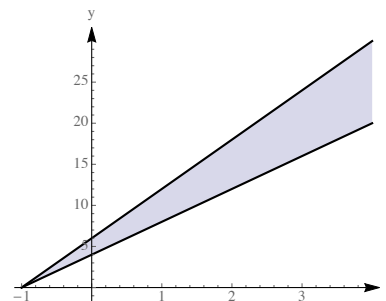
6.2.36

- a. The area of R_1 is $\int_0^{\pi/2} \cos y dy = \sin y \Big|_0^{\pi/2} = 1$.

- b. Together, R_1 and R_2 comprise a $1 \times \frac{\pi}{2}$ rectangle. So the area of R_2 is $\frac{\pi}{2} - 1$.

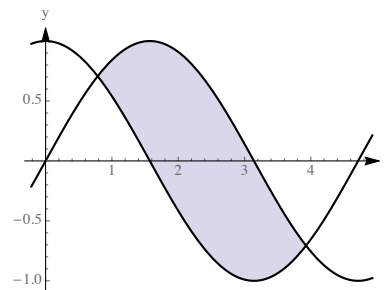
The nonvertical lines intersect when $4x+4 = 6x+6$, or $x = -1$. The vertical line $x = 4$ intersects both nonvertical lines when $x = 4$. The area is

6.2.37 given by $\int_{-1}^4 (6x+6 - (4x+4)) dx =$
 $\int_{-1}^4 (2x+2) dx = (x^2 + 2x) \Big|_{-1}^4 =$
 $16 + 8 - (1 - 2) = 25.$



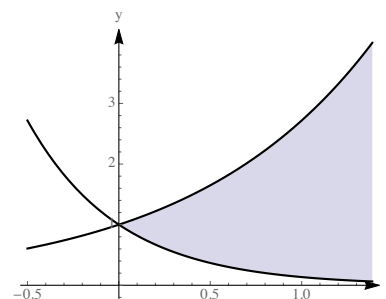
$\cos x = \sin x$ when $\tan x = 1$, or $x = \tan^{-1}(1) = \pi/4$. They also intersect at $\pi/4 + \pi = 5\pi/4$.

6.2.38 The area is given $\int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx =$
 $(-\cos x + \sin x) \Big|_{\pi/4}^{5\pi/4} = -((- \sqrt{2}/2 - \sqrt{2}/2) -$
 $(\sqrt{2}/2 + \sqrt{2}/2)) = 2\sqrt{2}.$



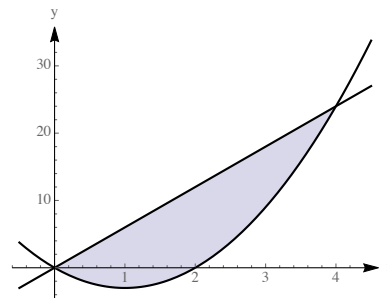
The curves e^x and e^{-2x} intersect when $e^x = e^{-2x}$, or $e^{3x} = 1$, which occurs only for $x = 0$. The vertical line $x = \ln 4$ clearly intersects the curves

6.2.39 for $x = \ln 4$. The area is given by $\int_0^{\ln 4} (e^x -$
 $e^{-2x}) dx = (e^x + e^{-2x}/2) \Big|_0^{\ln 4} = 4 + (1/32) - (1 +$
 $(1/2)) = 81/32.$



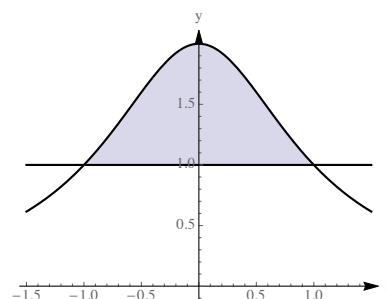
The curves intersect for $3x^2 - 6x = 6x$, or $3x^2 - 12x = 3x(x - 4) = 0$, so for $x = 0$ and $x = 4$.

6.2.40 The area is $\int_0^4 (6x - (3x^2 - 6x)) dx = \int_0^4 (12x - 3x^2) dx = (6x^2 - x^3) \Big|_0^4 = 96 - 64 = 32$.



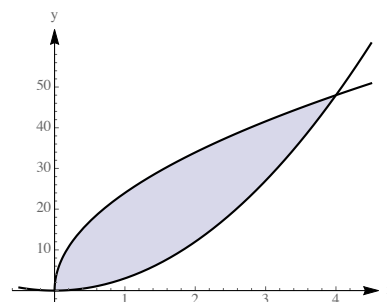
The curves intersect when $\frac{2}{1+x^2} = 1$, or $x^2 + 1 = 2$, or $x^2 - 1 = (x - 1)(x + 1) = 0$. So the intersections occur when $x = \pm 1$. Using

6.2.41 symmetry, the area is $2 \int_0^1 \left(\frac{2}{1+x^2} - 1 \right) dx = 2 \left(2 \tan^{-1} x - x \right) \Big|_0^1 = \pi - 2$.



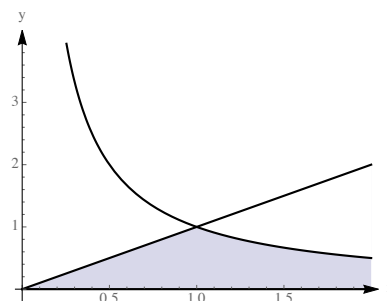
The curves intersect when $24\sqrt{x} = 3x^2$, so when either $x = 0$ or when $8 = x^{3/2}$. Thus happens for

6.2.42 $x = 8^{2/3} = 4$. We have $\int_0^4 (24\sqrt{x} - 3x^2) dx = (16x^{3/2} - x^3) \Big|_0^4 = 128 - 64 = 64$.



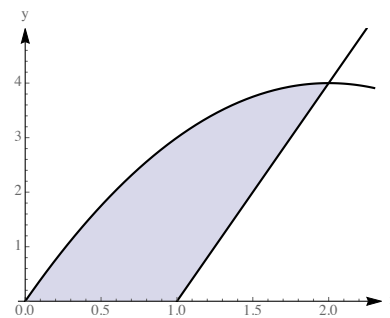
The curves intersect when $x = 1/x$, or $x^2 = 1$. This occurs in the first quadrant when $x = 1$. The

6.2.43 area is given by $\int_0^1 x dx + \int_1^2 1/x dx = (x^2/2) \Big|_0^1 + (\ln x) \Big|_1^2 = 1/2 + \ln 2$.



Note that the curves intersect when $4x - x^2 = 4x - 4$, or $x^2 - 4 = 0$. We see that the curves intersect for $x > 0$ when $x = 2$. The line $y = 4x - 4$ intersects the x -axis at $x = 1$. The area is given by $\int_0^1 (4x - x^2) dx + \int_1^2 (4x - x^2 - (4x - 4)) dx = \int_0^1 (4x - x^2) dx + \int_1^2 (4 - x^2) dx = (2x^2 - x^3/3) \Big|_0^1 + (4x - x^3/3) \Big|_1^2 = (2 - 1/3) + (8 - 8/3) - (4 - 1/3) = \frac{10}{3}$.

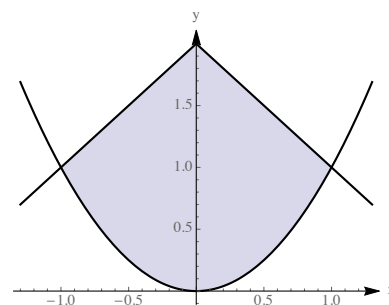
6.2.44



The curves intersect at $x = \pm 1$. Using symmetry, the area is

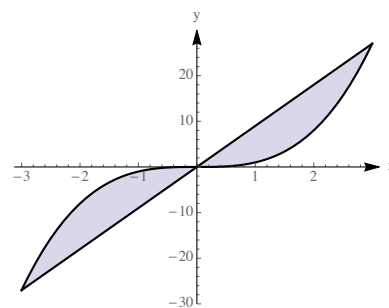
6.2.45

$$2 \int_0^1 (2 - x - x^2) dx = 2 \left(2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_0^1 = \frac{7}{3}.$$



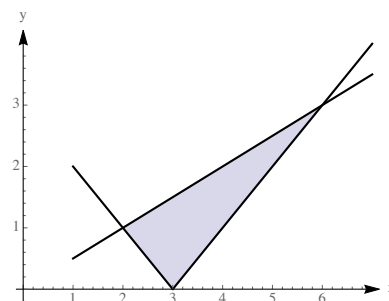
Using symmetry, the area is $2 \int_0^3 (9x - x^3) dx = 2 \left(9x^2/2 - x^4/4 \right) \Big|_0^3 = 2(81/2 - 81/4) = 81/2$.

6.2.46



6.2.47

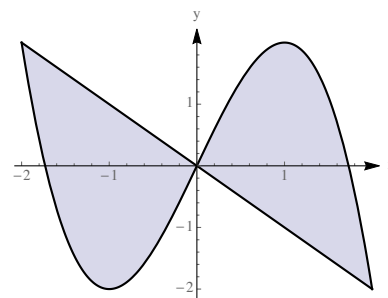
$$\begin{aligned} \int_2^6 (x/2 - |x - 3|) dx &= \int_2^3 (x/2 - (3 - x)) dx + \int_3^6 (x/2 - (x - 3)) dx \\ &= \int_2^3 (3x/2 - 3) dx + \int_3^6 (3 - x/2) dx \\ &= (3x^2/4 - 3x) \Big|_2^3 + (3x - x^2/4) \Big|_3^6 \\ &= 27/4 - 9 - (3 - 6) + (18 - 9 - (9 - 9/4)) = 3/4 + 9/4 = 3. \end{aligned}$$



The curves intersect for $3x - x^3 = -x$, or $x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2) = 0$, so for $x = 0$ and $x = \pm 2$. Using symmetry, the area is given by

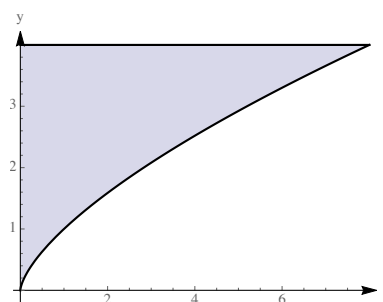
6.2.48

$$2 \int_0^2 (3x - x^3 + x) dx = 2 \left(2x^2 - \frac{x^4}{4} \right) \Big|_0^2 = 16 - 8 = 8.$$



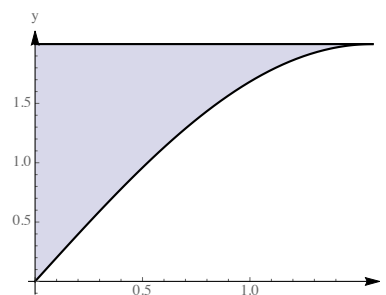
6.2.49

The area is given by $\int_0^8 (4 - x^{2/3}) dx = \left(4x - 3x^{5/3}/5 \right) \Big|_0^8 = 32 - 96/5 = 64/5$.



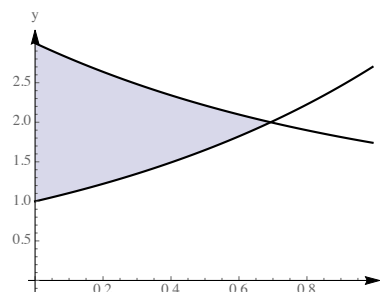
6.2.50

The area is given by $\int_0^{\pi/2} (2 - 2\sin x) dx = (2x + 2\cos x) \Big|_0^{\pi/2} = \pi - 2$.



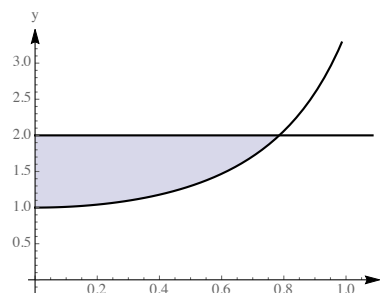
6.2.51

The area is given by $\int_0^{\ln 2} (2e^{-x} + 1 - e^x) dx = (-2e^{-x} + x - e^x) \Big|_0^{\ln 2} = -1 + \ln 2 - 2 - (-2 - 1) = \ln 2$.



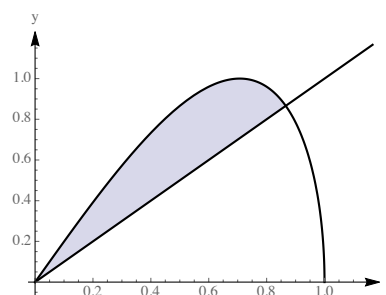
6.2.52

The area is given by $\int_0^{\pi/4} (2 - \sec^2 x) dx =$
 $(2x - \tan x) \Big|_0^{\pi/4} = \frac{\pi}{2} - 1.$



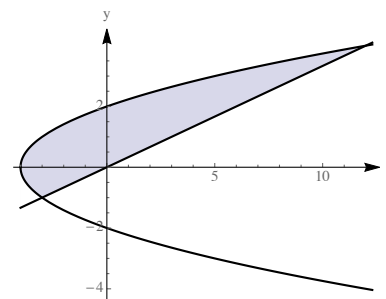
6.2.53

The area is given by $\int_0^{\sqrt{3}/2} (2x\sqrt{1-x^2} - x) dx =$
 $\left(-\frac{2}{3}(1-x^2)^{3/2} - \frac{x^2}{2} \right) \Big|_0^{\sqrt{3}/2} = \frac{-2}{24} - \frac{3}{8} + \frac{2}{3} = \frac{5}{24}.$



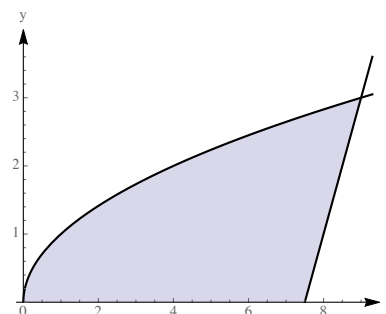
6.2.54

The area is given by $\int_{-1}^4 (3y - (y^2 - 4)) dy =$
 $\left(\frac{3y^2}{2} - \frac{y^3}{3} + 4y \right) \Big|_{-1}^4 = 24 - 64/3 + 16 - (3/2 +$
 $1/3 - 4) = 44 - 65/3 - 3/2 = \frac{125}{6}.$



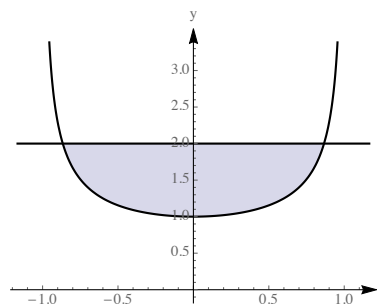
6.2.55

The curves intersect at the point (9,3). The
 area is given by $\int_0^3 \left(\frac{y}{2} + \frac{15}{2} - y^2 \right) dy =$
 $\left(\frac{y^2}{4} + \frac{15y}{2} - \frac{y^3}{3} \right) \Big|_0^3 = \frac{9}{4} + \frac{45}{2} - 9 = \frac{63}{4}.$



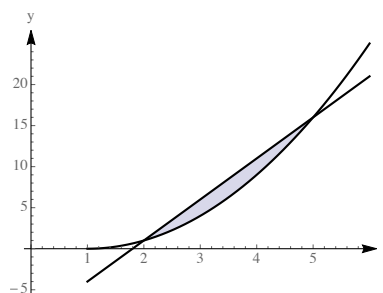
6.2.56

Note that the curves intersect for $2 = \frac{1}{\sqrt{1-x^2}}$, or $1 - x^2 = \frac{1}{4}$, or $x = \pm \frac{\sqrt{3}}{2}$. Using symmetry, the area is given by $2 \int_0^{\sqrt{3}/2} \left(2 - \frac{1}{\sqrt{1-x^2}} \right) dx = 2 \left(2x - \sin^{-1}(x) \right) \Big|_0^{\sqrt{3}/2} = 2\sqrt{3} - 2\pi/3$.



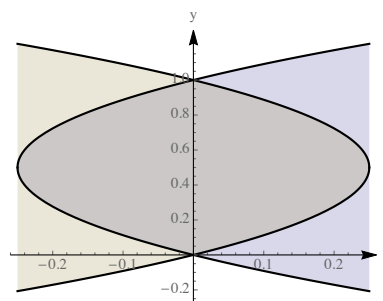
6.2.57

The curves intersect for $x^2 - 2x + 1 = 5x - 9$, or $x^2 - 7x + 10 = (x-5)(x-2) = 0$, so for $x=2$ and $x=5$. The area is given by $\int_2^5 (5x-9 - (x-1)^2) dx = (5x^2/2 - 9x - (x-1)^3/3) \Big|_2^5 = 125/2 - 45 - 64/3 - (10 - 18 - (1/3)) = 4.5$.



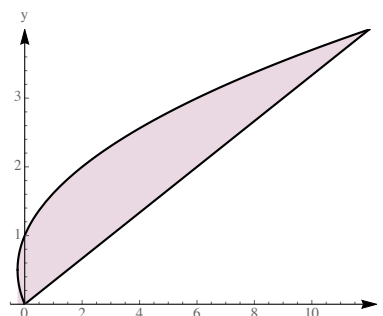
6.2.58

Using symmetry, this is $2 \int_0^1 (-y(y-1)) dy = 2 \left(-y^3/3 + y^2/2 \right) \Big|_0^1 = 1/3$.



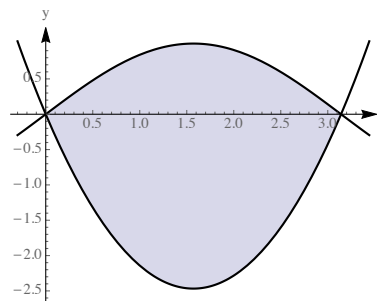
6.2.59

This is given by $\int_0^4 (3y - (y^2 - y)) dy = \int_0^4 (4y - y^2) dy = (2y^2 - y^3/3) \Big|_0^4 = 32 - 64/3 = 32/3$.



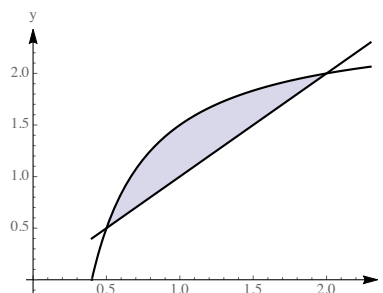
6.2.60

The area is given by $\int_0^\pi (\sin x - x^2 + \pi x) dx =$
 $\left. (-\cos x - x^3/3 + \pi x^2/2) \right|_0^\pi = 1 - \pi^3/3 + \pi^3/2 -$
 $(-1) = 2 + \pi^3/6.$



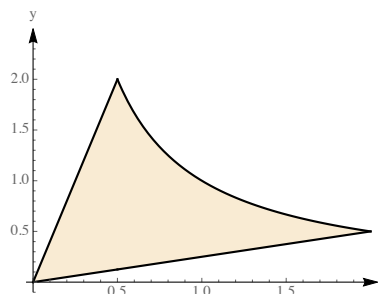
6.2.61

This area is given by $\int_{1/2}^2 (5/2 - 1/x - x) dx =$
 $\left. (5x/2 - \ln x - x^2/2) \right|_{1/2}^2 = 5 - \ln 2 - 2 - (5/4 -$
 $\ln(1/2) - 1/8) = 15/8 - 2 \ln 2.$



6.2.62

This area is given by $\int_0^{1/2} (4x - x/4) dx + \int_{1/2}^2 (1/x - x/4) dx =$
 $\left. (15x^2/8) \right|_0^{1/2} + \left. (\ln x - x^2/8) \right|_{1/2}^2 = 15/32 + \ln 2 -$
 $1/2 - (\ln 1/2 - 1/32) = 2 \ln 2.$



6.2.63 $y = 8x$ and $y = 9 - x^2$ intersect when $x^2 + 8x - 9 = (x + 9)(x - 1) = 0$, so at $x = 1$ in the first quadrant. $y = \frac{5}{2}x$ and $y = 9 - x^2$ intersect when $x^2 + \frac{5}{2}x - 9 = 0$, or $2x^2 + 5x - 18 = (2x + 9)(x - 2) = 0$, so at $x = 2$ in the first quadrant. Thus the area of the region is

$$\begin{aligned} A &= \int_0^1 \left(8x - \frac{5}{2}x \right) dx + \int_1^2 \left(9 - x^2 - \frac{5}{2}x \right) dx \\ &= \int_0^1 \frac{11}{2}x dx + \int_1^2 \left(9 - x^2 - \frac{5}{2}x \right) dx \\ &= \left. \left(\frac{11}{4}x^2 \right) \right|_0^1 + \left. \left(9x - \frac{x^3}{3} - \frac{5}{4}x^2 \right) \right|_1^2 \\ &= \frac{11}{4} + 18 - \frac{8}{3} - 5 - 9 + \frac{1}{3} + \frac{5}{4} = \frac{17}{3}. \end{aligned}$$

6.2.64

$$\begin{aligned}
 A &= \int_{1/2}^1 (4\sqrt{2x} - (-4x + 6)) dx + \int_1^2 (4\sqrt{2x} - 2x^2) dx \\
 &= \left(\frac{8\sqrt{2}}{3} x^{3/2} + 2x^2 - 6x \right) \Big|_{1/2}^1 + \left(\frac{8\sqrt{2}}{3} x^{3/2} - 2x^3/3 \right) \Big|_1^2 = 19/6.
 \end{aligned}$$

6.2.65

a. False. This can be done either with respect to x or with respect to y . For the latter, the relevant integral is $\int_0^1 (y - y^2) dy$.

b. False. On the interval $(0, \pi/4)$ the cosine function is greater, but on $(\pi/4, \pi/2)$ the sine function is greater. The area is $\int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$.

c. True. They both represent the area of the region in the first quadrant under $y = x$ and above $y = x^2$.

6.2.66 Given $\int_{-a}^a (f(x) - g(x)) dx = 10$, we have (by symmetry) that $\int_0^a (f(x) - g(x)) dx = 5$. Then, $\int_0^{\sqrt{a}} x(f(x^2) - g(x^2)) dx = \frac{1}{2} \int_0^a (f(u) - g(u)) du = \frac{5}{2}$. (Where we used the substitution $u = x^2$.)

6.2.67 $A = 2 \int_0^1 y^2 \sqrt{1 - y^3} dy = \frac{2}{3} \int_0^1 \sqrt{u} du$ (where $u = 1 - y^3$). So $A = \frac{2}{3} \left(\frac{2}{3} u^{3/2} \right) \Big|_0^1 = \frac{4}{9}$.

6.2.68 $A_n = \int_0^1 (x - x^n) dx = \left(x^2/2 - x^{n+1}/(n+1) \right) \Big|_0^1 = \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2n+2}$.

6.2.69 $A_n = \int_0^1 (x^{1/n} - x) dx = \left(\frac{nx^{(n+1)/n}}{n+1} - x^2/2 \right) \Big|_0^1 = \frac{n}{n+1} - \frac{1}{2} = \frac{n-1}{2n+2}$.

6.2.70 $A_n = \int_0^1 (x^{1/n} - x^n) dx = \left(\frac{nx^{(n+1)/n}}{n+1} - \frac{x^{n+1}}{n+1} \right) \Big|_0^1 = \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}$.

6.2.71 Using the result of the previous problem, $\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} \frac{n-1}{n+1} = 1$. As $n \rightarrow \infty$, the region in question approaches the 1×1 square over the interval $[0, 1]$, which has area 1.

6.2.72 R is a right triangle with both legs equal to 1, so its area is $\frac{1}{2}$. Integrating with respect to y , we are looking for k so that $\int_0^k (1 - y) dy$ is half the area of the triangle, or $\frac{1}{4}$. Now,

$$\int_0^k (1 - y) dy = \left(y - \frac{y^2}{2} \right) \Big|_0^k = k - \frac{k^2}{2}.$$

Setting $k - \frac{k^2}{2} = \frac{1}{4}$, we have $2k^2 - 4k + 1 = 0$, which has roots $k = \frac{2 \pm \sqrt{2}}{2}$. Only the negative sign choice gives a value between 0 and 1, so we want $k = \frac{2 - \sqrt{2}}{2} = 1 - \frac{\sqrt{2}}{2}$.

6.2.73 This is a triangle with base 2 and height 1, so it has area 1. Note that for $0 \leq x \leq 1$ the line which forms the top of the triangle is $y = x$, and for $1 \leq x \leq 2$, the line is $y = 2 - x$. Integrating with respect to y , we are looking for k so that $\int_0^k (2 - y - y) dy = \frac{1}{2}$. Now,

$$\int_0^k (2 - 2y) dy = (2y - y^2) \Big|_0^k = 2k - k^2.$$

Setting $2k - k^2 = \frac{1}{2}$, we have $2k^2 - 4k + 1 = 0$, which has roots $k = \frac{2 \pm \sqrt{2}}{2}$. Only the negative sign choice gives a value between 0 and 1, so we want $k = \frac{2 - \sqrt{2}}{2} = 1 - \frac{\sqrt{2}}{2}$.

6.2.74

- The proportion of the whole board which has the desired property is the same as the proportion of each “quarter board” with the desired property, so we can consider only the quarter board rather than the whole board.
- Let $P(x, y)$ be a point on the curve C . Let Q be the point on line segment $\overline{AB}$ so that $\overline{QP} \perp \overline{AB}$. Then, because the distance from O to P is the same as the distance from P to Q , we must have $\sqrt{x^2 + y^2} = 1 - y$, so $y = \frac{1}{2}(1 - x^2)$.
- The area of the region R is 1, and the area of $R_1 - R$ is

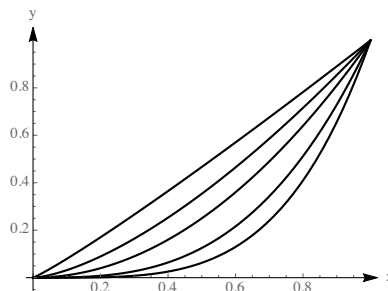
$$\begin{aligned} 2 \int_0^{\sqrt{2}-1} \left(\frac{1}{2}(1 - x^2) - x \right) dx &= \left(x - x^3/3 - x^2 \right) \Big|_0^{\sqrt{2}-1} \\ &= \sqrt{2} - 1 - (\sqrt{2} - 1)^3/3 - (\sqrt{2} - 1)^2 = \sqrt{2} - 1 + 7/3 - (5/3)\sqrt{2} + 2\sqrt{2} - 34\sqrt{2}/3 - 5/3. \end{aligned}$$

So the probability of landing in R_2 is $1 - (4\sqrt{2}/3 - 5/3) = \frac{8 - 4\sqrt{2}}{3} \approx .781$.

6.2.75

- The point (n, n) on the curve $y = x$ would represent the notion that the lowest $p\%$ of the society owns $p\%$ of the wealth, which would represent a form of equality.
- The function must be increasing and concave up because the poorest $p\%$ cannot own more than $p\%$ of the wealth.

- $y = x^{1.1}$ is closest to $y = x$, and $y = x^4$ is furthest from $y = x$.



- Note that $B = \int_0^1 L(x) dx$, and $A + B = 1/2$, so $A = \frac{1}{2} - \int_0^1 L(x) dx$. Then $G = \frac{A}{A+B} = \frac{A}{1/2} = 2A = 1 - 2 \int_0^1 L(x) dx$.

e. For $L(x) = x^p$, we have $G = 1 - 2 \int_0^1 x^p dx = 1 - 2 \left(\frac{x^{p+1}}{p+1} \right) \Big|_0^1 = 1 - \frac{2}{p+1} = \frac{p-1}{p+1}$. So we have

p	1.1	1.5	2	3	4
G	1/21	1/5	1/3	1/2	3/5

f. For $p = 1$ we have $G = 0$. Because $\lim_{p \rightarrow \infty} \frac{p-1}{p+1} = 1$, the largest possible value of G approaches 1.

g. For $L(x) = 5x^2/6 + x/6$, note that $L(0) = 0$, $L(1) = 1$, $L'(x) = 5x/3 + 1/6 > 0$ on $[0, 1]$, and $L''(x) = 5/3 > 0$ as well. The Gini index is

$$G = 1 - 2 \int_0^1 (5x^2/6 + x/6) dx = 1 - 2 (5x^3/18 + x^2/12) \Big|_0^1 = 1 - 2 (5/18 + 1/12) = 1 - 5/9 - 1/6 = 5/18.$$

6.2.76

a. In this case l_Q is $y = 2ax - a^2$, l_R is $y = 0$, and l_P is the line $y = -2ax - a^2$. We have $P' = (a/2, 0)$, $R' = (0, -a^2)$, and $Q' = (-a/2, 0)$. The area of triangle PQR is $\frac{1}{2} \cdot 2a \cdot a^2 = a^3$ and the area of triangle $P'Q'R'$ is $\frac{1}{2} \cdot (2(a/2)) \cdot (a^2) = a^3/2$.

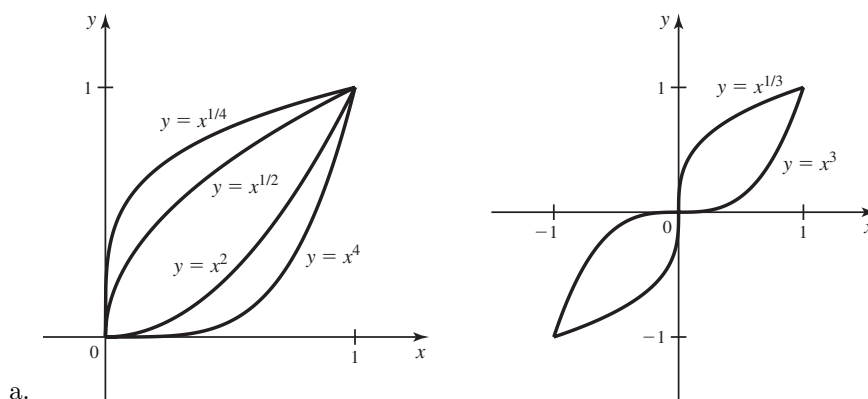
b. In this case l_Q is $y = 2bx - b^2$, l_R is $y = 0$, and l_P is $y = -2ax - a^2$. We have $P' = (b/2, 0)$, $R' = ((b-a)/2, -ab)$, and $Q' = (-a/2, 0)$. The area of triangle PQR is $ab(a+b)$, and the area of $P'Q'R'$ is $ab \cdot \frac{a+b}{2}$.

c. Let the coordinates of R be (c, c^2) . Assume $0 < a < c < b$ (the other cases can be handled similarly.)

In this case l_Q is $y = 2bx - b^2$, l_P is $y = -2ax - a^2$, and l_R is $y = 2cx - c^2$. We have $P' = ((b+c)/2, bc)$, $Q' = ((c-a)/2, -ca)$, and $R' = ((b-a)/2, -ab)$. The area of PQR is $\frac{1}{2}(a^2b + b^2c + ab^2 - ac^2 - a^2c - bc^2)$.

The area of triangle $P'Q'R'$ is $\frac{1}{4}(a^2b + b^2c + ab^2 - ac^2 - a^2c - bc^2)$.

6.2.77

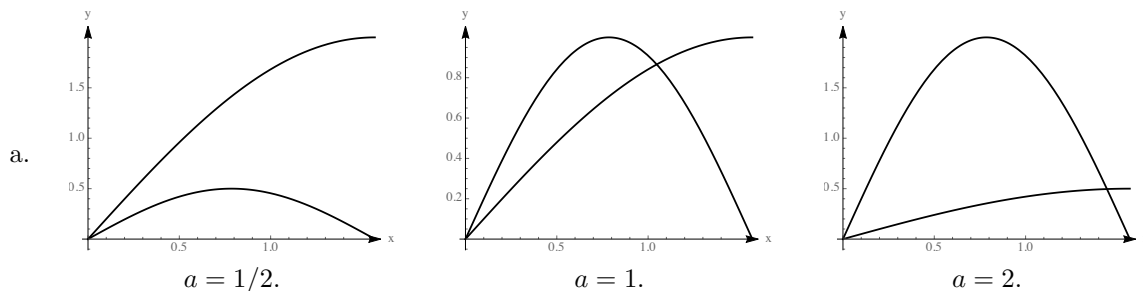


b. $A_n(x)$ is the net area of the region between the graphs of f and g from 0 to x .

c. Note that $\int_0^1 (f(s) - g(s)) ds = \left(\frac{s^{n+1}}{n+1} - \frac{s^{(1/n)+1}}{(1/n)+1} \right) \Big|_0^1 = \frac{1}{n+1} - \frac{n}{n+1} = \frac{1-n}{n+1} < 0$. So we seek the smallest

c so that $\int_1^c (f(s) - g(s)) ds = \frac{n-1}{n+1}$. This occurs when $\left(\frac{s^{n+1}}{n+1} - \frac{s^{(1/n)+1}}{(1/n)+1} \right) \Big|_1^c = \frac{n-1}{n+1}$, or $\frac{c^{n+1}}{n+1} = \frac{c^{(n+1)/n}}{(n+1)/n}$, or $c^{n+1} = nc^{(n+1)/n}$, or $c^{n-(1/n)} = n$, so $c = n^{n/(n^2-1)}$. As n increases, this root increases as well.

6.2.78



b. We seek a root of $a \sin 2x = (\sin x)/a$, or $2 \sin x \cos x = \frac{\sin x}{a^2}$, so $\cos x = \frac{1}{2a^2}$. This has a solution as long as $\frac{1}{2a^2} \leq 1$, which occurs when $a \geq \sqrt{1/2}$.

c.

$$\begin{aligned} A &= \int_0^{x^*} (\sin 2x - \sin x) dx = \left(\cos x - \frac{1}{2} \cos 2x \right) \Big|_0^{x^*} \\ &= \left(\cos x - \frac{1}{2} (2 \cos^2 x - 1) \right) \Big|_0^{x^*} = \left(\cos x - \cos^2 x + \frac{1}{2} \right) \Big|_0^{x^*} \\ &= \cos x^* - \cos^2 x^* + \frac{1}{2} - \left(1 - 1 + \frac{1}{2} \right) = \frac{1}{2a^2} - \frac{1}{4a^4} = \frac{2a^2 - 1}{4a^4}. \end{aligned}$$

When $a = 1$ this is equal to $\frac{1}{4}$.

d. Using the result of the previous calculation, we simply note that $\frac{2a^2-1}{4a^4} \rightarrow 0$ as $a \rightarrow 1/\sqrt{2}$.

6.3 Volume by Slicing

6.3.1 $A(x)$ is the area of the cross section through the solid at the point x .

6.3.2 The general slicing method would be used.

6.3.3

a. $A(x) = (\sqrt{3-x})^2 = 3-x$.

b. $\int_0^2 (3-x) dx$.

6.3.4 The cross sections are disks and $A(x)$ is the area of a disk.

6.3.5

a. The radius of a cross section is $\sqrt{\cos x}$.

b. $A(x) = \pi(\sqrt{\cos x})^2 = \pi \cos x$.

c. $V = \int_0^2 \pi \cos x dx$.

6.3.6

a. The radius of a cross section is $\cos y$.

b. $A(y) = \pi \cos^2 y$.

c. $V = \int_0^{\pi/2} \pi \cos^2 y dy$.

6.3.7

- a. The outer radius is given by $\sqrt{x} + 1$.
 b. The inner radius is 1.
 c. $A(x) = \pi \left((\sqrt{x} + 1)^2 - 1 \right)$.
 d. $V = \int_0^4 \pi \left((\sqrt{x} + 1)^2 - 1 \right) dx$.

6.3.9

- a. The radius is $\sqrt{x}$.
 b. $A(x) = \pi(\sqrt{x})^2 = \pi x$.
 c. $V = \int_0^4 \pi x dx$.

6.3.8

- a. The outer radius is 4.
 b. The inner radius is $(y - 1)^2$.
 c. $A(y) = \pi(16 - (y - 1)^4)$.
 d. $V = \int_1^3 \pi(16 - (y - 1)^4) dy$.

6.3.10

- a. The radius is $4 - (y - 1)^2$.
 b. $A(y) = \pi(4 - (y - 1)^2)^2$.
 c. $V = \int_1^3 \pi(4 - (y - 1)^2)^2 dy$.

6.3.11 The curves intersect when $\sqrt{1 - x^2} = 0$, or $x = \pm 1$.

$A(x) = (\sqrt{1 - x^2})^2 = 1 - x^2$. The volume is given by

$$\int_{-1}^1 (1 - x^2) dx = \left(x - x^3/3 \right) \Big|_{-1}^1 = \left(1 - \frac{1}{3} \right) - \left(-1 + \frac{1}{3} \right) = \frac{4}{3}.$$

6.3.12 The curves intersect when $2 - x^2 = x^2$, or $x = \pm 1$.

$A(x) = ((2 - x^2) - x^2)^2 = (2(1 - x^2))^2 = 4(1 - 2x^2 + x^4)$. The volume is given by

$$\int_{-1}^1 4(1 - 2x^2 + x^4) dx = 4 \left(x - 2x^3/3 + x^5/5 \right) \Big|_{-1}^1 = 4 \left(\left(1 - \frac{2}{3} + \frac{1}{5} \right) - \left(-1 + \frac{2}{3} - \frac{1}{5} \right) \right) = \frac{64}{15}.$$

6.3.13 The curves intersect when $\sqrt{\cos x} = 0$, or $x = \pm \pi/2$. $A(x) = \frac{1}{2} (\sqrt{\cos x})^2 = \frac{\cos x}{2}$. The volume is given by

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{2} dx = \frac{\sin x}{2} \Big|_{-\pi/2}^{\pi/2} = \frac{1}{2} - \left(-\frac{1}{2} \right) = 1.$$

6.3.14 $A(x) = (2\sqrt{25 - x^2})^2 = 100 - 4x^2$.

$$V = \int_0^5 A(x) dx = \int_0^5 (100 - 4x^2) dx = \left(100x - 4x^3/3 \right) \Big|_0^5 = 500 - \frac{500}{3} = \frac{1000}{3}.$$

6.3.15 For each value of x , the height of the triangle is $2 - x$, which is the diameter of the semicircle, so the area of that semicircle is

$$A(x) = \frac{\pi}{2} \left(\frac{2 - x}{2} \right)^2 = \frac{\pi}{8} (4 - 4x + x^2).$$

Thus the volume is

$$V = \int_0^2 A(x) dx = \int_0^2 \frac{\pi}{8} (4 - 4x + x^2) dx = \frac{\pi}{8} \left(4x - 2x^2 + x^3/3 \right) \Big|_0^2 = \frac{\pi}{3}.$$

6.3.16 The volume is $V = \int_{-1}^1 (1 - x^2)^2 dx = 2 \int_0^1 (1 - 2x^2 + x^4) dx = 2 \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_0^1 = 2 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{16}{15}$.

$$\mathbf{6.3.17} \quad V = \pi \int_0^3 4x^2 dx = 4\pi \left(x^3/3 \right) \Big|_0^3 = 36\pi.$$

$$\mathbf{6.3.18} \quad V = \pi \int_0^1 (2 - 2x)^2 dx = 4\pi \left(x - x^2 + x^3/3 \right) \Big|_0^1 = \frac{4\pi}{3}.$$

$$\mathbf{6.3.19} \quad V = \pi \int_0^{\ln 4} e^{-2x} dx = \frac{-\pi}{2} \left(e^{-2x} \right) \Big|_0^{\ln 4} = \frac{-\pi}{2} ((1/16) - 1) = \frac{15\pi}{32}.$$

$$\mathbf{6.3.20} \quad V = \pi \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx = \pi \sin^{-1} x \Big|_0^{1/2} = \frac{\pi^2}{6}.$$

$$\mathbf{6.3.21} \quad V = \pi \int_{-1}^1 \left(\frac{1}{\sqrt{1+y^2}} \right)^2 dy = \pi \tan^{-1} y \Big|_{-1}^1 = \pi(\pi/4 + \pi/4) = \pi^2/2.$$

$$\mathbf{6.3.22} \quad V = \pi \int_0^2 e^{2y} dy = \pi \frac{e^{2y}}{2} \Big|_0^2 = \frac{\pi(e^4 - 1)}{2}.$$

$$\mathbf{6.3.23} \quad V = \pi \int_0^4 ((2\sqrt{x})^2 - x^2) dx = \pi (2x^2 - x^3/3) \Big|_0^4 = \pi(32 - 64/3) = \frac{32\pi}{3}.$$

$$\mathbf{6.3.24} \quad \text{The curves intersect at } x = 0 \text{ and } x = 1. \text{ We have } V = \pi \int_0^1 ((\sqrt[4]{x^2}) - x^2) dx = \pi \int_0^1 (\sqrt{x} - x^2) dx = \pi \left(2x^{3/2}/3 - x^3/3 \right) \Big|_0^1 = \pi(2/3 - 1/3) = \pi/3.$$

$$\mathbf{6.3.25} \quad V = \pi \int_{\ln 2}^{\ln 3} ((e^{x/2})^2 - (e^{-x/2})^2) dx = \pi \int_{\ln 2}^{\ln 3} (e^x - e^{-x}) dx = \pi (e^x + e^{-x}) \Big|_{\ln 2}^{\ln 3} = \pi((3 + 1/3) - (2 + 1/2)) = 5\pi/6.$$

$$\mathbf{6.3.26} \quad V = \pi \int_0^4 ((x+2)^2 - x^2) dx = \pi \int_0^4 (4x + 4) dx = \pi (2x^2 + 4x) \Big|_0^4 = 48\pi.$$

$$\mathbf{6.3.27} \quad V = \pi \int_0^1 (1 - y^{2/3}) dy = \pi \left(y - \frac{3}{5} y^{5/3} \right) \Big|_0^1 = \frac{2\pi}{5}.$$

$$\mathbf{6.3.28} \quad V = \pi \int_0^6 (y^2 - y^2/4) dy = \frac{3\pi}{4} (y^3/3) \Big|_0^6 = 54\pi.$$

$$\mathbf{6.3.29} \quad V = \pi \int_0^{\pi/2} \cos^2 x dx = \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) dx = \frac{\pi}{2} \left(x + \frac{\sin 2x}{2} \right) \Big|_0^{\pi/2} = \frac{\pi^2}{4}.$$

$$\mathbf{6.3.30} \quad \text{Using symmetry, } V = 2\pi \int_0^5 (\sqrt{25 - x^2})^2 dx = 2\pi (25x - x^3/3) \Big|_0^5 = \frac{500\pi}{3}. \text{ The volume of a sphere of radius 5 is } \frac{4}{3}\pi \cdot 5^3 = \frac{500\pi}{3}.$$

$$\mathbf{6.3.31} \quad V = \pi \int_0^\pi \sin^2 x dx = \frac{\pi}{2} \int_0^\pi (1 - \cos 2x) dx = \frac{\pi}{2} \left(x - \frac{\sin 2x}{2} \right) \Big|_0^\pi = \frac{\pi^2}{2}.$$

$$\mathbf{6.3.32} \quad V = \pi \int_0^{\pi/4} \sec^2 x dx = \pi \tan x \Big|_0^{\pi/4} = \pi.$$

$$\mathbf{6.3.33} \quad V = \pi \int_0^{\pi/4} \sin^2 y \, dy = \frac{\pi}{2} \int_0^{\pi/4} (1 - \cos 2y) \, dy = \frac{\pi}{2} \left(y - \frac{\sin 2y}{2} \right) \Big|_0^{\pi/4} = \frac{\pi}{2} \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi(\pi - 2)}{8}.$$

$$\mathbf{6.3.34} \quad V = \pi \int_0^{\pi/2} (1 - \sin x) \, dx = \pi (x + \cos x) \Big|_0^{\pi/2} = \pi(\pi/2 - 1).$$

$$\begin{aligned} \mathbf{6.3.35} \quad V &= \pi \int_0^{\pi/2} (\sin x - \sin^2 x) \, dx = \pi \int_0^{\pi/2} \left(\sin x - \frac{1}{2}(1 - \cos 2x) \right) \, dx = \pi \left(\frac{\sin 2x}{4} - \cos x - \frac{x}{2} \right) \Big|_0^{\pi/2} \\ &= \pi \left(1 - \frac{\pi}{4} \right) = \frac{4\pi - \pi^2}{4}. \end{aligned}$$

$$\begin{aligned} \mathbf{6.3.36} \quad \text{Note that the curves intersect at } x = \pm 1. \text{ We have } V &= \pi \int_{-1}^0 ((2 - x^2)^2 - (-x)^2) \, dx + \pi \int_0^1 ((2 - x^2)^2 - x^2) \, dx \\ &= \pi \int_{-1}^1 (4 - 4x^2 + x^4 - x^2) \, dx = \pi \int_{-1}^1 (4 - 5x^2 + x^4) \, dx = \pi \left(4x - \frac{5x^3}{3} + \frac{x^5}{5} \right) \Big|_{-1}^1 \\ &= \pi \left(\left(4 - \frac{5}{3} + \frac{1}{5} \right) - \left(-4 + \frac{5}{3} - \frac{1}{5} \right) \right) = \frac{76\pi}{15}. \end{aligned}$$

$$\mathbf{6.3.37} \quad V = \pi \int_0^2 (16 - y^4) \, dy = \pi \left(16y - \frac{y^5}{5} \right) \Big|_0^2 = \frac{128\pi}{5}.$$

$$\mathbf{6.3.38} \quad V = \pi \int_0^4 (2^2 - (\sqrt{4 - y})^2) \, dy = \pi \int_0^4 y \, dy = \pi \frac{y^2}{2} \Big|_0^4 = \pi(8 - 0) = 8\pi.$$

$$\mathbf{6.3.39} \quad V = \pi \int_2^6 \left(\frac{1}{\sqrt{x}} \right)^2 \, dx = \pi (\ln x) \Big|_2^6 = \pi \ln 3.$$

$$\begin{aligned} \mathbf{6.3.40} \quad V &= \pi \int_0^1 ((2 - y)^2 - (\sqrt{y})^2) \, dy = \pi \int_0^1 (y^2 - 5y + 4) \, dy = \pi \left(\frac{y^3}{3} - \frac{5y^2}{2} + 4y \right) \Big|_0^1 \\ &= \pi \left(\frac{1}{3} - \frac{5}{2} + 4 \right) = \frac{11\pi}{6}. \end{aligned}$$

$$\mathbf{6.3.41} \quad V = \pi \int_0^2 e^{2x} \, dx = \frac{\pi}{2} (e^{2x}) \Big|_0^2 = \frac{\pi}{2} (e^4 - 1).$$

$$\mathbf{6.3.42} \quad V = \pi \int_0^{\ln 4} (e^{2x} - e^{-2x}) \, dx = \frac{\pi}{2} (e^{2x} + e^{-2x}) \Big|_0^{\ln 4} = \frac{\pi}{2} \left(16 + \frac{1}{16} - 2 \right) = \frac{225\pi}{32}.$$

$$\mathbf{6.3.43} \quad V = \pi \int_0^{\ln 8} (e^{2y} - e^y) \, dy = \pi (e^{2y}/2 - e^y) \Big|_0^{\ln 8} = \pi \left(32 - 8 - \left(\frac{1}{2} - 1 \right) \right) = \frac{49\pi}{2}.$$

$$\mathbf{6.3.44} \quad V(p) = \pi \int_0^p e^{-2x} \, dx = -\frac{\pi}{2} (e^{-2x}) \Big|_0^p = \frac{\pi}{2} (1 - e^{-2p}). \text{ The volume is bounded by } \pi/2 \text{ as } p \rightarrow \infty.$$

$$\mathbf{6.3.45} \quad \text{About the } x\text{-axis: } V_x = \pi \int_0^5 4x^2 \, dx = \pi (4x^3/3) \Big|_0^5 = \frac{500\pi}{3}.$$

$$\text{About the } y\text{-axis: } V_y = \pi \int_0^{10} (25 - y^2/4) \, dy = \pi (25y - y^3/12) \Big|_0^{10} = \frac{500\pi}{3}.$$

The volumes are the same.

$$\mathbf{6.3.46} \quad \text{About the } x\text{-axis: } V_x = \pi \int_0^2 (4 - 2x)^2 \, dx = 4\pi (4x - 2x^2 + x^3/3) \Big|_0^2 = \frac{32\pi}{3}.$$

$$\text{About the } y\text{-axis: } V_y = \pi \int_0^4 (2 - (y/2))^2 \, dy = \pi (4y - y^2 + y^3/12) \Big|_0^4 = \frac{16\pi}{3}.$$

The volume V_x is bigger.

6.3.47 About the x -axis: $V_x = \pi \int_0^1 (1 - x^3)^2 dx = \pi \int_0^1 (1 - 2x^3 + x^6) dx = \pi \left(x - x^4/2 + x^7/7 \right) \Big|_0^1 = \frac{9\pi}{14}$.

About the y -axis: $V_y = \pi \int_0^1 (\sqrt[3]{1-y})^2 dy = -\pi \left(\frac{3}{5}(1-y)^{5/3} \right) \Big|_0^1 = -\pi(0 - 3/5) = \frac{3\pi}{5}$. The volume V_x is bigger.

6.3.48 About the x -axis: $V_x = \pi \int_0^2 (8x - x^4) dx = \pi \left(4x^2 - x^5/5 \right) \Big|_0^2 = \frac{48\pi}{5}$.

About the y -axis: $V_y = \pi \int_0^4 (y - y^4/64) dy = \pi \left(y^2/2 - y^5/320 \right) \Big|_0^4 = \frac{24\pi}{5}$.

The volume V_x is bigger.

6.3.49 $V = \pi \int_0^1 (1 - (1 - \sqrt{x}))^2 dx = \pi \int_0^1 x dx = \pi \frac{x^2}{2} \Big|_0^1 = \frac{\pi}{2}$.

6.3.50 $V = \pi \int_0^1 (e^{-x/2} + 2 - 2)^2 dx = \pi \int_0^1 e^{-x} dx = -\pi e^{-x} \Big|_0^1 = \pi(1 - e^{-1})$.

6.3.51 $V = \pi \int_0^{\pi/3} (2 - (2 - \sec y))^2 dy = \pi \int_0^{\pi/3} \sec^2 y dy = \pi \tan y \Big|_0^{\pi/3} = \pi\sqrt{3}$.

6.3.52 $V = \pi \int_0^1 (1 - (1 - y)^2)^2 dy = \pi \int_0^1 (y^4 - 4y^3 + 4y^2) dy = \pi \left(\frac{y^5}{5} - y^4 + \frac{4y^3}{3} \right) \Big|_0^1 = \pi \left(\frac{1}{5} - 1 + \frac{4}{3} \right) = \frac{8\pi}{15}$.

6.3.53 Using the disk method, a disk located at x has a radius of $1 - \sqrt{x}$ when revolved about the line $y = 1$, so the volume is

$$V = \pi \int_0^1 (1 - \sqrt{x})^2 dx = \pi \int_0^1 (1 - 2\sqrt{x} + x) dx = \pi \left(x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 \right) \Big|_0^1 = \pi \left(1 - \frac{4}{3} + \frac{1}{2} \right) = \frac{\pi}{6}.$$

6.3.54 We use the washer method. Note that $x = y^2$, and for each value of y , the washer has outer radius 4 and inner radius $4 - y^2$, so that the volume is

$$V = \pi \int_0^2 (4^2 - (4 - y^2)^2) dy = \pi \int_0^2 (8y^2 - y^4) dy = \pi \left(\frac{8}{3}y^3 - \frac{1}{5}y^5 \right) \Big|_0^2 = \pi \left(\frac{64}{3} - \frac{32}{5} \right) = \frac{224}{15}\pi.$$

6.3.55 We use the washer method. For each x , the washer has outer radius $2 + 2\sin x$ and inner radius 2, so the volume is

$$\begin{aligned} V &= \pi \int_0^\pi ((2 + 2\sin x)^2 - 2^2) dx = \pi \int_0^\pi (8\sin x + 4\sin^2 x) dx \\ &= \pi \int_0^\pi (8\sin x + 2(1 - \cos 2x)) dx = \pi(-8\cos x + 2x - \sin 2x) \Big|_0^\pi \\ &= \pi(8 + 2\pi + 8) = 2\pi(\pi + 8). \end{aligned}$$

6.3.56 We use the washer method. First solve for x to obtain $x = e^y$; then for each value of y , the outer radius of a washer is $e^y + 1$ and the inner radius is 1, so the volume is

$$\begin{aligned} V &= \pi \int_0^1 ((e^y + 1)^2 - 1^2) dy = \pi \int_0^1 (2e^y + e^{2y}) dy = \pi \left(2e^y + \frac{1}{2}e^{2y} \right) \Big|_0^1 \\ &= \pi \left(2e + \frac{1}{2}e^2 - \frac{5}{2} \right) = \frac{\pi}{2} (e^2 + 4e - 5). \end{aligned}$$

6.3.57 We use the washer method. For each x , the outer radius is $1 + \sin x$ and the inner radius is $1 + (1 - \sin x) = 2 - \sin x$. Thus the volume is

$$\begin{aligned} V &= \pi \int_{\pi/6}^{5\pi/6} ((1 + \sin x)^2 - (2 - \sin x)^2) dx = \pi \int_{\pi/6}^{5\pi/6} (6 \sin x - 3) dx \\ &= \pi (-6 \cos x - 3x) \Big|_{\pi/6}^{5\pi/6} = \pi \left(3\sqrt{3} - \frac{5\pi}{2} + 3\sqrt{3} + \frac{\pi}{2} \right) = \pi (6\sqrt{3} - 2\pi). \end{aligned}$$

6.3.58 The lines $y = 1 + \frac{x}{2}$ and $y = x$ meet at the point $(2, 2)$. For each x , we obtain a washer with outer radius $3 - x$ and inner radius $3 - (1 + \frac{x}{2}) = 2 - \frac{x}{2}$. Hence the volume is

$$V = \pi \int_0^2 \left((3 - x)^2 - \left(2 - \frac{x}{2} \right)^2 \right) dx = \pi \int_0^2 \left(5 - 4x + \frac{3}{4}x^2 \right) dx = \pi \left(5x - 2x^2 + \frac{1}{4}x^3 \right) \Big|_0^2 = 4\pi.$$

6.3.59 The lines $x = 2 - y$ and $x = 1 - \frac{y}{2}$ meet at the point $(0, 2)$. For each y , we obtain a washer whose outer radius is $3 - (1 - \frac{y}{2}) = 2 + \frac{y}{2}$ and whose inner radius is $3 - (2 - y) = y + 1$. Thus the volume is

$$V = \pi \int_0^2 \left(\left(2 + \frac{y}{2} \right)^2 - (y + 1)^2 \right) dy = \pi \int_0^2 \left(-\frac{3}{4}y^2 + 3 \right) dy = \pi \left(-\frac{1}{4}y^3 + 3y \right) \Big|_0^2 = 4\pi.$$

6.3.60 We would expect the volume to be greater when revolved about the line $y = 2$, because the wider portion of the figure then moves through a larger radius, so it traces out a larger volume. When revolving about $y = 0$, we obtain a disk with radius $2x(2 - x)$, so the volume is

$$V = \pi \int_0^2 (2x(2 - x))^2 dx = \pi \int_0^2 (16x^2 - 16x^3 + 4x^4) dx = \left(\frac{16}{3}x^3 - 4x^4 + \frac{4}{5}x^5 \right) \Big|_0^2 = \frac{64\pi}{15}.$$

When revolving about the line $y = 2$, we have washers with outer radius 2 and inner radius $2 - 2x(2 - x)$, so the volume is

$$\begin{aligned} V &= \pi \int_0^2 (4 - (2 - 2x(2 - x))^2) dx = \pi \int_0^2 (8x(2 - x) - 4x^2(2 - x)^2) dx \\ &= \pi \int_0^2 (-4x^4 + 16x^3 - 24x^2 + 16x) dx = \pi \left(-\frac{4}{5}x^5 + 4x^4 - 8x^3 + 8x^2 \right) \Big|_0^2 = \frac{32}{5}\pi. \end{aligned}$$

The second calculation yields a larger result.

6.3.61

- False. The cross sections are not disks or washers.
- True. It is given by $V = \pi \int_0^R (R^2 - x^2) dx$.
- True. This is because if we shift the sine function horizontally by $\pi/2$ units, we obtain the cosine function.

6.3.62 The relationship between the height h of tetrahedron and the edge length l is $h = l\sqrt{2/3}$, which can be deduced using triangle geometry and the Pythagorean theorem.

Let z be the distance from the top vertex of the tetrahedron down toward the base perpendicularly. The cross sections perpendicular to this axis are all equilateral triangles with height z , so their side length is $\sqrt{3/2}z$, and their area is

$$A(z) = \frac{\sqrt{3}}{4} \left(z\sqrt{\frac{3}{2}} \right)^2 = \frac{3\sqrt{3}z^2}{8}.$$

The volume is thus

$$\int_0^{4\sqrt{2/3}} \frac{3\sqrt{3}}{8} z^2 dz = \frac{3\sqrt{3}}{8} (z^3/3) \Big|_0^{4\sqrt{2/3}} = \frac{16\sqrt{2}}{3}.$$

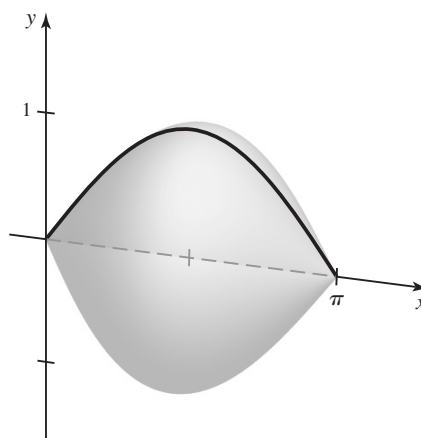
6.3.63 The volume V_S is $\pi \int_0^{\sqrt{a}} (y^2 - a)^2 dy = \pi \left(y^5/5 - 2ay^3/3 + a^2y \right) \Big|_0^{\sqrt{a}} = \frac{8\pi}{15} a^{5/2}$.

The volume V_T is $\frac{1}{3}\pi a^2 \cdot a^{1/2} = \frac{1}{3}\pi a^{5/2}$. The ratio of $V_S/V_T = \frac{8/15}{1/3} = \frac{8}{5}$.

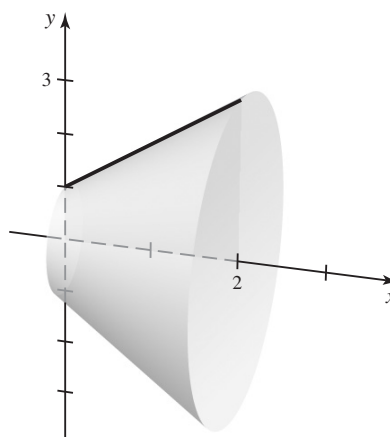
6.3.64 $V = \pi \int_0^2 x^2 dx + \pi \int_2^5 (2x - 2)^2 dx + \pi \int_5^6 (18 - 2x)^2 dx = \frac{8}{3}\pi + 84\pi + \frac{148\pi}{3} = 136\pi$.

6.3.65

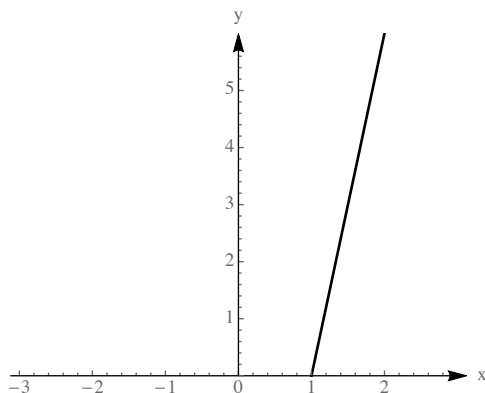
- a. This comes from revolving the region in the first quadrant bounded by $\sin x$ and the line $y = 0$ between $x = 0$ and $x = \pi$ around the x axis.



- b. This comes from revolving the region in the first quadrant bounded by $y = x + 1$ and the line $y = 0$ between $x = 0$ and $x = 2$ around the x axis.



6.3.66



By symmetry, the cup can be thought of as the result of revolving the line pictured about the y -axis. The line is given by $y = 6x - 6$ or $x = \frac{y}{6} + 1$. Using disks, we have

$$\pi \int_0^6 \left(\frac{y}{6} + 1\right)^2 dy = \pi \int_0^6 \left(\frac{y^2}{36} + \frac{y}{3} + 1\right) dy = \pi \left(\frac{y^3}{108} + \frac{y^2}{6} + y\right) \Big|_0^6 = \pi(2 + 6 + 6) = 14\pi.$$

6.3.67 At a given point x in $[0, 4]$, let $A(x)$ equal the area of a cross section of the solid of revolution that is perpendicular to the x -axis. Use the formula $V \approx \sum_{k=1}^n A(x_k^*) \Delta x$ to estimate the volume.

Left Riemann sum: $V \approx \pi(1^2 + 4^2 + 7^2 + 10^2) = 166\pi$.

Right Riemann sum: $V \approx \pi(4^2 + 7^2 + 10^2 + 12^2) = 309\pi$.

Midpoint Riemann sum: $V \approx \pi(3^2 + 5^2 + 8^2 + 11^2) = 219\pi$.

6.3.68 The volume is approximately

$$\pi \left((12.6/2)^2 + (14.0/2)^2 + (16.8/2)^2 + (25.2/2)^2 + (36.4/2)^2 + (42.0/2)^2 \right) \left(\frac{50}{6} \right) \approx 28542.7 \text{ cm}^3.$$

6.3.69

a. Think of the cone as being obtained by revolving the region under $y = x$ in the first quadrant from $x = 0$ to $x = R$ around the x -axis. The volume is $\pi \int_0^R x^2 dx = \pi R^3/3 = \frac{1}{3}V_C$.

b. Think of the hemisphere as being obtained by revolving the region under $y = \sqrt{R^2 - x^2}$ between $x = 0$ and $x = R$ around the x -axis. The volume is $\pi \int_0^R (R^2 - x^2) dx = \pi \left(R^2 x - x^3/3 \right) \Big|_0^R = \frac{2\pi R^3}{3} = \frac{2}{3}V_C$.

6.3.70 $V(h) = \pi \int_{-8}^{-8+h} (\sqrt{64 - y^2})^2 dy = \pi (64y - y^3/3) \Big|_{-8}^{-8+h} = 8\pi h^2 - \frac{\pi h^3}{3}$. Note that $V(0) = 0$ and $V(8) = \frac{2\pi}{3}8^3$.

6.3.71 $V = \pi \int_{-2}^2 ((3 + \sqrt{4 - y^2})^2 - (3 - \sqrt{4 - y^2})^2) dy = 24\pi \int_0^2 \sqrt{4 - y^2} dy = 24\pi(\pi) = 24\pi^2$. Note that the last integral evaluated represents 1/4 the area of a circle of radius 2

6.3.72 Around the x -axis: $V_x = \pi \int_0^1 (x - x^4) dx = \pi \left(x^2/2 - x^5/5 \right) \Big|_0^1 = \pi(1/2 - 1/5) = \frac{3\pi}{10}$.

Around the line $y = 1$: $V_1 = \pi \int_0^1 ((1 - x^2)^2 - (1 - \sqrt{x})^2) dx = \pi \int_0^1 (x^4 - 2x^2 - x + 2\sqrt{x}) dx = \pi \left(x^5/5 - 2x^3/3 - x^2/2 + 4x^{3/2}/3 \right) \Big|_0^1 = \pi(1/5 - 2/3 - 1/2 + 4/3) = \frac{11\pi}{30}$. Note that $V_1 > V_x$.

6.3.73

- a. By the general slicing method, $V(x) = \int_a^b A(x) dx$. Because the two figures have the same cross sections $A(x)$, they must therefore have the same volumes.
- b. According to Cavalieri's principle, the volume is $V = \pi r^2 h$ (the angle $\pi/4$ is irrelevant).

6.3.74

a.

$$\begin{aligned} V(n) &= \pi \int_0^1 (x^{2/n} - x^{2n}) dx = \pi \left(\frac{n}{n+2} x^{(n+2)/n} - \frac{1}{2n+1} x^{2n+1} \right) \Big|_0^1 \\ &= \pi \left(\frac{n}{n+2} - \frac{1}{2n+1} \right) = \frac{2\pi(n+1)(n-1)}{(n+2)(2n+1)} = \frac{2\pi(n^2-1)}{(n+2)(2n+1)}. \end{aligned}$$

b. $\lim_{n \rightarrow \infty} V(n) = \lim_{n \rightarrow \infty} \frac{2n^2 - 2}{2n^2 + 5n + 2} \cdot \pi = \pi.$

6.4 Volume by Shells

6.4.1 $V = 2\pi \int_a^b x(f(x) - g(x)) dx.$

6.4.2 ...revolved about the y axis. ...using the disk/washer method and integrating with respect to $\underline{y}$ or using the shell method and integrating with respect to $\underline{x}$.

6.4.3 ...revolved about the x axis. ...using the disk/washer method and integrating with respect to $\underline{x}$ or using the shell method and integrating with respect to $\underline{y}$.

6.4.4 In order to use washers, one would have to find the inner and outer radii of the washers, which would involve finding inverse functions for $y = x^2 - x^3$ over different portions of the interval $[0, 1]$.

6.4.5

- a. The radius is x .
- b. The height is $2 - x^2 - x$.
- c. The volume is $2\pi \int_0^1 x(2 - x^2 - x) dx.$

6.4.6

- a. The radius is y .
- b. The height is $4 - (2 - y)^2 = 4 - (4 - 4y + y^2) = 4y - y^2.$
- c. The volume is $2\pi \int_0^2 y(4y - y^2) dy.$

6.4.7

- a. The radius is $2 - y$.
- b. The height is $4 - (2 - y)^2 = 4 - (4 - 4y + y^2) = 4y - y^2.$
- c. The volume is $2\pi \int_0^2 (2 - y)(4y - y^2) dy.$

6.4.8

- a. The radius is $4 - x$.
- b. The height is $2 - (2 - \sqrt{x}) = \sqrt{x}$.
- c. The volume is $2\pi \int_0^4 (4 - x)\sqrt{x} \, dx$.

$$\mathbf{6.4.9} \quad V = 2\pi \int_0^1 x(x - x^2) \, dx = 2\pi \int_0^1 (x^2 - x^3) \, dx = 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{\pi}{6}.$$

$$\mathbf{6.4.10} \quad V = 2\pi \int_0^2 \frac{x}{1 + x^2} \, dx = \pi \ln(1 + x^2) \Big|_0^2 = \pi \ln 5.$$

$$\mathbf{6.4.11} \quad V = 2\pi \int_0^1 x(3 - 3x) \, dx = 2\pi \left(3x^2/2 - x^3 \right) \Big|_0^1 = \pi.$$

The volume of a cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 1^2 \cdot 3 = \pi$.

$$\mathbf{6.4.12} \quad V = 2\pi \int_1^4 x((-x^2 + 4x + 2) - (x^2 - 6x + 10)) \, dx = 2\pi \int_1^4 (-2x^3 + 10x^2 - 8x) \, dx =$$

$$2\pi \left(-\frac{x^4}{2} + \frac{10x^3}{3} - 4x^2 \right) \Big|_1^4 = 2\pi \left(\left(-128 + \frac{640}{3} - 64 \right) - \left(-\frac{1}{2} + \frac{10}{3} - 4 \right) \right) = 45\pi.$$

$$\mathbf{6.4.13} \quad V = 2\pi \int_0^2 y(4 - y^2) \, dy = 2\pi \left(2y^2 - y^4/4 \right) \Big|_0^2 = 8\pi.$$

$$\mathbf{6.4.14} \quad V = 2\pi \int_0^1 y(2 - y - y^2) \, dy = 2\pi \int_0^1 (2y - y^2 - y^3) \, dy = 2\pi \left(y^2 - \frac{y^3}{3} - \frac{y^4}{4} \right) \Big|_0^1 =$$

$$2\pi \left(1 - \frac{1}{3} - \frac{1}{4} \right) = \frac{5\pi}{6}.$$

$$\mathbf{6.4.15} \quad V = 2\pi \int_2^4 y(4 - y) \, dy = 2\pi \left(2y^2 - \frac{y^3}{3} \right) \Big|_2^4 = \frac{32\pi}{3}.$$

$$\mathbf{6.4.16} \quad V = 2\pi \int_1^{\sqrt{3}} y \left(\frac{4}{y + y^3} - \frac{1}{\sqrt{3}} \right) \, dy = 2\pi \int_1^{\sqrt{3}} \left(\frac{4}{1 + y^2} - \frac{y}{\sqrt{3}} \right) \, dy = 2\pi \left(4 \tan^{-1} y - \frac{y^2}{2\sqrt{3}} \right) \Big|_1^{\sqrt{3}} =$$

$$2\pi \left(\left(\frac{4\pi}{3} - \frac{3}{2\sqrt{3}} \right) - \left(\pi - \frac{1}{2\sqrt{3}} \right) \right) = 2\pi \left(\frac{\pi}{3} - \frac{1}{\sqrt{3}} \right) = \frac{2\pi}{3} (\pi - \sqrt{3}).$$

$$\mathbf{6.4.17} \quad V = 2\pi \int_0^{\sqrt{\pi/2}} x \cos x^2 \, dx = \pi (\sin x^2) \Big|_0^{\sqrt{\pi/2}} = \pi.$$

$$\mathbf{6.4.18} \quad V = 2\pi \int_2^4 x(6 - x) \, dx = 2\pi \left(3x^2 - \frac{x^3}{3} \right) \Big|_2^4 = \frac{104\pi}{3}.$$

$$\mathbf{6.4.19} \quad V = 2\pi \int_0^1 x(1 - x^2) \, dx = 2\pi \int_0^1 (x - x^3) \, dx = 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi}{2}.$$

$$\mathbf{6.4.20} \quad V = 2\pi \int_0^1 x\sqrt{x} \, dx = 2\pi \left(\frac{2x^{5/2}}{5} \right) \Big|_0^1 = \frac{4\pi}{5}.$$

$$\mathbf{6.4.21} \quad V = 2\pi \int_0^3 y(y^2) dy = 2\pi \int_0^3 y^3 dy = 2\pi \left(\frac{y^4}{4} \right) \Big|_0^3 = 81\pi/2.$$

$$\mathbf{6.4.22} \quad V = 2\pi \int_0^1 y(\sqrt[3]{y}) dy = 2\pi \left(\frac{3}{7} y^{7/3} \right) \Big|_0^1 = \frac{6\pi}{7}.$$

6.4.23 Note that the lines intersect at $(0, 0)$, $(2, 0)$ and $(1, 1)$.

$$V = 2\pi \int_0^1 y((2-y) - y) dy = 2\pi \int_0^1 (2y - 2y^2) dy = 2\pi \left(y^2 - \frac{2y^3}{3} \right) \Big|_0^1 = \frac{2\pi}{3}.$$

$$\mathbf{6.4.24} \quad V = 2\pi \int_0^{\sqrt{2}} x\sqrt{4-2x^2} dx. \text{ Let } u = 4 - 2x^2, \text{ so that } du = -4x dx. \text{ Substituting gives}$$

$$-\frac{\pi}{2} \int_4^0 u^{1/2} du = -\frac{\pi}{2} \cdot \frac{2}{3} u^{3/2} \Big|_4^0 = \frac{8\pi}{3}.$$

$$\mathbf{6.4.25} \quad V = 2\pi \int_1^2 \frac{x}{(x^2+1)^2} dx. \text{ Let } u = x^2 + 1, \text{ so that } du = 2x dx. \text{ Substituting gives}$$

$$V = \pi \int_2^5 u^{-2} du = -\pi \cdot \frac{1}{u} \Big|_2^5 = -\pi \left(\frac{1}{5} - \frac{1}{2} \right) = \frac{3\pi}{10}.$$

$$\mathbf{6.4.26} \quad \text{Note that the line } x = 4 \text{ intersects the curve } x = y^2 \text{ at } (4, 2). \quad V = 2\pi \int_0^2 y(4 - y^2) dy = 2\pi \int_0^2 (4y - y^3) dy = 2\pi \left(2y^2 - y^4/4 \right) \Big|_0^2 = 8\pi.$$

$$\mathbf{6.4.27} \quad V = 2\pi \int_2^{16} y(2/y)^{2/3} dy = 2^{5/3}\pi \int_2^{16} y^{1/3} dy = 2^{5/3}\pi \left(3y^{4/3}/4 \right) \Big|_2^{16} = 2^{5/3}\pi(3 \cdot 2^{10/3} - 3 \cdot 2^{-2/3}) = 3\pi(2^5 - 2) = 90\pi.$$

$$\mathbf{6.4.28} \quad V = 2\pi \int_0^{\sqrt{\pi/2}} y \sin y^2 dy = -\pi \cos y^2 \Big|_0^{\sqrt{\pi/2}} = -\pi(0 - 1) = \pi.$$

$$\mathbf{6.4.29} \quad V = 2\pi \int_1^2 x(e^x/x) dx = 2\pi e^x \Big|_1^2 = 2\pi(e^2 - e).$$

$$\mathbf{6.4.30} \quad V = 2\pi \int_1^3 \frac{\ln x}{x} dx = 2\pi \left(\frac{\ln^2 x}{2} \right) \Big|_1^3 = \pi(\ln 3)^2.$$

$$\mathbf{6.4.31} \quad \text{Note that } y = \sqrt{\cos^{-1} x} \text{ intersects the axes at } (0, \sqrt{\pi/2}) \text{ and } (1, 0). \quad V = 2\pi \int_0^{\sqrt{\pi/2}} y \cos y^2 dy = \pi \sin y^2 \Big|_0^{\sqrt{\pi/2}} = \pi(1 - 0) = \pi.$$

6.4.32

$$V = 2\pi \int_0^{5\sqrt{2}} y\sqrt{(50-y^2)/2} dy = \sqrt{2}\pi \int_0^{5\sqrt{2}} y\sqrt{50-y^2} dy = \sqrt{2}\pi \cdot \frac{1}{2} \int_0^{50} u^{1/2} du,$$

where $u = 50 - y^2$. Thus,

$$V = \frac{\sqrt{2}\pi}{2} \left(2u^{3/2}/3 \right) \Big|_0^{50} = \frac{\sqrt{2}\pi}{2} \left(2 \cdot 50 \cdot 5\sqrt{2}/3 \right) = \frac{500\pi}{3}.$$

6.4.33 Note that the line $y = 1$ intersects the curve $y = x^3 - x^8 + 1$ at $x = 0$ and $x = 1$. $V = 2\pi \int_0^1 x(x^3 - x^8 + 1 - 1) dx = 2\pi \int_0^1 (x^4 - x^9) dx = 2\pi \left(\frac{x^5}{5} - \frac{x^{10}}{10} \right) \Big|_0^1 = 2\pi \left(\frac{1}{5} - \frac{1}{10} \right) = \frac{\pi}{5}$.

6.4.34

$$V = 2\pi \int_0^2 y(e^{y^2} - e^{y^2/3}) dy = \pi \left(e^{y^2} - 3e^{y^2/3} \right) \Big|_0^2 = \pi(e^4 - 3e^{4/3} - (1 - 3)) = \pi(2 + e^4 - 3e^{4/3}).$$

6.4.35 With washers we have

$$V = \pi \int_0^1 (x^{2/3} - x^2) dx = \pi \left(\frac{3x^{5/3}}{5} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{4\pi}{15}.$$

With shells we have

$$V = 2\pi \int_0^1 y(y - y^3) dy = 2\pi \left(\frac{y^3}{3} - \frac{y^5}{5} \right) \Big|_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{4\pi}{15}.$$

6.4.36 Using washers we have

$$V = \pi \int_0^2 \left(\left(1 - \frac{x}{3}\right)^2 - \frac{1}{(x+1)^2} \right) dx = \pi \left(x - \frac{x^2}{3} + \frac{x^3}{27} + \frac{1}{(x+1)} \right) \Big|_0^2 = \frac{8\pi}{27}.$$

Using shells we have

$$\begin{aligned} V &= 2\pi \int_{1/3}^1 y \left(3 - 3y - \left(\frac{1}{y} - 1 \right) \right) dy = 2\pi \int_{1/3}^1 (4y - 3y^2 - 1) dy = 2\pi (2y^2 - y^3 - y) \Big|_{1/3}^1 \\ &= 2\pi \left(0 - \left(\frac{2}{9} - \frac{1}{27} - \frac{1}{3} \right) \right) = \frac{8\pi}{27}. \end{aligned}$$

6.4.37 Using washers we have

$$\begin{aligned} V &= \pi \int_{-10}^{25} (\sqrt[3]{y+2} + 2)^2 dy = \pi \int_{-8}^{27} (u^{1/3} + 2)^2 du = \pi \int_{-8}^{27} (u^{2/3} + 4u^{1/3} + 4) du \\ &= \pi \left(3u^{5/3}/5 + 3u^{4/3} + 4u \right) \Big|_{-8}^{27} = \pi(3^6/5 + 3^5 + 108 - (-96/5 + 48 - 32)) = \pi \left(\frac{2484}{5} + \frac{16}{5} \right) = 500\pi. \end{aligned}$$

Using shells we have

$$\begin{aligned} V &= 2\pi \int_0^5 x(25 - (x-2)^3 + 2) dx = 2\pi \int_0^5 (27x - x(x-2)^3) dx = 2\pi \int_0^5 (-x^4 + 6x^3 - 12x^2 + 35x) dx \\ &= 2\pi \left(-\frac{x^5}{5} + \frac{3x^4}{2} - 4x^3 + \frac{35x^2}{2} \right) \Big|_0^5 = 500\pi \end{aligned}$$

6.4.38 Using shells, we have

$$\begin{aligned} V &= 2\pi \int_0^2 x(8 - (2x+2)) dx = 2\pi \int_0^2 x(6 - 2x) dx \\ &= 2\pi \int_0^2 (6x - 2x^2) dx = 2\pi \left(3x^2 - \frac{2x^3}{3} \right) \Big|_0^2 = 2\pi \left(12 - \frac{16}{3} \right) = \frac{40\pi}{3}. \end{aligned}$$

Using disks, we have

$$\begin{aligned} V &= \pi \int_2^6 \left(\frac{1}{2}y - 1 \right)^2 dy + \pi \int_6^8 4 dy = \pi \int_2^6 \left(\frac{y^2}{4} - y + 1 \right) dy + 8\pi \\ &= \pi \left(\frac{y^3}{12} - \frac{y^2}{2} + y \right) \Big|_2^6 + 8\pi = \pi \left((18 - 18 + 6) - \left(\frac{8}{12} - 2 + 2 \right) \right) + 8\pi \\ &= \frac{16\pi}{3} + 8\pi = \frac{40\pi}{3}. \end{aligned}$$

$$\mathbf{6.4.39} \quad V = 2\pi \int_0^1 (x+2)x^2 dx = 2\pi \int_0^1 (x^3 + 2x^2) dx = \pi \left(\frac{x^4}{4} + \frac{2x^3}{3} \right) \Big|_0^1 = \frac{11\pi}{6}.$$

$$\mathbf{6.4.40} \quad V = 2\pi \int_0^1 (1-x)x^2 dx = 2\pi \int_0^1 (x^2 - x^3) dx = 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = \frac{\pi}{6}.$$

$$\mathbf{6.4.41} \quad V = 2\pi \int_0^1 (2-x)x^2 dx = 2\pi \int_0^1 (2x^2 - x^3) dx = 2\pi \left(\frac{2x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = 2\pi \left(\frac{2}{3} - \frac{1}{4} \right) = \frac{5\pi}{6}.$$

6.4.42

$$\begin{aligned} V &= 2\pi \int_0^1 (1-y)(1-\sqrt{y}) dy = 2\pi \int_0^1 (1-\sqrt{y}-y+y^{3/2}) dy \\ &= 2\pi \left(y - \frac{2y^{3/2}}{3} - \frac{y^2}{2} + \frac{2y^{5/2}}{5} \right) \Big|_0^1 = 2\pi \left(1 - \frac{2}{3} - \frac{1}{2} + \frac{2}{5} \right) = \frac{7\pi}{15} \end{aligned}$$

6.4.43

$$\begin{aligned} V &= 2\pi \int_0^1 (y+2)(1-\sqrt{y}) dy = 2\pi \int_0^1 (y+2-y^{3/2}-2y^{1/2}) dy = 2\pi \left(\frac{y^2}{2} + 2y - \frac{2y^{5/2}}{5} - \frac{4y^{3/2}}{3} \right) \Big|_0^1 \\ &= 2\pi \left(\frac{1}{2} + 2 - \frac{2}{5} - \frac{4}{3} \right) = \frac{23\pi}{15}. \end{aligned}$$

6.4.44

$$V = 2\pi \int_0^1 (1-\sqrt{y})(2-y) dy = 2\pi \int_0^1 (2-y-2y^{1/2}+y^{3/2}) dy = 2\pi \left(2y - \frac{y^2}{2} - \frac{4y^{3/2}}{3} + \frac{2y^{5/2}}{5} \right) \Big|_0^1 = \frac{17\pi}{15}.$$

6.4.45 Using washers, we have

$$\begin{aligned} V &= \pi \int_0^1 (3^2 - (x^2 + 2)^2) dx = \pi \int_0^1 (9 - x^4 - 4x^2 - 4) dx = \pi \int_0^1 (5 - x^4 - 4x^2) dx \\ &= \pi \left(5x - \frac{x^5}{5} - \frac{4x^3}{3} \right) \Big|_0^1 = \pi \left(5 - \frac{1}{5} - \frac{4}{3} \right) = \frac{52\pi}{15}. \end{aligned}$$

Using shells, we have

$$\begin{aligned} V &= 2\pi \int_0^1 (y+2)\sqrt{y} dy = 2\pi \int_0^1 (y^{3/2} + 2y^{1/2}) dy = 2\pi \left(\frac{2y^{5/2}}{5} + \frac{4y^{3/2}}{3} \right) \Big|_0^1 \\ &= 2\pi \left(\frac{2}{5} + \frac{4}{3} \right) = \frac{52\pi}{15} \end{aligned}$$

6.4.46 Using washers, we have

$$V = \pi \int_0^1 ((\sqrt{y}+1)^2 - 1^2) dy = \pi \int_0^1 (y+2\sqrt{y}) dy = \pi \left(\frac{y^2}{2} + \frac{4y^{3/2}}{3} \right) \Big|_0^1 = \frac{11\pi}{6}.$$

Using shells, we have

$$\begin{aligned} V &= 2\pi \int_0^1 (x+1)(1-x^2) dx = 2\pi \int_0^1 (x-x^3+1-x^2) dx = 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4} + x - \frac{x^3}{3} \right) \Big|_0^1 \\ &= 2\pi \left(\frac{1}{2} - \frac{1}{4} + 1 - \frac{1}{3} \right) = \frac{11\pi}{6}. \end{aligned}$$

6.4.47 Using washers, we have

$$V = \pi \int_0^1 ((6-x^2)^2 - 5^2) dx = \pi \int_0^1 (x^4 - 12x^2 + 11) dx = \pi \left(\frac{x^5}{5} - 4x^3 + 11x \right) \Big|_0^1 = \frac{36\pi}{5}.$$

Using shells, we have

$$V = 2\pi \int_0^2 (6-y)\sqrt{y} dy = 2\pi \int_0^1 (6y^{1/2} - y^{3/2}) dy = 2\pi \left(4y^{3/2} - \frac{2y^{5/2}}{5} \right) \Big|_0^1 = 2\pi \left(4 - \frac{2}{5} \right) = \frac{36\pi}{5}.$$

6.4.48 Using washers, we have

$$V = \pi \int_0^1 (4 - (2 - \sqrt{y})^2) dy = \pi \int_0^1 (4\sqrt{y} - y) dy = \pi \left(\frac{8y^{3/2}}{3} - \frac{y^2}{2} \right) \Big|_0^1 = \frac{13\pi}{6}.$$

Using shells, we have

$$\begin{aligned} V &= 2\pi \int_0^1 (2-x)(1-x^2) dx = 2\pi \int_0^1 (2-2x^2-x+x^3) dx \\ &= 2\pi \left(2x - \frac{2x^3}{3} - \frac{x^2}{2} + \frac{x^4}{4} \right) \Big|_0^1 = 2\pi \left(2 - \frac{2}{3} - \frac{1}{2} + \frac{1}{4} \right) = \frac{13\pi}{6}. \end{aligned}$$

6.4.49

a. The volume is $V = 2\pi \int_0^r y(2\sqrt{r^2 - y^2}) dy$. Let $u = r^2 - y^2$, so that $du = -2y dy$. Substituting gives

$$2\pi \int_{r^2}^0 -\sqrt{u} du = 2\pi \int_0^{r^2} u^{1/2} du = 2\pi \frac{2u^{3/2}}{3} \Big|_0^{r^2} = \frac{4}{3}\pi r^3.$$

b. The volume is

$$\begin{aligned} V &= \pi \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx = \pi \int_{-r}^r (r^2 - x^2) dx = \pi \left(r^2x - \frac{x^3}{3} \right) \Big|_{-r}^r \\ &= \pi \left(r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{(-r)^3}{3} \right) \right) = 2\pi \left(r^3 - \frac{r^3}{3} \right) = \frac{4}{3}\pi r^3. \end{aligned}$$

6.4.50

a. Using washers, the volume is

$$V = \pi \int_{-a}^a b^2 \left(1 - \frac{x^2}{a^2} \right) dx = \pi b^2 \left(x - \frac{x^3}{3a^2} \right) \Big|_{-a}^a = \pi b^2 \left(a - \frac{1}{3}a - \left(-a + \frac{(-a)^3}{3} \right) \right) = \frac{4}{3}\pi ab^2.$$

Using shells, the volume is $V = 2\pi \int_0^b y \left(2a\sqrt{1 - \frac{y^2}{b^2}} \right) dy$. Let $u = 1 - \frac{y^2}{b^2}$. Then $du = \frac{-2y}{b^2} dy$.

Substituting gives

$$V = -2\pi ab^2 \int_1^0 \sqrt{u} du = 2\pi ab^2 \int_0^1 u^{1/2} du = 2\pi ab^2 \left(\frac{2u^{3/2}}{3} \right) \Big|_0^1 = \frac{4}{3}\pi ab^2.$$

b. For the rugby union football, we have $\frac{4}{3}\pi \cdot 6 \cdot (3.82)^2 \approx 366.75 \text{ in}^3$. For the American football, we have $\frac{4}{3}\pi \cdot 5.62 \cdot (3.38)^2 \approx 268.94 \text{ in}^3$.

c. $\frac{366.74}{269.94} \approx 1.36.$

6.4.51

a. $V = 2\pi \int_1^5 2x\sqrt{4 - (x-3)^2} dx.$

b. $V = 2\pi \int_0^2 ((3 + \sqrt{4 - y^2})^2 - (3 - \sqrt{4 - y^2})^2) dy = 24\pi \int_0^2 \sqrt{4 - y^2} dy.$

c. Note that the last integral in part b represents $\frac{1}{4}$ of the area of the circle of radius 2 centered at the origin, so its value is $\frac{1}{4}\pi \cdot 4 = \pi$. So the volume is $24\pi \cdot \pi = 24\pi^2$.

6.4.52

a. $V = \pi \int_0^h (rx/h)^2 dx = \frac{\pi r^2}{h^2} (x^3/3) \Big|_0^h = \frac{\pi r^2 h}{3}.$

b. $V = 2\pi \int_0^r y(h - (h/r)y) dy = 2\pi h (y^2/2 - y^3/(3r)) \Big|_0^r = \frac{\pi r^2 h}{3}.$

6.4.53 First, note that the curve and the line intersect at $(0, 0)$ and $(1, 0)$.

We choose the washer method:

$$\begin{aligned} V &= \pi \int_0^1 (x - x^4)^2 dx = \pi \int_0^1 (x^2 - 2x^5 + x^8) dx \\ &= \pi (x^3/3 - x^6/3 + x^9/9) \Big|_0^1 = \pi/9 \end{aligned}$$

6.4.54 First, note that the curve and the line intersect at $(0, 0)$ and $(1, 0)$.

We choose the shell method:

$$\begin{aligned} V &= 2\pi \int_0^1 x(x - x^4) dx = 2\pi \int_0^1 (x^2 - x^5) dx \\ &= 2\pi (x^3/3 - x^6/6) \Big|_0^1 = 2\pi(1/3 - 1/6) = \pi/3. \end{aligned}$$

6.4.55 Using washers:

$$V = 2\pi \int_0^1 ((2 - x^2)^2 - (x^2)^2) dx = 2\pi \int_0^1 (4 - 4x^2) dx = 2\pi (4x - 4x^3/3) \Big|_0^1 = \frac{16\pi}{3}.$$

6.4.56 Using washers:

$$\begin{aligned} V &= \pi \int_{\pi/6}^{5\pi/6} (\sin^2 x - (1 - \sin x)^2) dx = \pi \int_{\pi/6}^{5\pi/6} (2\sin x - 1) dx \\ &= -\pi (2\cos x + x) \Big|_{\pi/6}^{5\pi/6} = 2\sqrt{3}\pi - \frac{2\pi^2}{3}. \end{aligned}$$

6.4.57 Using shells:

$$V = 2\pi \int_2^6 x(2x + 2 - x) dx = 2\pi \int_2^6 (x^2 + 2x) dx = 2\pi (x^3/3 + x^2) \Big|_2^6 = \frac{608\pi}{3}.$$

6.4.58 Using disks:

$$V = \pi \int_0^2 (x^3)^2 dx = \pi \left(x^7/7 \right) \Big|_0^2 = \frac{128\pi}{7}.$$

6.4.59 Using shells:

$$V = 2\pi \int_0^1 y(e^{y^2} - e^{y^2/2}) dy = 2\pi \left(e^{y^2}/2 - e^{y^2/2} \right) \Big|_0^1 = \pi(e - 2\sqrt{e} - (1 - 2)) = \pi(\sqrt{e} - 1)^2.$$

6.4.60 Using shells:

$$V = 2\pi \int_0^6 x(2x + 2 - 2) dx = 2\pi \left(2x^3/3 \right) \Big|_0^6 = 288\pi.$$

6.4.61 The curves intersect when $x^2 = 2 - x$, or $x^2 + x - 2 = (x - 1)(x + 2) = 0$, so for $x = 1$ and $x = -2$. Only $x = 1$ is in the first quadrant; the corresponding y coordinate is $y = 1$. Also, $2 - x \geq x^2$ on $[0, 1]$.

Using shells, the height of each shell is $2 - x - x^2$, so the volume is

$$V = 2\pi \int_0^1 x(2 - x - x^2) dx = 2\pi \left(x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = \frac{5\pi}{6}.$$

6.4.62 Using disks:

$$V = \pi \int_0^4 (\sqrt{x})^2 dx = \pi \left(x^2/2 \right) \Big|_0^4 = 8\pi.$$

6.4.63

- True. Otherwise, we wouldn't have shells!
- False. Either method can be used when revolving around either axis.
- True.

6.4.64 Consider the rectangle in the first quadrant bounded by $x = 2$, $x = 4$, $y = 6$ and $y = 0$. The solid in question is formed by revolving this region around the y -axis. The volume is

$$V = 2\pi \int_2^4 6x dx = 2\pi \left(3x^2 \right) \Big|_2^4 = 72\pi.$$

6.4.65 Consider the region in the first quadrant bounded by the coordinate axes and the line $y = 8 - (8/3)x$. We can generate the desired cone by revolving this region around the y -axis. We then have

$$V = 2\pi \int_0^3 x(8 - (8/3)x) dx = \frac{16\pi}{3} \int_0^3 (3x - x^2) dx = \frac{16\pi}{3} \left(3x^2/2 - x^3/3 \right) \Big|_0^3 = 24\pi.$$

6.4.66 Consider the region in the first and second quadrants bounded by $x = 3$, the circle $x^2 + y^2 = 36$. The solid in question is formed by revolving this region around the y -axis. The volume is $V = 4\pi \int_3^6 x\sqrt{36 - x^2} dx$. This integral can be computed using the substitution $u = 36 - x^2$. We then have

$$V = 2\pi \int_0^{27} u^{1/2} du = 2\pi \left(2u^{3/2}/3 \right) \Big|_0^{27} = 108\sqrt{3}\pi.$$

6.4.67 Consider the triangle in the first quadrant bounded by $y = 0$, $x = 3$, and $y = 9 - (3/2)x$. The solid in question is formed by revolving this region around the y -axis. The volume is

$$V = 2\pi \int_3^6 x(9 - (3/2)x) dx = 2\pi \left(9x^2/2 - x^3/2 \right) \Big|_3^6 = 54\pi.$$

6.4.68

$$V = 2\pi \int_r^R x \cdot 6(1 - x^2/R^2) dx = 12\pi \left(x^2/2 - x^4/4R^2 \right) \Big|_r^R = \frac{3\pi(R^2 - r^2)^2}{R^2}.$$

6.4.69

a. $V_1 = \pi \int_0^1 (ax^2 + 1)^2 dx = \pi \int_0^1 (a^2x^4 + 2ax^2 + 1) dx = \pi(a^2/5 + 2a/3 + 1).$

$$V_2 = 2\pi \int_0^1 x(ax^2 + 1) dx = 2\pi(a/4 + 1/2) = \pi a/2 + \pi.$$

- b. These are equal when $\pi a/2 + \pi = \pi a^2/5 + 2a\pi/3 + \pi$, or when $\pi a^2/5 + a\pi/6 = 0$, which occurs when $a = 0$ and when $a = -5/6$.

6.4.70

a. $V = \pi \int_{r-h}^r (r^2 - y^2) dy = \pi \left(r^2y - y^3/3 \right) \Big|_{r-h}^r = \pi(2r^3/3 - (r^2(r-h) - (r-h)^3/3)) = \frac{\pi h^2}{3}(3r-h).$

b.

$$\begin{aligned} V &= 2\pi \int_0^{\sqrt{2rh-h^2}} x(\sqrt{r^2-x^2} - (r-h)) dx = 2\pi \left(\frac{-1}{3}(\sqrt{r^2-x^2})^3 - (r-h)x^2/2 \right) \Big|_0^{\sqrt{2rh-h^2}} \\ &= 2\pi \left(\frac{-1}{3}(\sqrt{r^2-2rh+h^2})^3 - (r-h)(2rh-h^2)\frac{1}{2} + \frac{1}{3}r^3 \right) \\ &= 2\pi \left(-(r-h)^3/3 - r^2h + rh^2/2 + rh^2 - h^3/2 + r^3/3 \right) = \pi h^2r - \frac{\pi h^3}{3}. \end{aligned}$$

- c. If we take slices perpendicular to the y -axis, we have circles whose area $A(y) = \pi(r^2 - y^2)$. So $V = \pi \int_{r-h}^r (r^2 - y^2) dy$, which is exactly the integral computed in part (a) above.

Note that all three approaches led to the same result, and the result is consistent with other formulas.

For example, when $h = 0$ we have no figure, so our volume is 0. When $h = r$, we have $V = \frac{2\pi r^3}{3}$, the volume of a hemisphere.

6.4.71 $V = 2\pi \int_0^2 xf(x^2) dx = \pi \int_0^4 f(u) du$, where $u = x^2$. Thus, $V = 10\pi$.

6.4.72

- a. Consider the region in the first quadrant bounded by the coordinate axes and $y = 8 - 2x$. The integral on the left represents the volume of the solid obtained when this region is revolved around the x -axis, using the disk/washer method, and the integral on the right represents the same volume calculated using the shell method. Hence, they are equal.
- b. Consider the region in the first quadrant bounded by the line $x = 0$, the line $y = 5$, and the curve $y = x^2 + 1$. The integral on the left represents $\frac{1}{\pi}$ times the volume of the solid obtained when this region is revolved around the x -axis, using the disk/washer method, and the integral on the right represents $\frac{1}{\pi}$ times the same volume calculated using the shell method. Hence, they are equal.

6.4.73

- a. The longest diagonal of the cube is equal to the diameter of the sphere, which is $\sqrt{r^2 + r^2 + r^2} = \sqrt{3}r$. Thus, if R is the radius of the sphere, we must have $R = \frac{\sqrt{3}}{2}r$.

Now analyzing the cone which contains the sphere, we see that the height h of the cone is $3R$, so $h = \frac{3\sqrt{3}}{2}r$.

Now the volume of the cylinder is $V = \pi \left(\frac{3r}{2}\right)^2 \left(\frac{3\sqrt{3}r}{2}\right) = \frac{27\sqrt{3}\pi r^3}{8}$.

- b. Imagine the cone with its vertex up. Consider the plane which contains the vertex of the cone and two non-adjacent vertices of the cube's bottom face. The cross section of this plane with the cone and cube consists of an $r \times \sqrt{2}r$ rectangle (where r is the side length of the cube) and an isosceles triangle of base 2 and height 3, with the rectangle inscribed in the triangle, and the longer side of the rectangle lying on the base of the triangle. Using similar triangles, we have $\frac{r}{3} = \frac{1 - (r/\sqrt{2})}{1}$, so $r = \frac{3\sqrt{2}}{3 + \sqrt{2}}$,

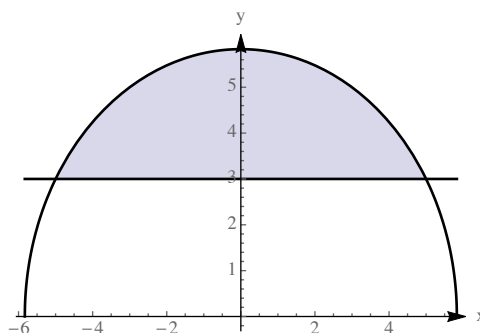
and the volume of the cube is $\frac{54\sqrt{2}}{(3 + \sqrt{2})^3}$.

- c. Imagine the sphere with the hole as being obtained by revolving the region pictured around the x -axis, where the relevant curves are $y = r$ and $y = \sqrt{R^2 - x^2}$ where r is the radius of the hole and R is the radius of the sphere. Note that if you draw the triangle with vertices $(0, 0)$, $(5, 0)$, and $(5, r)$, you have a right triangle with legs of length 5 and r , and hypotenuse of length R , so $25 + r^2 = R^2$.

The volume we are interested in is

$$V = 2\pi \int_0^5 ((R^2 - x^2) - r^2) dx = 2\pi \int_0^5 (25 - x^2) dx = 2\pi (25x - x^3/3) \Big|_0^5 = \frac{500\pi}{3}.$$

Note that (surprisingly!), the result doesn't depend on the radius of the original sphere.



6.4.74 Vertical slices of the wedge are triangles with area $\frac{1}{2}xh$, where x is the base and h is the height. Note that $h = x \tan \theta$, so the triangles have area $\frac{1}{2}x^2 \tan \theta$. Now if we think of the curved part of the base of the wedge as having equation $x^2 + y^2 = a^2$, then we have $x^2 = a^2 - y^2$. So the volume of the wedge is given by

$$\int_{-a}^a \frac{1}{2} \tan \theta (a^2 - y^2) dy = \int_0^a \tan \theta (a^2 - y^2) dy = \tan \theta (a^2 y - y^3/3) \Big|_0^a = \frac{2a^3}{3} \tan \theta.$$

6.4.75

- a. The cross sections of such an object are washers, with inner radius given by $g(x) - y_0$ and outer radius given by $f(x) - y_0$, so the volume is given by $\pi \int_a^b ((f(x) - y_0)^2 - (g(x) - y_0)^2) dx$.
- b. The cross section of this object are also washers, and this time the inner radius is $y_0 - f(x)$ and the outer radius is $y_0 - g(x)$, so the volume is given by $\pi \int_a^b ((y_0 - g(x))^2 - (y_0 - f(x))^2) dx$.

6.4.76

- a. The result of revolving a slice around the line $x = x_0$ is a shell with the same height as before, but with the radius being $\bar{x}_k - x_0$ rather than $\bar{x}_k$. Thus, when the volumes of the shells are added and the limit is taken, the resulting integral is $\int_a^b 2\pi(x - x_0)(f(x) - g(x)) dx$ instead of $\int_a^b 2\pi(x)(f(x) - g(x)) dx$.
- b. When $x_0 > b$, the radius of a typical shell is given by $x_0 - \bar{x}_k$, so the volume is given by

$$\int_a^b 2\pi(x_0 - x)(f(x) - g(x)) dx.$$

6.5 Length of Curves

6.5.1 Given $f(x)$ and a and b , compute $f'(x)$ and then compute $\int_a^b \sqrt{1 + f'(x)^2} dx$.

6.5.2 Given $g(y)$ and c and d , compute $g'(y)$ and then compute $\int_c^d \sqrt{1 + g'(y)^2} dy$.

6.5.3 Because $f'(x) = 3x^2$, the arc length is $\int_{-2}^5 \sqrt{1 + f'(x)^2} dx = \int_{-2}^5 \sqrt{1 + 9x^4} dx$.

6.5.4 Because $f'(x) = -6 \sin 3x$, the arc length is $\int_{-\pi}^{\pi} \sqrt{1 + f'(x)^2} dx = \int_{-\pi}^{\pi} \sqrt{1 + 36 \sin^2 3x} dx$.

6.5.5 Because $f'(x) = -2e^{-2x}$, the arc length is $\int_0^2 \sqrt{1 + f'(x)^2} dx = \int_0^2 \sqrt{1 + 4e^{-4x}} dx$.

6.5.6 Because $f'(x) = \frac{1}{x}$, the arc length is $\int_1^{10} \sqrt{1 + f'(x)^2} dx = \int_1^{10} \sqrt{1 + \frac{1}{x^2}} dx = \int_1^{10} \frac{\sqrt{x^2 + 1}}{x} dx$.

6.5.7 $L = \int_1^5 \sqrt{1 + 2^2} dx = 4\sqrt{5}$. Using the fact that this is a straight line segment between the points $(1, 3)$ and $(5, 11)$, we can calculate the arc length using the distance formula in the plane:

$$L = \sqrt{(5 - 1)^2 + (11 - 3)^2} = \sqrt{16 + 64} = \sqrt{80} = 4\sqrt{5}.$$

6.5.8 $L = \int_{-3}^2 \sqrt{1 + (y')^2} = \int_{-3}^2 \sqrt{1 + (-3)^2} dx = \int_{-3}^2 \sqrt{10} dx = 5\sqrt{10}$. Using the fact that this is a straight line segment between the points $(-3, 13)$ and $(2, -2)$, we can calculate the arc length using the distance formula in the plane:

$$L = \sqrt{(2 - (-3))^2 + (-2 - 13)^2} = \sqrt{25 + 225} = \sqrt{250} = 5\sqrt{10}.$$

6.5.9 $L = \int_{-2}^6 \sqrt{1 + (y')^2} dx = \int_{-2}^6 \sqrt{1 + (-8)^2} dx = \int_{-2}^6 \sqrt{65} dx = 8\sqrt{65}$.

6.5.10 $y' = \frac{1}{2}(e^x - e^{-x})$, so $1 + (y')^2 = 1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = \left(\frac{1}{2}(e^x + e^{-x})\right)^2$. Thus,

$$L = \int_{-\ln 2}^{\ln 2} \frac{1}{2}(e^x + e^{-x}) dx = \frac{1}{2}(e^x - e^{-x}) \Big|_{-\ln 2}^{\ln 2} = \frac{1}{2}(2 - 1/2 - (1/2 - 2)) = \frac{3}{2}.$$

6.5.11 $y' = \sqrt{x}/2$, so $1 + (y')^2 = 1 + x/4$. Thus,

$$L = \int_0^{60} \sqrt{1 + (x/4)} dx = \int_0^{60} \frac{1}{2} \sqrt{x+4} dx = \left(\frac{1}{3} (x+4)^{3/2} \right) \Big|_0^{60} = \frac{504}{3} = 168.$$

6.5.12 $y' = 3/x - x/(12)$, so $1 + (y')^2 = 1 + \frac{9}{x^2} - \frac{1}{2} + \frac{x^2}{144} = \left(\frac{3}{x} + \frac{x}{12} \right)^2$. Thus,

$$L = \int_1^6 \left(\frac{3}{x} + \frac{x}{12} \right) dx = (3 \ln x + x^2/24) \Big|_1^6 = 3 \ln 6 + \frac{35}{24}.$$

6.5.13 $y' = x(x^2 + 2)^{1/2}$, so $1 + (y')^2 = 1 + x^2(x^2 + 2) = x^4 + 2x^2 + 1 = (x^2 + 1)^2$. Thus,

$$L = \int_0^1 (x^2 + 1) dx = (x^3/3 + x) \Big|_0^1 = \frac{4}{3}.$$

6.5.14 $y' = \sqrt{x}/2 - 1/(2\sqrt{x})$, so $1 + (y')^2 = 1 + \frac{x}{4} - \frac{1}{2} + \frac{1}{4x} = \left(\frac{\sqrt{x}}{2} + \frac{1}{2\sqrt{x}} \right)^2$. Thus,

$$L = \int_4^{16} (x^{1/2}/2 + (1/2)x^{-1/2}) dx = (x^{3/2}/3 + x^{1/2}) \Big|_4^{16} = 64/3 + 4 - (8/3 + 2) = 2 + 56/3 = \frac{62}{3}.$$

6.5.15 $y' = x^3 - \frac{1}{4x^3}$, so $1 + (y')^2 = 1 + x^6 - \frac{1}{2} + \frac{1}{16x^6} = \left(x^3 + \frac{1}{4x^3} \right)^2$. Thus,

$$L = \int_1^2 (x^3 + \frac{1}{4x^3}) dx = \left(x^4/4 + \frac{-1}{8} x^{-2} \right) \Big|_1^2 = 4 - (1/32) - (1/4 - (1/8)) = \frac{123}{32}.$$

6.5.16 $y' = x^{1/2} - (1/(4x^{1/2}))$, so $1 + (y')^2 = 1 + x - 1/2 + \frac{1}{16x} = \left(x^{1/2} + \frac{1}{4x^{1/2}} \right)^2$. Thus,

$$L = \int_1^9 \left(x^{1/2} + \frac{1}{4x^{1/2}} \right) dx = \left(2x^{3/2}/3 + \sqrt{x}/2 \right) \Big|_1^9 = 18 + 3/2 - (2/3 + 1/2) = \frac{55}{3}.$$

6.5.17 $\frac{dx}{dy} = y^3 - 1/(4y^3)$, so $1 + \left(\frac{dx}{dy} \right)^2 = 1 + y^6 - 1/2 + 1/(16y^6) = \left(y^3 + \frac{1}{4y^3} \right)^2$. So

$$L = \int_1^2 (y^3 + \frac{1}{4} \cdot y^{-3}) dy = \left(y^4/4 - \frac{1}{8y^2} \right) \Big|_1^2 = 4 - (1/32) - (1/4 - 1/8) = \frac{123}{32}.$$

6.5.18 $\frac{dx}{dy} = 2\sqrt{2}e^{\sqrt{2}y} - \frac{\sqrt{2}}{16}e^{-\sqrt{2}y}$, so

$$1 + \left(\frac{dx}{dy} \right)^2 = 1 + 8e^{2\sqrt{2}y} - (1/2) + \frac{2}{16^2}e^{-2\sqrt{2}y} = \left(2\sqrt{2}e^{\sqrt{2}y} + \frac{\sqrt{2}}{16}e^{-\sqrt{2}y} \right)^2.$$

Then

$$L = \int_0^{(\ln 2)/\sqrt{2}} (2\sqrt{2}e^{\sqrt{2}y} + \frac{\sqrt{2}}{16}e^{-\sqrt{2}y}) dy = \left(2e^{\sqrt{2}y} - \frac{1}{16}e^{-\sqrt{2}y} \right) \Big|_0^{(\ln 2)/\sqrt{2}} = (4 - (1/32) - (2 - (1/16))) = \frac{65}{32}.$$

6.5.19 $\frac{dx}{dy} = 2$, so $1 + \left(\frac{dx}{dy} \right)^2 = 1 + 4 = 5$, so $L = \int_{-3}^4 \sqrt{5} dy = 7\sqrt{5}$.

6.5.20 $y = \ln(x - \sqrt{x^2 - 1})$, so $e^y = x - \sqrt{x^2 - 1}$, and $e^{-y} = \frac{1}{x - \sqrt{x^2 - 1}} = \frac{x + \sqrt{x^2 - 1}}{x^2 - (x^2 - 1)} = x + \sqrt{x^2 - 1}$.

Thus $\frac{e^y + e^{-y}}{2} = x$.

We have $\frac{dx}{dy} = \frac{e^y - e^{-y}}{2}$, so $1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{2y}/4 - 1/2 + e^{-2y}/4 = \left(\frac{e^y + e^{-y}}{2}\right)^2$. Thus,

$$L = \int_{\ln(\sqrt{2}-1)}^0 \frac{e^y + e^{-y}}{2} dy = \left(\frac{e^y - e^{-y}}{2}\right) \Big|_{\ln(\sqrt{2}-1)}^0 = 0 - \frac{\sqrt{2} - 1 - \frac{1}{\sqrt{2}-1}}{2} \cdot \frac{\sqrt{2} - 1}{\sqrt{2} - 1} = \frac{-2 + 2\sqrt{2} - 1 + 1}{2(\sqrt{2} - 1)} = 1.$$

6.5.21

a. $y' = 2x$, so $1 + (y')^2 = 1 + 4x^2$, so $L = \int_{-1}^1 \sqrt{1 + 4x^2} dx$.

b. $L = \int_{-1}^1 \sqrt{1 + 4x^2} dx \approx 2.958$.

6.5.22

a. $y' = \cos x$, so $1 + (y')^2 = 1 + \cos^2 x$, so $L = \int_0^\pi \sqrt{1 + \cos^2 x} dx$.

b. $L = \int_0^\pi \sqrt{1 + \cos^2 x} dx \approx 3.820$.

6.5.23

a. $y' = 1/x$, so $1 + (y')^2 = 1 + (1/x)^2 = \frac{x^2 + 1}{x^2}$, so $L = \int_1^4 \frac{\sqrt{x^2 + 1}}{x} dx$.

b. $L = \int_1^4 \frac{\sqrt{x^2 + 1}}{x} dx \approx 3.343$.

6.5.24

a. $y' = x^2$, so $1 + (y')^2 = 1 + x^4$, so $L = \int_{-1}^1 \sqrt{1 + x^4} dx$.

b. $L = \int_{-1}^1 \sqrt{1 + x^4} dx \approx 2.179$.

6.5.25

a. $x' = \frac{1}{2\sqrt{y-2}}$, so $1 + (x')^2 = 1 + \frac{1}{4(y-2)} = \frac{4y-7}{4y-8}$, so $L = \int_3^4 \sqrt{\frac{4y-7}{4y-8}} dy$.

b. $L = \int_3^4 \sqrt{\frac{4y-7}{4y-8}} dy \approx 1.083$.

6.5.26

a. $y' = -\frac{16}{x^3}$, so $1 + (y')^2 = 1 + \frac{16^2}{x^6} = \frac{x^6 + 16^2}{x^6}$, so $L = \int_1^4 \sqrt{\frac{x^6 + 16^2}{x^6}} dx$.

b. $L = \int_1^4 \frac{\sqrt{x^6 + 16^2}}{x^3} dx \approx 8.708$.

6.5.27

a. $y' = -2 \sin 2x$, so $1 + (y')^2 = 1 + 4 \sin^2 2x$, so $L = \int_0^\pi \sqrt{1 + 4 \sin^2 2x} \, dx$.

b. $L = \int_0^\pi \sqrt{1 + 4 \sin^2 2x} \, dx \approx 5.270$.

6.5.28

a. $y' = 4 - 2x$, so $1 + (y')^2 = 1 + 16 - 16x + 4x^2 = 4x^2 - 16x + 17$, so $L = \int_0^4 \sqrt{4x^2 - 16x + 17} \, dx$.

b. $L = \int_0^4 \sqrt{4x^2 - 16x + 17} \, dx \approx 9.294$.

6.5.29

a. $y' = -1/x^2$, so $1 + (y')^2 = 1 + \frac{1}{x^4} = \frac{x^4 + 1}{x^4}$. Thus, $L = \int_1^{10} \frac{\sqrt{x^4 + 1}}{x^2} \, dx$.

b. $L = \int_1^{10} \frac{\sqrt{x^4 + 1}}{x^2} \, dx \approx 9.153$.

6.5.30

a. $x' = \frac{-2y}{(y^2 + 1)^2}$, so $1 + (x')^2 = 1 + \frac{4y^2}{(y^2 + 1)^4} = \frac{(y^2 + 1)^4 + 4y^2}{(y^2 + 1)^4}$, so $L = \int_{-5}^5 \frac{\sqrt{(y^2 + 1)^4 + 4y^2}}{(y^2 + 1)^2} \, dy$.

b. $L = \int_{-5}^5 \frac{\sqrt{(y^2 + 1)^4 + 4y^2}}{(y^2 + 1)^2} \, dy \approx 10.369$.

6.5.31 $f'(x) = 0.00074x$, so $L = \int_{-640}^{640} \sqrt{1 + (0.00074x)^2} \, dx \approx 1326.4$ meters.

6.5.32 $f'(x) = \frac{-630}{239.2} \sinh(x/239.2)$, where $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

So

$$L = \int_{-315}^{315} \sqrt{1 + \left(\frac{630}{239.2} \sinh(x/239.2) \right)^2} \, dx \approx 1472.17 \text{ ft.}$$

6.5.33

a. False. For example, if $f(x) = x^2$, the first integrand is $\sqrt{1 + 4x^2}$ and the second is $1 + 2x$, which clearly yield different values for (for example) $a = 0$ and $b = 1$.

b. True. They are both equal to $\int_a^b \sqrt{1 + f'(x)^2} \, dx$.

c. False. Because $\sqrt{1 + f'(x)^2} > 0$, arc length can't be negative.

6.5.34 $y' = m$, so $1 + (y')^2 = 1 + m^2$, and $L = \int_a^b \sqrt{1 + m^2} \, dx = (b - a)\sqrt{1 + m^2}$.

To see that this is the same result as the one given by the distance formula, consider the points $(a, ma + c)$ and $(b, mb + c)$. The distance between them is

$$\sqrt{(b - a)^2 + (mb + c - (ma + c))^2} = \sqrt{(b - a)^2 + m^2(b - a)^2} = \sqrt{(b - a)^2} \sqrt{1 + m^2} = (b - a)\sqrt{1 + m^2}.$$

6.5.35

- a. We are seeking functions $f(x)$ so that $f'(x) = \pm 4x^2$, so any function of the form $f(x) = \pm 4x^3/3 + C$ will work.
- b. We are seeking functions $f(x)$ so that $f'(x) = \pm 6\cos(2x)$, so any function of the form $f(x) = \pm 3\sin(2x) + C$ will work.

6.5.36 Because $f'(x) = \pm 4x^{-3}$, we have $f(x) = \frac{\pm 2}{x^2} + C$. Because $f(1) = 5$, we must have either $f(x) = 7 - \frac{2}{x^2}$ or $f(x) = 3 + \frac{2}{x^2}$.

6.5.37 The length of the parabola is $\int_{-1}^1 \sqrt{1 + 4x^2} dx \approx 2.9597$.

The length of the given cosine function is $\int_{-1}^1 \sqrt{1 + \frac{\pi^2}{4} \sin^2(\pi x/2)} dx \approx 2.924$, so the parabola is longer.

6.5.38 $f'(x) = \sin x$, so the arc length is given by $\int_0^\pi \sqrt{1 + \sin^2 x} dx$.

6.5.39

- a. Let $u = 2x$. Then the given integral is equal to $\frac{1}{2} \int_a^b \sqrt{1 + f'(u)^2} du = \frac{L}{2}$.
- b. Let $u = cx$. Then the given integral is equal to $\frac{1}{c} \int_a^b \sqrt{1 + f'(u)^2} du = \frac{L}{c}$.

6.5.40 If f is odd, then the symmetry of f assures that the portion of the curve from 0 to b matches exactly the portion of the curve from $-b$ to 0, as can be seen by rotating the original curve about the y -axis and then about the x -axis. The same is true for f even, although the two portions match up after just rotating about the y -axis.

Suppose f is even, and recall that this means that f' is odd. Consider

$$L_- = \int_{-b}^0 \sqrt{1 + f'(x)^2} dx.$$

Let $u = -x$. The integral becomes

$$L_- = \int_0^b \sqrt{1 + f'(-x)^2} dx = \int_0^b \sqrt{1 + (-f'(x))^2} dx = \int_0^b \sqrt{1 + f'(x)^2} dx = L_+.$$

If f is odd, so that f' is even, we have $L_- = \int_{-b}^0 \sqrt{1 + f'(x)^2} dx$. Let $u = -x$. Then the integral becomes

$$L_- = \int_0^b \sqrt{1 + f'(-x)^2} dx = \int_0^b \sqrt{1 + f'(x)^2} dx = L_+.$$

6.5.41

- a. $f'(x) = Aae^{ax} - \frac{1}{4Aa}e^{-ax}$, so $1 + f'(x)^2 = 1 + (Aae^{ax})^2 - \frac{1}{2} + \left(\frac{1}{4Aa}e^{-ax}\right)^2 = (Aae^{ax})^2 + \frac{1}{2} + \left(\frac{1}{4Aa}e^{-ax}\right)^2 = \left(Aae^{ax} + \frac{1}{4Aa}e^{-ax}\right)^2$. So the arc length for $c \leq x \leq d$ is

$$L = \int_c^d \left(Aae^{ax} + \frac{1}{4Aa}e^{-ax}\right) dx = \left(Ae^{ax} - \frac{1}{4Aa^2}e^{-ax}\right) \Big|_c^d.$$

b. Applying the previous result with $c = 0$ and $d = \ln 2$, we have

$$L = \left(Ae^{ax} - \frac{1}{4Aa^2} e^{-ax} \right) \Big|_0^{\ln 2} = A2^a - \frac{1}{4Aa^2} 2^{-a} - A + \frac{1}{4Aa^2} = A(2^a - 1) - \frac{1}{4a^2 A} (2^{-a} - 1).$$

6.5.42

a. $y' = \frac{2n+1}{2n} x^{1/(2n)}$, so $1 + (y')^2 = 1 + \left(\frac{2n+1}{2n} \right)^2 x^{1/n}$.

So

$$L = \int_0^a \sqrt{1 + \left(\frac{2n+1}{2n} \right)^2 x^{1/n}} dx.$$

b. For the sake of simplicity, let $r = \frac{2n+1}{2n}$, and let $d = \sqrt{1 + r^2 a^{1/n}}$. If we let $u^2 = 1 + r^2 x^{1/n}$, then $x = \left(\frac{u^2 - 1}{r^2} \right)^n$, and $dx = \frac{n}{r^{2n}} (u^2 - 1)^{n-1} \cdot 2u du$, and our integral becomes

$$\int_1^d u \cdot \frac{n}{r^{2n}} (u^2 - 1)^{n-1} \cdot 2u du = \frac{2n}{r^{2n}} \int_1^d u^2 (u^2 - 1)^{n-1} du.$$

c. The binomial theorem assures us that

$$(u^2 - 1)^{n-1} = u^{2n-2} - \binom{n-1}{1} u^{2n-4} + \binom{n-1}{2} u^{2n-6} - \dots + (-1)^{n-1}.$$

Thus

$$\begin{aligned} u^2 (u^2 - 1)^{n-1} &= u^2 \left(u^{2n-2} - \binom{n-1}{1} u^{2n-4} + \binom{n-1}{2} u^{2n-6} - \dots + (-1)^{n-1} \right) \\ &= u^{2n} - \binom{n-1}{1} u^{2n-2} + \binom{n-1}{2} u^{2n-4} - \dots + (-1)^{n-1} u^2. \end{aligned}$$

Then

$$\begin{aligned} L &= \frac{2n}{r^{2n}} \left(u^{2n+1}/(2n+1) - \binom{n-1}{1} u^{2n-1}/(2n-1) \right. \\ &\quad \left. + \binom{n-1}{2} u^{2n-3}/(2n-3) - \dots + (-1)^{n+1} u^3/3 \right) \Big|_1^d. \end{aligned}$$

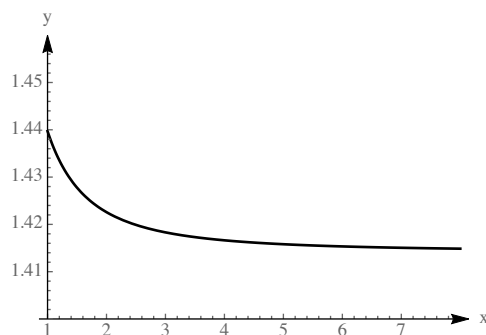
d. For $n = 2$ and $a = 1$ we have $r = (5/4)$ and $d = \sqrt{41}/4$. Using the previous result, we have

$$L = \frac{4^5}{5^4} \left(u^5/5 - u^3/3 \right) \Big|_1^{\sqrt{41}/4} = \frac{2048}{9375} + \frac{1763\sqrt{41}}{9375} \approx 1.423.$$

For $n = 3$ and $a = 1$ we have $r = 7/6$ and $d = \sqrt{85}/6$. Using the previous results, we have

$$L = \frac{6^7}{7^6} \left(u^7/7 - 2u^5/5 + u^3/3 \right) \Big|_1^{\sqrt{85}/6} = \frac{142885\sqrt{85}}{823543} - \frac{746496}{4117715} \approx 1.418.$$

e.



6.6 Surface Area

6.6.1 $S = \pi r \sqrt{r^2 + h^2} = 3\pi \sqrt{9 + 16} = 15\pi.$

6.6.2 By the formula developed in the section: $S = \pi(4)(6^2 - 2^2)\sqrt{1 + 4^2} = 128\sqrt{17}\pi.$ Or it can be computed via integration:

$$S = 2\pi \int_2^6 4x\sqrt{1 + 4^2} dx = 8\sqrt{17}\pi \left(x^2/2 \right) \Big|_2^6 = 8\sqrt{17}\pi(18 - 2) = 128\sqrt{17}\pi.$$

6.6.3 Evaluate $\int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2} dx.$

6.6.4 Evaluate $\int_c^d 2\pi g(y)\sqrt{1 + g'(y)^2} dy.$

6.6.5

a. $S = \int_0^2 2\pi(2 - x)\sqrt{1 + (-1)^2} dx = 2\sqrt{2}\pi \int_0^2 (2 - x) dx = 2\sqrt{2}\pi \left(2x - \frac{x^2}{2} \right) \Big|_0^2 = 4\sqrt{2}\pi.$

b. The surface is a cone with $\ell = 2\sqrt{2}$ and $r = 2$; using the formula for the lateral surface area of a cone, we have $s = \pi r \ell = 4\sqrt{2}\pi.$

6.6.6

a. $S = \int_c^d 2\pi g(y)\sqrt{1 + g'(y)^2} dy = \int_0^8 2\pi \cdot 3\sqrt{1 + 0} dy = 6\pi \int_0^8 dy = 48\pi.$

b. This surface is right circular cylinder of radius $r = 3$ and height $h = 8$; using the formula for the lateral surface area of a cylinder we have $S = 2\pi r h = 2\pi \cdot 3 \cdot 8 = 48\pi.$

6.6.7 $S = 2\pi \int_0^6 (3x + 4)\sqrt{1 + 9} dx = 2\sqrt{10}\pi \left(3x^2/2 + 4x \right) \Big|_0^6 = 2\sqrt{10}\pi(54 + 24) = 156\sqrt{10}\pi.$

6.6.8 $S = 2\pi \int_1^3 (12 - 3x)\sqrt{1 + 9} dx = 2\sqrt{10}\pi \left(12x - 3x^2/2 \right) \Big|_1^3 = 2\sqrt{10}\pi(36 - 27/2 - (12 - 3/2)) = 24\sqrt{10}\pi.$

6.6.9

$$\begin{aligned} S &= 2\pi \int_9^{20} 8\sqrt{x}\sqrt{1 + (16/x)} dx = 16\pi \int_9^{20} \sqrt{x + 16} dx \\ &= 16\pi \left(\frac{2(x + 16)^{3/2}}{3} \right) \Big|_9^{20} = \frac{32\pi}{3} (216 - 125) = \frac{2912\pi}{3}. \end{aligned}$$

6.6.10 $S = 2\pi \int_0^1 x^3 \sqrt{1+9x^4} dx$. Let $u = 1 + 9x^4$ so that $du = 36x^3 dx$. Substituting gives

$$S = \frac{\pi}{18} \int_1^{10} \sqrt{u} du = \frac{\pi}{18} \left(\frac{2u^{3/2}}{3} \right) \Big|_1^{10} = \frac{\pi}{27} (10\sqrt{10} - 1).$$

6.6.11 Note that $x = y^3/3$ for $0 \leq y \leq 2$. The surface area is $S = 2\pi \int_0^2 (y^3/3) \sqrt{1+y^4} dy$. Let $u = 1 + y^4$ so that $du = 4y^3 dy$. Substituting yields $S = 2\pi \int_1^{17} \frac{1}{12} u^{1/2} du = \frac{\pi}{9} \left(u^{3/2} \right) \Big|_1^{17} = \frac{\pi}{9} (17\sqrt{17} - 1)$.

6.6.12 Note that $x = 2\sqrt{y}$ for $1 \leq y \leq 4$. The surface area is $S = 2\pi \int_1^4 2\sqrt{y} \sqrt{1+1/y} dy = 4\pi \int_1^4 \sqrt{1+y} dy = \frac{8\pi}{3} \left((1+y)^{3/2} \right) \Big|_1^4 = \frac{8\pi}{3} (5\sqrt{5} - 2\sqrt{2})$.

6.6.13

$$\begin{aligned} S &= 2\pi \int_{-1/2}^{1/2} \sqrt{1-x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}} \right)^2} dx \\ &= 4\pi \int_0^{1/2} \sqrt{1-x^2} \sqrt{1 + \frac{x^2}{1-x^2}} dx = 4\pi \int_0^{1/2} \sqrt{1-x^2} \sqrt{\frac{1}{1-x^2}} dx = 4\pi \int_0^{1/2} dx = 2\pi. \end{aligned}$$

6.6.14

$$\begin{aligned} S &= 2\pi \int_3^{3.5} \sqrt{-y^2+6y-8} \sqrt{1 + \left(\frac{-2y+6}{2\sqrt{-y^2+6y-8}} \right)^2} dy \\ &= 2\pi \int_3^{3.5} \sqrt{-y^2+6y-8} \sqrt{1 + \left(\frac{-y+3}{\sqrt{-y^2+6y-8}} \right)^2} dy \\ &= 2\pi \int_3^{3.5} \sqrt{-y^2+6y-8+y^2-6y+9} dy = 2\pi \int_3^{3.5} 1 dy = \pi. \end{aligned}$$

6.6.15 Note that $x = \frac{y+1}{4}$ for $3 \leq y \leq 15$. The surface area is

$$\begin{aligned} S &= 2\pi \int_3^{15} \frac{y+1}{4} \sqrt{1 + \frac{1}{16}} dy = \frac{\pi\sqrt{17}}{8} \int_3^{15} (y+1) dy \\ &= \frac{\pi\sqrt{17}}{8} \left(y^2/2 + y \right) \Big|_3^{15} = \frac{\pi\sqrt{17}}{8} (225/2 + 15 - (9/2 + 3)) = \frac{\pi\sqrt{17}}{8} (120) = 15\sqrt{17}\pi. \end{aligned}$$

6.6.16

$$\begin{aligned} S &= 2\pi \int_0^5 \sqrt{4x+6} (\sqrt{1 + (2/\sqrt{4x+6})^2}) dx = 2\pi \int_0^5 \sqrt{4x+6} \sqrt{\left(\frac{4x+6+4}{4x+6} \right)} dx = 2\pi \int_0^5 \sqrt{4x+10} dx \\ &= \frac{4\pi}{3} \left((4x+10)^{3/2} / 4 \right) \Big|_0^5 = \frac{\pi}{3} (30\sqrt{30} - 10\sqrt{10}) = \frac{10\sqrt{10}\pi}{3} (3\sqrt{3} - 1). \end{aligned}$$

6.6.17

$$\begin{aligned}
S &= 2\pi \int_{-2}^2 (1/4)(e^{2x} + e^{-2x}) \sqrt{1 + \frac{(2e^{2x} - 2e^{-2x})^2}{16}} dx = \frac{\pi}{2} \int_{-2}^2 (e^{2x} + e^{-2x}) \sqrt{\frac{16 + 4e^{4x} - 8 + 4e^{-4x}}{16}} dx \\
&= \frac{\pi}{8} \int_{-2}^2 (e^{2x} + e^{-2x}) \sqrt{(2e^{2x} + 2e^{-2x})^2} dx = \frac{\pi}{8} \int_{-2}^2 (e^{2x} + e^{-2x})(2e^{2x} + 2e^{-2x}) dx \\
&= \frac{\pi}{4} \int_{-2}^2 (e^{4x} + 2 + e^{-4x}) dx = \frac{\pi}{4} \left(\frac{e^{4x}}{4} + 2x - \frac{e^{-4x}}{4} \right) \Big|_{-2}^2 \\
&= \frac{\pi}{4} \left(\frac{e^8}{4} + 4 - \frac{e^{-8}}{4} - \left(\frac{e^{-8}}{4} - 4 - \frac{e^8}{4} \right) \right) = \frac{\pi}{4} \left(\frac{e^8}{2} + 8 - \frac{e^{-8}}{2} \right).
\end{aligned}$$

6.6.18

$$\begin{aligned}
S &= 2\pi \int_1^4 \sqrt{5x - x^2} \sqrt{1 + \frac{(5 - 2x)^2}{4(5x - x^2)}} dx = 2\pi \int_1^4 \sqrt{5x - x^2} \sqrt{\frac{20x - 4x^2 + 25 - 20x + 4x^2}{4(5x - x^2)}} dx \\
&= 2\pi \int_1^4 \sqrt{5x - x^2} \sqrt{\frac{25}{4(5x - x^2)}} dx = 2\pi \int_1^4 5/2 dx = 5\pi (x) \Big|_1^4 = 15\pi
\end{aligned}$$

6.6.19

$$\begin{aligned}
S &= 2\pi \int_2^{10} \sqrt{12y - y^2} \sqrt{1 + \frac{(12 - 2y)^2}{4(12y - y^2)}} dy = 2\pi \int_2^{10} \sqrt{12y - y^2} \sqrt{\frac{48y - 4y^2 + 144 - 48y + 4y^2}{4(12y - y^2)}} dy \\
&= 2\pi \int_2^{10} 6 dy = 12\pi(10 - 2) = 96\pi
\end{aligned}$$

6.6.20 Note that $x = \sqrt{1 - (y - 1)^2} = \sqrt{1 - (y^2 - 2y + 1)} = \sqrt{2y - y^2}$.

$$\begin{aligned}
S &= 2\pi \int_1^{3/2} \sqrt{2y - y^2} \sqrt{1 + \frac{(2 - 2y)^2}{4(2y - y^2)}} dy = 2\pi \int_1^{3/2} \sqrt{2y - y^2} \sqrt{\frac{8y - 4y^2 + 4 - 8y + 4y^2}{4(2y - y^2)}} dy \\
&= 2\pi \int_1^{3/2} 1 dy = 2\pi(3/2 - 1) = \pi.
\end{aligned}$$

6.6.21 $S = 2\pi \int_1^7 \sqrt{8x - x^2} \sqrt{1 + \frac{(8 - 2x)^2}{4(8x - x^2)}} dx = 2\pi \int_1^7 \sqrt{8x - x^2} \sqrt{\frac{32x - 4x^2 + 64 - 32x + 4x^2}{4(8x - x^2)}} dx =$
 $2\pi \int_1^7 \sqrt{8x - x^2} \sqrt{\frac{64}{4(8x - x^2)}} dx = 2\pi \int_1^7 4 dx = 8\pi(7 - 1) = 48\pi$. Because the surface area is 48π square meters, the volume of paint required to cover the surface to a thickness of 0.0015 meters is $48\pi(.0015) \approx .226$ cubic meters. This is about 59.75 gallons.

6.6.22

$$\begin{aligned}
S &= 2\pi \int_{-8}^8 \sqrt{100 - x^2} \sqrt{1 + \frac{(-2x)^2}{4(100 - x^2)}} dx \\
&= 2\pi \int_{-8}^8 \sqrt{100 - x^2} \sqrt{\frac{400 - 4x^2 + 4x^2}{4(100 - x^2)}} dx = 2\pi \int_{-8}^8 10 dx = 20\pi(8 - (-8)) = 320\pi.
\end{aligned}$$

Because the surface area is 320π square meters, the volume of paint required to cover the surface to a thickness of 0.0015 meters is $320\pi(0.0015) \approx 1.5$ cubic meters. This is about 398.36 gallons.

6.6.23

- False. One would need to find $x = f^{-1}(y)$ and compute the corresponding integral using f^{-1} .
- False. For example, the curve given in number 14 above isn't one-to-one on the given interval, but the surface is still defined.
- True. Because the curve is symmetric about the y -axis, the surface generated by revolving half the curve is half the surface generated by revolving the whole curve.
- False. This curve is symmetric about the y -axis, so the surface generated by revolving the whole curve is the same as the surface generated by revolving the portion over $[0, 4]$.

6.6.24

- $S = 2\pi \int_0^1 x^5 \sqrt{1 + 25x^8} dx.$
- $S \approx 3.362.$

6.6.25

- $S = 2\pi \int_0^{\pi/2} \cos x \sqrt{1 + \sin^2 x} dx.$
- $S \approx 7.21.$

6.6.26

- $S = 2\pi \int_1^e \ln y \sqrt{1 + \frac{1}{y^2}} dy.$
- $S \approx 7.05.$

6.6.27

- $S = 2\pi \int_0^{\pi/4} \tan x \sqrt{1 + \sec^4 x} dx.$
- $S \approx 3.84.$

6.6.28

- $S = 2\pi \int_1^{\sqrt{e}} \ln x^2 \sqrt{1 + 4/x^2} dx.$
- $S \approx 3.845.$

6.6.29 Let $y = f(x) = (a^{2/3} - x^{2/3})^{3/2}$. Note that $f'(x) = -x^{-1/3} \sqrt{a^{2/3} - x^{2/3}}$, so

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{a^{2/3} - x^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3} + a^{2/3} - x^{2/3}}{x^{2/3}}} = \frac{a^{1/3}}{x^{1/3}}.$$

Thus (using symmetry) $S = 4\pi \int_0^a (a^{2/3} - x^{2/3})^{3/2} (a^{1/3} x^{-1/3}) dx$. Let $u = (a^{2/3} - x^{2/3})$, so that $du = (-2x^{-1/3}/3) dx$. Substituting gives

$$S = -6\pi a^{1/3} \int_{a^{2/3}}^0 u^{3/2} du = 6\pi a^{1/3} \int_0^{a^{2/3}} u^{3/2} du = \frac{12\pi}{5} a^{1/3} \left(u^{5/2} \right) \Big|_0^{a^{2/3}} = \frac{12\pi a^2}{5}.$$

6.6.30 The surface area of the described cylinder is $2\pi rh$. The surface area of the described cone is $\pi r \sqrt{h^2 + r^2}$. The cone's area is $\frac{2h}{\sqrt{h^2 + r^2}}$ of the cylinder's.

6.6.31

$$\begin{aligned}
S &= 2\pi \int_1^2 \left(x^{3/2} - \frac{\sqrt{x}}{3} \right) \sqrt{1 + (3\sqrt{x}/2 - x^{-1/2}/6)^2} dx = 2\pi \int_1^2 (x^{3/2} - \sqrt{x}/3) \sqrt{1 + \left(\frac{9x-1}{6\sqrt{x}} \right)^2} dx \\
&= 2\pi \int_1^2 (x^{3/2} - \sqrt{x}/3) \sqrt{\frac{36x + 81x^2 - 18x + 1}{36x}} dx = 2\pi \int_1^2 (x^{3/2} - \sqrt{x}/3) \sqrt{\frac{(1+9x)^2}{36x}} dx \\
&= 2\pi \int_1^2 \left(\frac{(1+9x)x}{6} - \frac{1+9x}{18} \right) dx = \frac{2\pi}{18} \int_1^2 (3x + 27x^2 - 1 - 9x) dx \\
&= \frac{2\pi}{18} \int_1^2 (27x^2 - 6x - 1) dx = \frac{\pi}{9} (9x^3 - 3x^2 - x) \Big|_1^2 = \frac{\pi}{9} (72 - 12 - 2 - (9 - 3 - 1)) = \frac{53\pi}{9}
\end{aligned}$$

6.6.32

$$\begin{aligned}
S &= 2\pi \int_1^2 (x^4/8 + x^{-2}/4) \sqrt{1 + (x^3/2 - x^{-3}/2)^2} dx = 2\pi \int_1^2 (x^4/8 + x^{-2}/4) \sqrt{1 + x^6/4 - 1/2 + x^{-6}/4} dx \\
&= 2\pi \int_1^2 (x^4/8 + x^{-2}/4) \sqrt{x^6/4 + 1/2 + x^{-6}/4} dx = 2\pi \int_1^2 (x^4/8 + x^{-2}/4) \sqrt{(x^3/2 + x^{-3}/2)^2} dx \\
&= 2\pi \int_1^2 (x^4/8 + x^{-2}/4) (x^3/2 + x^{-3}/2) dx = 2\pi \int_1^2 (x^7/16 + x/16 + x/8 + x^{-5}/8) dx \\
&= \frac{\pi}{8} \int_1^2 (x^7 + 3x + 2x^{-5}) dx = \frac{\pi}{8} (x^8/8 + 3x^2/2 - x^{-4}/2) \Big|_1^2 \\
&= \frac{\pi}{8} (32 + 6 - (1/32) - (1/8 + 3/2 - 1/2)) = \frac{1179\pi}{256}
\end{aligned}$$

6.6.33

$$\begin{aligned}
S &= 2\pi \int_{1/2}^2 (x^3/3 + x^{-1}/4) \sqrt{1 + (x^2 - x^{-2}/4)^2} dx \\
&= 2\pi \int_{1/2}^2 (x^3/3 + x^{-1}/4) \sqrt{1 + x^4 - 1/2 + x^{-4}/16} dx = 2\pi \int_{1/2}^2 (x^3/3 + x^{-1}/4) \sqrt{x^4 + 1/2 + x^{-4}/16} dx = \\
&= 2\pi \int_{1/2}^2 (x^3/3 + x^{-1}/4) \sqrt{(x^2 + x^{-2}/4)^2} dx = 2\pi \int_{1/2}^2 (x^3/3 + x^{-1}/4) (x^2 + x^{-2}/4) dx \\
&= 2\pi \int_{1/2}^2 (x^5/3 + x/12 + x/4 + x^{-3}/16) dx = 2\pi \int_{1/2}^2 (x^5/3 + x/3 + x^{-3}/16) dx \\
&= 2\pi (x^6/18 + x^2/6 - x^{-2}/32) \Big|_{1/2}^2 = 2\pi (32/9 + 2/3 - 1/128 - (1/1152 + 1/24 - 1/8)) = \frac{275\pi}{32}
\end{aligned}$$

6.6.34 We can rewrite the given curve as $2x + \sqrt{4x^2 - 1} = e^{2y}$. Squaring yields $4x^2 + 4x\sqrt{4x^2 - 1} + 4x^2 - 1 = e^{4y}$. So $e^{4y} + 1 = 8x^2 + 4x\sqrt{4x^2 - 1} = 4x(2x + \sqrt{4x^2 - 1}) = 4x(e^{2y})$. Thus

$$x = \frac{e^{4y} + 1}{4e^{2y}} = \frac{e^{2y} + e^{-2y}}{4}.$$

The portion of the curve in question is for $0 \leq y \leq \ln 2$.

The surface area is

$$\begin{aligned}
S &= 2\pi \int_0^{\ln 2} \frac{e^{2y} + e^{-2y}}{4} \sqrt{1 + (e^{2y}/2 - e^{-2y}/2)^2} dy = 2\pi \int_0^{\ln 2} \frac{e^{2y} + e^{-2y}}{4} \sqrt{e^{4y}/4 + 1/2 + e^{-2y}/4} dy \\
&= 2\pi \int_0^{\ln 2} \frac{e^{2y} + e^{-2y}}{4} \frac{\sqrt{(e^{2y} + e^{-2y})^2}}{2} dy = \frac{\pi}{4} \int_0^{\ln 2} (e^{4y} + 2 + e^{-4y}) dy = \frac{\pi}{4} (e^{4y}/4 + 2y - e^{-4y}/4) \Big|_0^{\ln 2} \\
&= \frac{\pi}{4} (4 + 2\ln 2 - 1/64 - (1/4 + 0 - 1/4)) = \frac{\pi}{4} \left(\frac{255}{64} + 2\ln 2 \right)
\end{aligned}$$

6.6.35

$$\begin{aligned}
S &= 2\pi \int_1^4 (4y^{3/2} - y^{1/2}/12) \sqrt{1 + (6y^{1/2} - y^{-1/2}/24)^2} dy \\
&= 2\pi \int_1^4 (4y^{3/2} - y^{1/2}/12) \sqrt{1 + 36y - 1/2 + 1/(576y)} dy \\
&= 2\pi \int_1^4 (4y^{3/2} - y^{1/2}/12) \sqrt{36y + 1/2 + 1/(576y)} dy \\
&= 2\pi \int_1^4 (4y^{3/2} - y^{1/2}/12) \sqrt{(6y^{1/2} + y^{-1/2}/24)^2} dy \\
&= 2\pi \int_1^4 (4y^{3/2} - y^{1/2}/12) (6y^{1/2} + y^{-1/2}/24) dy \\
&= 2\pi \int_1^4 (24y^2 - y/3 - 1/288) dy \\
&= 2\pi \left(8y^3 - y^2/6 - y/288 \right) \Big|_1^4 = 2\pi \left(512 - \frac{8}{3} - \frac{1}{72} \left(8 - \frac{1}{6} - \frac{1}{288} \right) \right) = \frac{48143\pi}{48}.
\end{aligned}$$

6.6.36 Let $y = f(x) = a + \sqrt{r^2 - x^2}$. Revolving this curve around the x -axis will give a portion of the surface of the torus. The other comes from revolving $y = g(x) = a - \sqrt{r^2 - x^2}$ around the x -axis. We will compute the surface area for the first portion first.

Note that

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}} \right)^2} = \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}} = \frac{r}{\sqrt{r^2 - x^2}}.$$

So (using symmetry) the surface area for this portion of the torus is

$$\begin{aligned}
4\pi \int_0^r (a + \sqrt{r^2 - x^2}) \left(\frac{r}{\sqrt{r^2 - x^2}} \right) dx &= 4\pi ar \int_0^r \frac{1}{\sqrt{r^2 - x^2}} dx + 4\pi r \int_0^r 1 dx \\
&= 4\pi ar (\sin^{-1}(x/r)) \Big|_0^r + 4\pi r^2 = 4\pi ar (\pi/2 - 0) + 4\pi r^2 = 2\pi^2 ar + 4\pi r^2.
\end{aligned}$$

If we revolve $y = g(x) = a - \sqrt{r^2 - x^2}$ around the x -axis, the surface area will be $2\pi^2 ar - 4\pi r^2$ by a very similar calculation. Thus, the total surface area of the torus is

$$S = 2\pi^2 ar + 4\pi r^2 + 2\pi^2 ar - 4\pi r^2 = 4\pi^2 ar.$$

6.6.37 By symmetry, we can let $y = f(x) = \sqrt{r^2 - x^2}$, and imagine the surface obtained by revolving this curve around the x -axis for $a \leq x \leq a + h$. The surface area is

$$S = 2\pi \int_a^{a+h} \sqrt{r^2 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}} \right)^2} dx.$$

This can be written as

$$\begin{aligned}
&2\pi \int_a^{a+h} \sqrt{r^2 - x^2} \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}} dx \\
&= 2\pi \int_a^{a+h} \sqrt{r^2 - x^2} \frac{\sqrt{r^2}}{\sqrt{r^2 - x^2}} dx = 2\pi \int_a^{a+h} r dx = 2\pi r(a + h - a) = 2\pi rh.
\end{aligned}$$

6.6.38

a. Let $y = f(x) = \frac{b}{a} \sqrt{a^2 - x^2}$. Then $f'(x) = \frac{b}{a} \cdot \frac{-x}{\sqrt{a^2 - x^2}}$, so

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{b^2}{a^2} \cdot \frac{x^2}{a^2 - x^2}} = \sqrt{\frac{a^4 - a^2 x^2 + b^2 x^2}{a^2}} \cdot \frac{1}{\sqrt{a^2 - x^2}}.$$

Thus $f(x)\sqrt{1+f'(x)^2}$ is

$$\frac{b}{a}\sqrt{a^2-x^2}\sqrt{\frac{a^4-a^2x^2+b^2x^2}{a^2}} \cdot \frac{1}{\sqrt{a^2-x^2}} = \frac{b}{a}\sqrt{a^2-\frac{(a^2-b^2)x^2}{a^2}} = \frac{b}{a}\sqrt{a^2-\left(1-\frac{b^2}{a^2}\right)x^2}.$$

Letting $c^2 = 1 - \frac{b^2}{a^2}$ and using symmetry gives

$$S = \frac{4\pi b}{a} \int_0^a \sqrt{a^2 - c^2 x^2} dx.$$

b. Let $u = cx$, so that $du = c dx$. Note that when $x = 0$, u is also 0, and when $x = a$, we have $u = ca = \sqrt{1 - \frac{b^2}{a^2}}\sqrt{a^2} = \sqrt{a^2 - b^2}$. Thus,

$$S = \frac{4\pi b}{ac} \int_0^{\sqrt{a^2-b^2}} \sqrt{a^2 - u^2} du = \frac{4\pi b}{\sqrt{a^2-b^2}} \int_0^{\sqrt{a^2-b^2}} \sqrt{a^2 - u^2} du.$$

c. S is

$$\frac{2\pi b}{\sqrt{a^2-b^2}} \left(u\sqrt{a^2-u^2} + a^2 \sin^{-1}(u/a) \right) \Big|_0^{\sqrt{a^2-b^2}} = \frac{2\pi b}{\sqrt{a^2-b^2}} \left(b\sqrt{a^2-b^2} + a^2 \sin^{-1} \left(\frac{\sqrt{a^2-b^2}}{a} \right) \right).$$

Multiplying through by the factor of $\sqrt{a^2-b^2}$ in the denominator gives

$$S = 2\pi b \left(b + \frac{a^2}{\sqrt{a^2-b^2}} \sin^{-1} \left(\frac{\sqrt{a^2-b^2}}{a} \right) \right).$$

d. S would be measured in square meters.

e. Returning to our work in the first part of this problem, if $a = b$ then $f(x)\sqrt{1+f'(x)^2} = \frac{b}{a}\sqrt{a^2-0} = b$.

So the surface area is $S = 4\pi \int_0^a b dx = 4\pi b(a-0) = 4\pi ab = 4\pi a^2$.

6.6.39

a. Using R for the radius of the sphere, The ratio is $\frac{6R^2}{R^3} = \frac{6}{R}$.

b. The ratio is $\frac{4\pi R^2}{(4/3)\pi R^3} = \frac{3}{R}$.

c. We have an ellipsoid whose axes have lengths $2R$, R , and R , where $R = \ell\sqrt[3]{4}$. The volume of the ellipsoid is

$$V = \frac{4\pi}{3} \cdot R \cdot \frac{R}{2} \cdot \frac{R}{2} = \frac{\pi R^3}{3} = \frac{4\pi \ell^3}{3}.$$

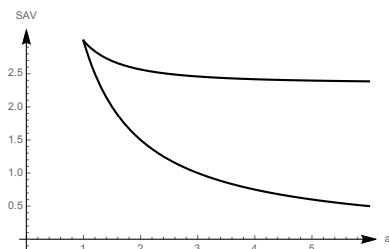
The surface area is

$$\begin{aligned} S &= 2\pi \cdot \frac{R}{2} \left(\frac{R}{2} + \frac{R^2}{\sqrt{R^2 - (R^2/4)}} \sin^{-1} \frac{\sqrt{R^2 - (R^2/4)}}{R} \right) \\ &= \pi R \left(\frac{R}{2} + \frac{2R^2}{\sqrt{3}R} \sin^{-1} \left(\frac{\sqrt{3}R}{2R} \right) \right) \\ &= \pi \ell \sqrt[3]{4} \left(\frac{\ell \sqrt[3]{4}}{2} + \frac{2\ell \sqrt[3]{4}}{\sqrt{3}} \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) \right) \\ &= \pi \ell \sqrt[3]{4} \left(\frac{\ell \sqrt[3]{4}}{2} + \frac{2\ell \sqrt[3]{4}}{\sqrt{3}} \cdot \frac{\pi}{3} \right) \\ &= \pi \ell^2 4^{2/3} \left(\frac{1}{2} + \frac{2\pi}{3\sqrt{3}} \right). \end{aligned}$$

The SAV ratio is therefore

$$\frac{3\pi\ell^2 4^{2/3} \left(\frac{1}{2} + \frac{2\pi}{3\sqrt{3}}\right)}{4\pi\ell^3} = \frac{3(3\sqrt{3} + 4\pi)}{4^{1/3}\ell 6\sqrt{3}} = \frac{9 + 4\sqrt{3}\pi}{6 \cdot 4^{1/3}\ell}.$$

d.



The lower curve is for the sphere and the upper for the ellipsoid.

e. More spherical is better for minimizing heat loss.

6.6.40 Let $c = \frac{g(b) - g(a)}{b - a}$.

The slant height ℓ can be written as $\sqrt{(g(b) - g(a))^2 + (b - a)^2}$, so the quantity $\pi(g(b) + g(a))\ell$ can be written as

$$\frac{b - a}{\sqrt{(b - a)^2}} \pi(g(b) + g(a)) \sqrt{(g(b) - g(a))^2 + (b - a)^2} = (b - a) \pi(g(b) + g(a)) \sqrt{c^2 + 1}.$$

We will show that the surface area of the frustum is this quantity.

The line between $(a, g(a))$ and $(b, g(b))$ has slope $\frac{g(b) - g(a)}{b - a} = c$. Then the equation of the line is $y = f(x) = g(a) + c(x - a)$. The surface area is

$$\begin{aligned} S &= 2\pi \int_a^b (g(a) + c(x - a)) \sqrt{1 + c^2} dx = 2\pi \sqrt{1 + c^2} \int_a^b (g(a) + c(x - a)) dx \\ &= 2\pi \sqrt{1 + c^2} \left(g(a)x + \frac{c(x - a)^2}{2} \right) \Big|_a^b = 2\pi \sqrt{1 + c^2} \left(g(a)(b - a) + \frac{c(b - a)^2}{2} \right) \\ &= \pi \sqrt{1 + c^2} (2g(a)(b - a) + (g(b) - g(a))(b - a)) = \pi \sqrt{1 + c^2} (b - a) (2g(a) + g(b) - g(a)) \\ &= \pi \sqrt{1 + c^2} (b - a) (g(a) + g(b)) \end{aligned}$$

as desired.

6.6.41

a. Note that $g(x) = cf(x)$, so that $g'(x) = cf'(x)$.

$$2\pi \int_a^b g(x) \sqrt{c^2 + g'(x)^2} dx = 2\pi \int_a^b cf(x) \sqrt{c^2 + c^2 f'(x)^2} dx = 2c^2 \pi \int_a^b f(x) \sqrt{1 + f'(x)^2} dx = c^2 A.$$

b. Note that $h'(x) = cf'(cx)$. Consider

$$2\pi \int_{a/c}^{b/c} f(cx) \sqrt{c^2 + c^2 f'(cx)^2} dx = 2c\pi \int_{a/c}^{b/c} f(cx) \sqrt{1 + f'(cx)^2} dx.$$

Let $u = cx$ so that $du = c dx$. Then our integral is equal to

$$2\pi \int_a^b f(u) \sqrt{1 + f'(u)^2} du = A.$$

6.6.42 The area obtained by revolving $f(x) + C$ around the x -axis is

$$2\pi \int_a^b (f(x) + C) \sqrt{1 + f'(x)^2} dx = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} dx + 2\pi C \int_a^b \sqrt{1 + f'(x)^2} dx = S + 2\pi CL,$$

as desired.

6.7 Physical Applications

6.7.1 $m = \rho_1 \cdot l_1 + \rho_2 \cdot l_2 = 1 \text{ g/cm} \cdot 50 \text{ cm} + 2 \text{ g/cm} \cdot 50 \text{ cm} = 150 \text{ g}.$

6.7.2 The mass is given by $m = \int_a^b \rho(x) dx.$

6.7.3 The work is the product of 5 Newtons and 5 meters, which is 25 J.

6.7.4 If the force is not constant, the interval must be divided up into pieces, and on each small piece the work can be approximated by assuming a constant force. These approximations are then added up and then the sum is refined through a limiting process, which leads to a definite integral.

6.7.5 Different volumes of water are moved different distances.

6.7.6 Different parts of the dam have different depths and thus different amount of pressure.

6.7.7 $F = \rho gh = 1000 \cdot 9.8 \cdot 4 = 39,200 \text{ N/m}^2.$

6.7.8 Along a thin horizontal strip, the pressure is the same, because the depth is constant.

6.7.9 $W = \int_5^{10} 25\pi\rho g(15 - y) dy.$

6.7.10 $W = \int_0^5 25\pi\rho g(15 - y) dy.$

6.7.11 $W = \int_0^{10} 25\pi\rho g(10 - y) dy.$

6.7.12 $W = \int_0^3 25\pi\rho g(10 - y) dy.$

6.7.13 $m = \int_0^\pi (1 + \sin x) dx = (x - \cos x) \Big|_0^\pi = \pi - (-1) - (0 - 1) = \pi + 2.$

6.7.14 $m = \int_0^1 (1 + x^3) dx = (x + x^4/4) \Big|_0^1 = \frac{5}{4}.$

6.7.15 $m = \int_0^2 (2 - x/2) dx = (2x - x^2/4) \Big|_0^2 = 3.$

6.7.16 $m = \int_0^4 5e^{-2x} dx = ((-5/2)e^{-2x}) \Big|_0^4 = \frac{5}{2}(1 - e^{-8}).$

6.7.17 $m = \int_0^1 x\sqrt{2 - x^2} dx = \left(\frac{-1}{3}(2 - x^2)^{3/2} \right) \Big|_0^1 = (-1/3) + (2\sqrt{2}/3) = \frac{2\sqrt{2} - 1}{3}.$

$$\mathbf{6.7.18} \quad m = \int_0^2 1 \, dx + \int_2^3 2 \, dx = 2 + 2 = 4.$$

$$\mathbf{6.7.19} \quad m = \int_0^2 1 \, dx + \int_2^4 (1+x) \, dx = 2 + \left(x + x^2/2 \right) \Big|_2^4 = 2 + (4+8) - (2+2) = 10.$$

$$\mathbf{6.7.20} \quad m = \int_0^1 x^2 \, dx + \int_1^2 x(2-x) \, dx = \left(x^3/3 \right) \Big|_0^1 + \left(x^2 - x^3/3 \right) \Big|_1^2 = \frac{1}{3} + 4 - \frac{8}{3} - \left(1 - \frac{1}{3} \right) = 1.$$

$$\mathbf{6.7.21} \quad W = \int_0^3 2x \, dx = x^2 \Big|_0^3 = 9 \text{ J.}$$

$$\mathbf{6.7.22} \quad W = \int_1^3 2/x^2 \, dx = \left(-2/x \right) \Big|_1^3 = -\frac{2}{3} - (-2) = \frac{4}{3} \text{ J.}$$

6.7.23

a. Because $f(0.2) = 0.2k = 30$, we have $k = 150$.

$$\text{b. } W = \int_0^{-0.4} 150x \, dx = 75x^2 \Big|_0^{-0.4} = 75 \cdot 0.16 = 12 \text{ J.}$$

$$\text{c. } W = \int_0^{0.3} 150x \, dx = 75x^2 \Big|_0^{0.3} = 75 \cdot 0.09 = 6.75 \text{ J.}$$

$$\text{d. } W = \int_{0.2}^{0.4} 150x \, dx = 75x^2 \Big|_{0.2}^{0.4} = 75(0.16 - 0.04) = 9 \text{ J.}$$

6.7.24

a. $f(0.25) = 0.25k = 15$, so $k = 60$.

$$\text{b. } W = \int_0^{-0.2} 60x \, dx = 30x^2 \Big|_0^{-0.2} = 30 \cdot 0.04 = 1.2 \text{ J.}$$

$$\text{c. } W = \int_{0.25}^{0.55} 60x \, dx = 30x^2 \Big|_{0.25}^{0.55} = 30(0.3025 - 0.0625) = 30 \cdot 0.24 = 7.2 \text{ J.}$$

6.7.25

a. $f(x) = kx$, and $f(0.5) = 50$, so $k(0.5) = 50$, so $k = 100$.

$$\text{Therefore } W = \int_0^{1.5} 100x \, dx = \left(50x^2 \right) \Big|_0^{1.5} = 112.5 \text{ J.}$$

$$\text{b. } \int_0^{-0.5} 100x \, dx = \left(50x^2 \right) \Big|_0^{-0.5} = 12.5 \text{ J.}$$

6.7.26 $f(x) = kx$, and $f(0.02) = 0.02k = 500 \cdot 9.8 = 4900$, so $k = 245000$.

$$W = \int_0^{0.04} 245000x \, dx = \left(122500x^2 \right) \Big|_0^{0.04} = 196 \text{ J.}$$

6.7.27

$$\text{a. } f(0.2) = 0.2k = 50, \text{ so } k = 250. \quad W = \int_0^{0.5} 250x \, dx = 125x^2 \Big|_0^{0.5} = 125 \cdot 0.25 = 31.25 \text{ J.}$$

b. $\int_0^{0.2} kx \, dx = kx^2/2 \Big|_0^{0.2} = 0.02k = 50$, so $k = 2500$. $W = \int_0^{0.5} 2500x \, dx = 1250x^2 \Big|_0^{0.5} = 1250 \cdot 0.25 = 312.5$ J.

6.7.28

a. $f(0.1) = 0.1k = 50$, so $k = 500$. $W = \int_0^{0.4} 500x \, dx = 250x^2 \Big|_0^{0.4} = 250 \cdot 0.16 = 40$ J.

b. $\int_0^{0.1} kx \, dx = kx^2/2 \Big|_0^{0.1} = 0.005k = 2$, so $k = 400$. $W = \int_0^{0.4} 400x \, dx = 200x^2 \Big|_0^{0.4} = 200 \cdot 0.16 = 32$ J.

6.7.29

a. We have $\int_0^{0.5} kx \, dx = \frac{k}{2}x^2 \Big|_0^{0.5} = 0.125k = 100$, so that $k = 800$. Then

$$W = \int_0^{1.25} 800x \, dx = 400x^2 \Big|_0^{1.25} = 625 \text{ J.}$$

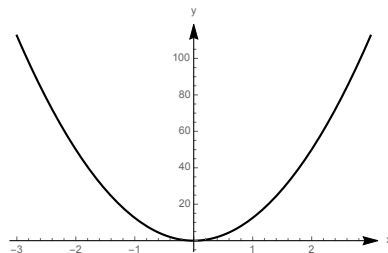
b. $f(0.5) = 0.5k = 250$, so $k = 500$ and thus

$$W = \int_0^{1.25} 500x \, dx = 250x^2 \Big|_0^{1.25} = 390.625 \text{ J.}$$

6.7.30

$$W(x) = \int_0^x 25t \, dt = (25t^2/2) \Big|_0^x = 25x^2/2.$$

Note that W is an even function, so that $W(-x) = W(x)$, and thus the work is the same to compress or stretch the spring a given distance from its equilibrium position.

**6.7.31**

a. $W_1 = \int_0^{30} \rho g(30 - y) \, dy = 5g(30y - y^2/2) \Big|_0^{30} = 2250g = 2250 \cdot 9.8 = 22050$ J.

b. $W = W_1 + W_2$, where W_2 is the work to just lift the block. $W_2 = 50g \cdot 30 = 1500g$, so $W = 2250g + 1500g = 3750g = 3750 \cdot 9.8 = 36750$ J.

6.7.32 $W = \int_0^{60} \frac{55}{1000}g(60 - y) \, dy = \frac{55g}{1000}(60y - y^2/2) \Big|_0^{60} = 99g = 99 \cdot 9.8 = 970.2$ J.

6.7.33 The bottom half of the chain weighs 25 kg, so this problem is equivalent to determining the sum of the work done by the winch in winding up a 25-kg, 10-m chain and lifting a 25-kg mass a distance of 10 m. So

$$\begin{aligned} W &= \int_{10}^{20} 2.5(9.8)(20 - y) \, dy + 25(9.8)(10) = 2450 + 24.5 \int_{10}^{20} (20 - y) \, dy = 2450 + 24.5 \left(20y - \frac{y^2}{2} \right) \Big|_{10}^{20} \\ &= 2450 + 24.5(400 - 200 - (200 - 50)) = 2450 + 1225 = 3675 \text{ J.} \end{aligned}$$

6.7.34 $F(x) = 9.8 \left(4 - \frac{1}{5}y\right)$, so

$$W = \int_0^{10} 9.8 \left(4 - \frac{1}{5}y\right) dy = 9.8 \left(4y - \frac{y^2}{10}\right) \Big|_0^{10} = 9.8(40 - 10) = 294 \text{ J.}$$

6.7.35 $W = \int_0^{2.5} \rho g A(y)(2.5 - y) dy = 1000 \cdot 9.8 \cdot 25 \cdot 15 \int_0^{2.5} (2.5 - y) dy = 3675000 \left(2.5y - \frac{y^2}{2}\right) \Big|_0^{2.5} = 3675000 \cdot \frac{(2.5)^2}{2} = 11,484,375 \text{ J.}$

6.7.36

a. $W = \int_0^8 \rho g \pi \cdot 2^2(8 - y) dy = 4\pi \rho g (8y - y^2/2) \Big|_0^8 \approx 3.941 \times 10^6 \text{ J.}$

b. Not true. For pumping half the water from a full tank the work is

$$\int_4^8 4\rho \pi g(8 - y) dy = 4\pi \rho g (8y - y^2/2) \Big|_4^8 = 32\pi \rho g \approx 985203 \text{ J.}$$

To empty a half-full tank, the work is

$$\int_0^4 4\rho \pi g(8 - y) dy = 4\pi \rho g (8y - y^2/2) \Big|_0^4 = 96\pi \rho g \approx 2.95561 \times 10^6 \text{ J.}$$

6.7.37 $W = \int_0^4 \rho \pi g 2^2(10 - y) dy = 4\pi \rho g \left(10y - \frac{y^2}{2}\right) \Big|_0^4 = 4\pi \rho g(40 - 8) = 128\pi \rho g \approx 3.941 \times 10^6 \text{ J.}$

6.7.38 $W = \int_0^2 \rho g A(y)(3 - y) dy = 1000 \cdot 9.8 \cdot 25 \cdot 15 \int_0^2 (3 - y) dy = 3675000 \left(3y - \frac{y^2}{2}\right) \Big|_0^2 = (3675000) \cdot 4 = 1.47 \times 10^7 \text{ J.}$

6.7.39

a. Let the vertex of the cone be at $(0,0)$, with the y -axis vertically oriented. Note that the area of a horizontal slice at height y is $\pi y^2/16$, and it must move $6 - y$ meters to get to the top.

$$W = \int_0^6 \rho g \pi \frac{y^2}{16}(6 - y) dy = \frac{\pi \rho g}{16} \int_0^6 (6y^2 - y^3) dy = \frac{\pi \rho g}{16} \left(2y^3 - \frac{y^4}{4}\right) \Big|_0^6 = \frac{\pi \rho g}{16}(108) = 66,150\pi \text{ J.}$$

b. Not true.

$$\int_0^3 \rho g \pi \frac{y^2}{16}(6 - y) dy = \frac{\pi \rho g}{16} \int_0^3 (6y^2 - y^3) dy = \frac{\pi \rho g}{16} \left(2y^3 - \frac{y^4}{4}\right) \Big|_0^3 = \frac{\pi \rho g}{16} \cdot 33.75 \approx 20672\pi \text{ J,}$$

less than half the amount from part (a). Note that while the water must be raised further than water in the top half, due to the shape of the tank, there is far less water in the bottom half than in the top.

6.7.40

a. $W = \pi \rho g \int_4^8 9(10 - y) dy = \pi \rho g (90y - 9y^2/2) \Big|_4^8 = (720 - 288 - 360 + 72)\pi \rho g = 144\pi \rho g \approx 4.433 \times 10^6 \text{ J.}$

b. $W = \pi \rho g \int_0^4 9(10 - y) dy = \pi \rho g (90y - 9y^2/2) \Big|_0^4 = (360 - 72)\pi \rho g = 288\pi \rho g \approx 4.43342 \times 10^6 \approx 8.867 \times 10^6 \text{ J.}$

c. The water in the lower part of the tank has to travel farther, so it requires more work to pump it out.

6.7.41 A vertical cross section of the tank that passes through the center of the tank is a circle of radius 8. If we place the origin of the xy -plane at the center of this circle, then we have

$$\begin{aligned} W &= \int_{-8}^8 1000(9.8)(64 - y^2)\pi(10 + y) dy = 9800\pi \int_{-8}^8 (640 + 64y - 10y^2 - y^3) dy \\ &= 9800\pi \left(640y + 32y^2 - \frac{10y^3}{3} - \frac{y^4}{4} \right) \Big|_{-8}^8 \\ &= 9800\pi \left(5120 + 2048 - \frac{5120}{3} - 1024 - \left(-5120 + 2048 + \frac{5120}{3} - 1024 \right) \right) \\ &= 9800\pi \left(10240 - \frac{10240}{3} \right) = 9800\pi \left(\frac{20480}{3} \right) \approx 210,176,737 \text{ J.} \end{aligned}$$

6.7.42

a. Orient the axes so that $(0, 0)$ is the south pole of the semicircle at one end of the trough. The equation of the semicircle is $x^2 + (y - (1/4))^2 = (1/4)^2$. At a height of y , a slice has area $2x \cdot 3 = 6\sqrt{(1/2)y - y^2}$. The distance it must travel to the top is $1/4 - y$.

$$W = \int_0^{1/4} \rho g 6\sqrt{(1/2)y - y^2}((1/4) - y) dy.$$

Let $u = (1/2)y - y^2$, so that $du = ((1/2) - 2y) dy = 2((1/4) - y) dy$. Then we have

$$W = 3\rho g \int_0^{1/16} u^{1/2} du = 2\rho g \left(u^{3/2} \right) \Big|_0^{1/16} = \rho g/32 = 306.25 \text{ J.}$$

b. Yes. If we double the length of the trough, the area of a slice is doubled, and the work integral is doubled.

c. No. If the radius is doubled, the work is more than doubled. (There are “more slices,” and each must travel farther to get to the top of the tank.)

6.7.43

a. Orient the axes so that the lower corners of the trough are at $(-0.25, 0)$ and at $(0.25, 0)$. Then the upper corners are at $(-0.5, 1)$ and at $(0.5, 1)$. Note that the line between $(0.25, 0)$ and $(0.5, 1)$ is given by $y = 4x - 1$. The area of a slice at height y is $2x \cdot 10 = 20 \cdot \frac{1}{4}(y + 1) = 5(y + 1)$. Thus,

$$W = \rho g \int_0^1 5(y + 1)(1 - y) dy = 5\rho g \int_0^1 (1 - y^2) dy = 5\rho g \left(y - y^3/3 \right) \Big|_0^1 = \frac{10\rho g}{3} \approx 32,667 \text{ J.}$$

b. Yes. If the length is doubled, the area of each slice is doubled, so the work integral is doubled as well.

6.7.44 Note that this tank is full of water, rather than gasoline.

$$W = 1000 \cdot 9.8 \int_{-5}^5 20\sqrt{25 - y^2}(10 - y) dy = 196,000 \int_{-5}^5 \sqrt{25 - y^2}(10 - y) dy.$$

The integral can be divided into two integrals as in the example. We have

$$10 \int_{-5}^5 \sqrt{25 - y^2} dy - \int_{-5}^5 y\sqrt{25 - y^2} dy.$$

The first represents 10 times the area of a half-circle of radius 5, so the value is $(10)\frac{25\pi}{2} = 125\pi$. The second integral can be computed with the substitution $u = 25 - y^2$, $du = -2y dy$. The resulting integral is $\frac{1}{2} \int_0^0 \sqrt{u} du = 0$. So the total work is

$$W = 196000 \cdot 125\pi \approx 7.697 \times 10^7 \text{ J.}$$

6.7.45 Let the vertex of the cone be at $(0, 0)$, with the y -axis vertically oriented. Note that the area of a horizontal slice at height y is $\pi y^2/16$, and it must move $3 - y$ meters to get to the point 1 meter above the top.

$$\begin{aligned} W &= \int_0^2 \rho g \pi \frac{y^2}{16} (3 - y) dy = (\pi \rho g / 16) \int_0^2 (3y^2 - y^3) dy \\ &= (\pi \rho g / 16) \left(y^3 - y^4/4 \right) \Big|_0^2 = (\pi \rho g / 16)(4) = \frac{\pi \rho g}{4} \approx 7696.9 \text{ J.} \end{aligned}$$

6.7.46 $F = \int_0^{10} \rho g (10 - y) \cdot 40 dy = 40 \rho g (10y - y^2/2) \Big|_0^{10} = 200 \rho g = 1.960 \times 10^6 \text{ N.}$

6.7.47 Orient the axes so that the lower corners of the trapezoid are at $(5, 0)$ and $(-5, 0)$, and the upper corners are at $(10, 15)$ and $(-10, 15)$. Note that the line between the corners for $x > 0$ is given by $y = 3(x - 5)$, so at level y , we have a width of $2x = \frac{2y}{3} + 10$.

$$\begin{aligned} F &= \rho g \int_0^{15} (15 - y) \left(\frac{2y + 30}{3} \right) dy = \frac{2\rho g}{3} \int_0^{15} (225 - y^2) dy \\ &= \frac{2\rho g}{3} \left(225y - y^3/3 \right) \Big|_0^{15} = \frac{2\rho g}{3} \cdot 2250 = 1500 \rho g = 1.470 \times 10^7 \text{ N.} \end{aligned}$$

6.7.48 Orient the axes so that $(0, 0)$ is at the “south pole” of the semicircle. The center of the circle is $(0, 20)$ and the radius is 20, so the equation is $x^2 + (y - 20)^2 = 20^2$, so $x = \sqrt{40y - y^2}$. The width is $2x = 2\sqrt{40y - y^2}$.

$$F = 2\rho g \int_0^{20} (20 - y) \sqrt{40y - y^2} dy.$$

Let $u = 40y - y^2$, so that $du = 2(20 - y) dy$. Then

$$F = \rho g \int_0^{400} \sqrt{u} du = \rho g \left(2u^{3/2}/3 \right) \Big|_0^{400} = \frac{16000}{3} \rho g = 5.227 \times 10^7 \text{ N.}$$

6.7.49 Orient the axes so that the bottom vertex is at $(0, 0)$. The other vertices are at $(\pm 10, 30)$, and the line between $(0, 0)$ and $(10, 30)$ is given by $y = 3x$. Thus, a slice at height y has width $2x = 2y/3$.

$$F = \int_0^{30} \rho g (30 - y) \frac{2y}{3} dy = \frac{2\rho g}{3} \left(15y^2 - y^3/3 \right) \Big|_0^{30} = \frac{2\rho g}{3} (4500) = 3000 \rho g = 2.940 \times 10^7 \text{ N.}$$

6.7.50 At a height of y , the width of a slice is $2x = 2(4\sqrt{y}) = 8y^{1/2}$. This gives

$$F = 8\rho g \int_0^{25} (25 - y)y^{1/2} dy = 8\rho g \left(50y^{3/2}/3 - 2y^{5/2}/5 \right) \Big|_0^{25} = 8\rho g \left(\frac{2500}{3} \right) = 6.533 \times 10^7 \text{ N.}$$

6.7.51 The width of the plate at depth y is $2 - y$, so the force on the plate is

$$F = \int_1^2 \rho g (2 - y)y dy = \rho g \left(y^2 - \frac{y^3}{3} \right) \Big|_1^2 = \rho g \cdot \frac{2}{3} \approx 6533 \text{ N.}$$

$$\mathbf{6.7.52} \quad F = \int_0^{1/2} \rho g(4-y) \cdot 0.5 \, dy = \rho g \left(2y - y^2/4 \right) \Big|_0^{1/2} = \frac{15\rho g}{16} = 9187.5 \text{ N}.$$

$$\mathbf{6.7.53} \quad F = \int_1^{1.5} \rho g(4-y) \cdot 0.5 \, dy = \rho g \left(2y - y^2/4 \right) \Big|_1^{1.5} = \frac{11\rho g}{16} = 6737.5 \text{ N}.$$

6.7.54 Orient the axes so that $(0, 0)$ is at the bottom of the circle at the bottom of the pool. Then the equation of the circle is $x^2 + (y - 1/2)^2 = (1/2)^2$, so the width of a slice at a height of y is $2x = 2\sqrt{y - y^2}$. Thus,

$$\begin{aligned} F &= 2\rho g \int_0^1 (4-y)(\sqrt{y-y^2}) \, dy = 2\rho g \int_0^1 \frac{7}{2} \sqrt{y-y^2} \, dy + 2\rho g \int_0^1 \left(\frac{1}{2} - y\right) \sqrt{y-y^2} \, dy \\ &= 7\rho g \left(\frac{\pi}{8}\right) + 2\rho g \left(\frac{1}{3}(y-y^2)^{3/2}\right) \Big|_0^1 = \frac{7\pi\rho g}{8} + 0 \approx 2.694 \times 10^4 \text{ N}. \end{aligned}$$

$$\mathbf{6.7.55} \quad F = \int_0^{50} (150 + 2y) \cdot 80 \, dy = 80(150y + y^2) \Big|_0^{50} = 8 \times 10^5 \text{ N}.$$

6.7.56 Arrange the axes so that the origin is in the center of the circular end of the tank. The equation of the circle is $x^2 + y^2 = 25$, so the width of the tank at height y is $2\sqrt{25 - y^2}$, and the depth is $5 - y$. Thus the force is given by

$$\begin{aligned} \rho g \int_{-5}^5 (4-y)2\sqrt{25-y^2} \, dy &= 14445.2 \int_{-5}^5 (5-y)\sqrt{25-y^2} \, dy \\ &= 14445.2 \left(\int_{-5}^5 5\sqrt{25-y^2} \, dy + \int_{-5}^5 y\sqrt{25-y^2} \, dy \right). \end{aligned}$$

The first integral is five times the area of a semicircle of radius 5, while the second integral is zero since it is a symmetric integral of an odd function. So the total force is $14445.2 \cdot \frac{125\pi}{2} \approx 2.836 \times 10^6 \text{ N}$.

6.7.57

a. True. $m = \int_a^b \rho(x) \, dx = \frac{1}{b-a} \int_a^b \rho(x) \, dx \cdot (b-a) = \bar{\rho} \cdot L.$

b. True. $\int_0^L kx \, dx = \frac{kL^2}{2} = \int_0^{-L} kx \, dx.$

c. True. This follows because work is force times distance.

d. False. Although they have the same geometry, they are placed at different depths of the water, so the force is different.

6.7.58

a. $m_1 = \int_0^L 4e^{-x} \, dx = (-4e^{-x}) \Big|_0^L = 4(1 - e^{-L}).$

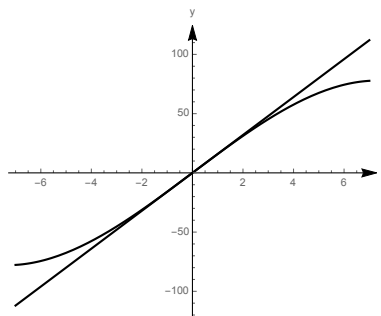
$$m_2 = \int_0^L 6e^{-2x} \, dx = (-3e^{-2x}) \Big|_0^L = 3(1 - e^{-2L}).$$

These are the same when $3(e^{-L})^2 - 4e^{-L} + 1 = 0$, or $(e^{-L} - 1)(3e^{-L} - 1) = 0$, so $L = \ln 3$. m_2 is bigger on $(0, \ln 3)$ and m_1 is bigger for $L > \ln 3$.

b. No. $\lim_{L \rightarrow \infty} m_1 = 4$ and $\lim_{L \rightarrow \infty} m_2 = 3.$

6.7.59

- a. Compared to the linear spring $F(x) = 16x$, the restoring force is less for large displacements.



$$\text{b. } W = \int_0^{1.5} (16x - 0.1x^3) dx = \left(8x^2 - .025x^4 \right) \Big|_0^{1.5} = 17.87 \text{ J.}$$

$$\text{c. } W = \int_0^{-2} (16x - 0.1x^3) dx = \left(8x^2 - .025x^4 \right) \Big|_0^{-2} = 31.6 \text{ J.}$$

6.7.60

- a. $f(x) = kx$, and $f(2) = 2k = 10g$, so $k = 5g$. To compress the spring:

$$W = g \int_{1.5}^2 5x dx = g \left(5x^2/2 \right) \Big|_{1.5}^2 = g(10 - 45/8) = 35g/8 = 42.875 \text{ J.}$$

To move the mass: $10 \cdot 0.5 \cdot 9.8 = 49 \text{ J}$. So the total work is $42.875 + 49 = 91.875 \text{ J}$.

- b. To stretch the spring:

$$W = g \int_2^{2.5} 5x dx = g \left(5x^2/2 \right) \Big|_2^{2.5} = g(125/8 - 10) = 45g/8 = 55.125 \text{ J.}$$

To move the mass: $10 \cdot 0.5 \cdot 9.8 = 49 \text{ J}$. So the total work is $55.125 + 49 = 104.125 \text{ J}$.

6.7.61 Dividing the leak rate by the lifting rate, we find that the bucket is leaking at 20 kg/m . So the force exerted by the bucket after it has been raised y meters is $F(y) = 9.8(5000 - 20y)$. Therefore, the work required to lift the bucket 30 m is

$$W = \int_0^{30} 9.8(5000 - 20y) dy = 9.8 \left(5000y - 10y^2 \right) \Big|_0^{30} = 9.8(150,000 - 9000) = 1,381,800 \text{ J.}$$

6.7.62 Orient the y -axis vertically, with the point $(10, 0)$ representing a point on the bottom of the pool in one corner of the deep end, and $(-10, 1)$ representing a point on the bottom of the pool in the shallow end on the same side of the pool. Note that the straight line between these two points is given by $y = \frac{1}{20}(10 - x)$. So in the first 1 meter of depth, at a height of y the length of the pool is $10 - x = 20y$, so the area of a slice is $20y \cdot 10$. In the 2nd meter of depth, the slices are uniformly of area 200 square meters.

$$\begin{aligned} W &= \int_0^1 \rho g 200y(2.2 - y) dy + \int_1^2 \rho g 200(2.2 - y) dy = 200\rho g \left(\left(1.1y^2 - y^3/3 \right) \Big|_0^1 + \left(2.2y - y^2/2 \right) \Big|_1^2 \right) \\ &= 200\rho g (1.1 - (1/3) + (4.4 - 2 - (2.2 - 1/2))) = \frac{880}{3} \rho g \approx 2.87467 \times 10^6 \text{ J.} \end{aligned}$$

6.7.63 Orient the axes so that $(0, 0)$ is in the middle of the bottom of the cup. Note that the line between $(0.02, 0)$ and $(0.025, 0.15)$ is given by $y = 30x - 3/5$, so $x = \frac{y}{30} + \frac{1}{50}$. The area of a cross section at height y is given by $\pi x^2 = \pi \left(\frac{5y+3}{150} \right)^2$. Note that the distance the slice must travel is $0.2 - y$, because it must go 0.05 above the top of the glass. Thus,

$$\begin{aligned} W &= \pi \rho g \int_0^{0.15} \left(\frac{5y+3}{150} \right)^2 (0.2 - y) dy = \frac{\pi \rho g}{5 \cdot 150^2} \int_0^{0.15} (5y+3)^2 (1-5y) dy \\ &= \frac{\pi \rho g}{5 \cdot 150^2} \int_0^{0.15} (-125y^3 - 125y^2 - 15y + 9) dy \\ &= \frac{\pi \rho g}{5 \cdot 150^2} \left(-125y^4/4 - 125y^3/3 - 15y^2/2 + 9y \right) \Big|_0^{0.15} \\ &= \frac{\pi \rho g}{5 \cdot 150^2} \cdot 1.0248 \approx 0.280 \text{ J.} \end{aligned}$$

6.7.64

- a. The acceleration due to gravity is $F = mg$, and is in the vertical direction. The tangent direction to the curve is perpendicular to the normal, which makes an angle of θ with the vertical. If we form a right triangle with hypotenuse of length mg , and angle θ , then the two legs must have lengths $mg \sin \theta$ (parallel to the curve) and $mg \cos \theta$ (normal to the curve).
- b. Note that the angle t is $t = S/L$, where S is arc length. Then $S = Lt$ and $dS = L dt$. So

$$W = \int_0^\theta F ds = \int_0^\theta mg \sin \theta \cdot L d\theta = mgL (-\cos \theta) \Big|_0^\theta = mg(L - L \cos \theta) = mgh.$$

6.7.65

- a. $F = \int_1^3 \rho g(4-y) \cdot 2 dy = \rho g(8y - y^2) \Big|_1^3 = 8\rho g = 78,400 \text{ N}$. This is less than 90,000 N., so the window can withstand the force.
- b. $F(h) = \int_1^3 \rho g(h-y) \cdot 2 dy = \rho g(2hy - y^2) \Big|_1^3 = \rho g(6h - 9 - (2h - 1)) = 4\rho g(h - 2)$. This is less than or equal to 90,000 Newtons when $h \leq 4.296$ meters.

6.7.66 The plate pictured on the left should have more force, because it has its wider part lower in the pool.

For each plate, orient the axes so that the vertical line through the vertex pointing up (or down) is the y -axis, and the x -axis lies along the horizontal base of the triangle. For the plate on the left, the line through $(1/2, 0)$ and $(0, \sqrt{3}/2)$ is given by $y = \sqrt{3}((1/2) - x)$, so $x = \frac{1}{2} - (\sqrt{3}/3)y$, and the width at height y is given by $1 - \frac{2\sqrt{3}}{3}y$.

For the plate on the left,

$$\begin{aligned} F &= \int_0^{\sqrt{3}/2} \rho g(1 + (\sqrt{3}/2) - y)(1 - \frac{2}{\sqrt{3}}y) dy \\ &= \rho g \int_0^{\sqrt{3}/2} \left(\frac{2y^2}{\sqrt{3}} - \frac{2y}{\sqrt{3}} - 2y + \frac{\sqrt{3}}{2} + 1 \right) dy \\ &= \rho g \left(\frac{1}{6} \left(\frac{4y^3}{\sqrt{3}} - 2(3 + \sqrt{3})y^2 + 3(2 + \sqrt{3})y \right) \right) \Big|_0^{\sqrt{3}/2} = \frac{1}{4} + \frac{\sqrt{3}}{4} \approx 0.683 \text{ N.} \end{aligned}$$

For the plate on the right, the line through $(0, 0)$ and $(1/2, \sqrt{3}/2)$ is given by $y = \sqrt{3}x$, so $x = \frac{y}{\sqrt{3}}$, and the width at height y is $\frac{2y}{\sqrt{3}}$.

Thus,

$$\begin{aligned} F &= \int_0^{\sqrt{3}/2} \rho g (1 + \sqrt{3}/2 - y) \frac{2y}{\sqrt{3}} dy = \rho g \int_0^{\sqrt{3}/2} \left(-\frac{2y^2}{\sqrt{3}} + \frac{2y}{\sqrt{3}} + y \right) dy \\ &= \rho g \left(-\frac{2y^3}{3\sqrt{3}} + \frac{y^2}{\sqrt{3}} + \frac{y^2}{2} \right) \Big|_0^{\sqrt{3}/2} = \frac{1 + 2\sqrt{3}}{8} \approx 0.558 \text{ N}. \end{aligned}$$

6.7.67 The plate on the left has more than half of its area below the horizontal line which is $3/2$ below the surface, while the plate on the right has exactly half its area below that line, so the plate on the left should have more force than the plate on the right.

- a. Note that the line in the first quadrant from $(0, 0)$ to $(\sqrt{2}/2, \sqrt{2}/2)$ is given by $y = x$, while the line from $(\sqrt{2}/2, \sqrt{2}/2)$ to $(0, \sqrt{2})$ is given by $y = -x + \sqrt{2}$. So the width of a slice at height y in the lower part of the region is $2y$, and in the upper part of the region is $2(\sqrt{2} - y)$.

$$\begin{aligned} F &= \rho g \int_0^{\sqrt{2}/2} (\sqrt{2} + 1 - y)(2y) dy + \rho g \int_{\sqrt{2}/2}^{\sqrt{2}} (\sqrt{2} + 1 - y)2(\sqrt{2} - y) dy \\ &= \rho g \int_0^{\sqrt{2}/2} (-2y^2 + 2\sqrt{2}y + 2y) dy + \rho g \int_{\sqrt{2}/2}^{\sqrt{2}} (2y^2 - 4\sqrt{2}y - 2y + 2\sqrt{2} + 4) dy \\ &= \rho g \left(-\frac{2y^3}{3} + \sqrt{2}y^2 + y^2 \right) \Big|_0^{\sqrt{2}/2} + \rho g \left(2 \left(\frac{y^3}{3} - \frac{1}{2} (1 + 2\sqrt{2}) y^2 + \sqrt{2}y + 2y \right) \right) \Big|_{\sqrt{2}/2}^{\sqrt{2}} \\ &= \rho g \left(\frac{1}{2} + \frac{\sqrt{2}}{3} + \frac{1}{2} + \frac{1}{3\sqrt{2}} \right) = \rho g \left(1 + \frac{\sqrt{2}}{2} \right) \text{ N}. \end{aligned}$$

This is 16,730 N.

b. $F = \int_0^1 \rho g (2 - y) \cdot 1 dy = \rho g (2y - y^2/2) \Big|_0^1 = \frac{3\rho g}{2} \text{ N}.$ This is 14,700 N.

6.7.68

a. $W = \int_{x_0}^{x_f} mx'' dx = \int_0^{x(5)} 2 \cdot 8 dx = 16(x(5)) = 1600 \text{ J}.$

b. $W = \int_{x_0}^{x_f} mx'' dx = \int_0^5 2 \cdot 8 \frac{dx}{dt} dt = \int_0^5 16 \cdot 8t dt = (64t^2) \Big|_0^5 = 64 \cdot 25 = 1600 \text{ J}.$

6.7.69

a.

$$\begin{aligned} W &= \int_0^{2500000} \frac{GMm}{(x+R)^2} dx = GMm \left(\frac{-1}{x+R} \right) \Big|_0^{2500000} \\ &= GMm \left(\frac{1}{R} - \frac{1}{R+2500000} \right) = \frac{GMm2500000}{R(R+2500000)} \approx 8.87435 \times 10^9 \text{ J}. \end{aligned}$$

b. $W(x) = \int_0^x \frac{GMm}{(t+R)^2} dt = GMm \left(\frac{-1}{t+R} \right) \Big|_0^x = GMm \left(\frac{1}{R} - \frac{1}{R+x} \right) = \frac{GMmx}{R(R+x)} = \frac{500GMx}{R(R+x)}.$

$$\text{c. } \lim_{x \rightarrow \infty} \frac{GMmx}{R(R+x)} = \lim_{x \rightarrow \infty} \frac{GMm}{R\left(\left(\frac{R}{x}\right) + 1\right)} = \frac{GMm}{R}.$$

$$\text{d. Suppose } \frac{GMmx}{R(R+x)} = \frac{1}{2}mv^2, \text{ then } v^2 = \frac{2GMx}{R(R+x)}, \text{ and as } x \rightarrow \infty \text{ we have } v^2 = \frac{2GM}{R}, \text{ so } v = \sqrt{\frac{2GM}{R}}.$$

6.7.70 When the box is fully submerged in the water, the buoyant force is $8g\rho_w$. The force required is $F = 8g\rho_w - 8g\rho = 8g\rho_w - 8g(\rho_w/2) = 4g\rho_w = 39,200$ N.

Chapter Six Review

1

- True. A vertical slice would lead to shells, while a horizontal slice would lead to either disks or washers.
- True. In order to find position, you would also need to know either its initial position, or at least its position at some time.
- True. If dV/dt is constant, then V is a linear function of time.

2

- $v(t) = 0$ when $\frac{1}{2} = \cos\left(\frac{\pi t}{3}\right)$, which occurs on the given interval for $t = 1$. For $0 \leq t < 1$, we have $v(t) < 0$, so the motion is in the negative direction, and for $1 < t \leq 3$, $v(t) > 0$, so the motion is in the positive direction.

$$\text{b. The displacement is given by } \int_0^3 \left(1 - 2\cos\left(\frac{\pi t}{3}\right)\right) dt = \left(t - \frac{6\sin(\pi t/3)}{\pi}\right) \Big|_0^3 = 3 \text{ m.}$$

- The distance traveled is

$$\begin{aligned} & \int_1^3 \left(1 - 2\cos\left(\frac{\pi t}{3}\right)\right) dt - \int_0^1 \left(1 - 2\cos\left(\frac{\pi t}{3}\right)\right) dt \\ &= \left(t - \frac{6\sin(\pi t/3)}{\pi}\right) \Big|_1^3 - \left(t - \frac{6\sin(\pi t/3)}{\pi}\right) \Big|_0^1 \\ &= (3 - 0 - (1 - 3\sqrt{3}/\pi)) - (1 - 3\sqrt{3}/\pi - (0 - 0)) \\ &= 1 + \frac{6\sqrt{3}}{\pi} = \frac{6\sqrt{3} + \pi}{\pi}. \end{aligned}$$

$$\text{d. } s(t) = s(0) + \int_0^t (1 - 2\cos \pi x/3) dx = \left(x - \frac{6\sin(\pi x/3)}{\pi}\right) \Big|_0^t = t - \frac{6}{\pi} \sin \frac{\pi t}{3}.$$

With the antiderivative method, we would have

$$s(t) = \int v(t) dt = \int \left(1 - 2\cos\left(\frac{\pi t}{3}\right)\right) dt = \left(t - \frac{6\sin(\pi t/3)}{\pi}\right) + C,$$

and because $s(0) = 0$, we would have $C = 0$. Therefore $s(t) = t - \frac{6}{\pi} \sin \frac{\pi t}{3}$.

3

- $12t^2 - 30t + 12 = 6(2t^2 - 5t + 2) = 6(2t - 1)(t - 2)$, which is zero for $t = 2$ and $t = \frac{1}{2}$. $v(t) > 0$ for $0 \leq t < \frac{1}{2}$ and for $2 < t \leq 3$, so the motion is in the positive direction on those intervals, while $v < 0$ for $\frac{1}{2} < t < 2$, so the motion is in the negative direction on that interval.

b. The displacement is given by $\int_0^3 (12t^2 - 30t + 12) dt = (4t^3 - 15t^2 + 12t) \Big|_0^3 = 108 - 135 + 36 = 9$ m.

c. The distance traveled is

$$\begin{aligned} & \int_0^{1/2} (12t^2 - 30t + 12) dt + \int_2^3 (12t^2 - 30t + 12) dt - \int_{1/2}^2 (12t^2 - 30t + 12) dt \\ &= (4t^3 - 15t^2 + 12t) \Big|_0^{1/2} + (4t^3 - 15t^2 + 12t) \Big|_2^3 - (4t^3 - 15t^2 + 12t) \Big|_{1/2}^2 \\ &= (1/2 - 15/4 + 6) + (108 - 135 + 36) - (32 - 60 + 24) - ((32 - 60 + 24) - (1/2 - 15/4 + 6)) \\ &= 22.5 \text{ m.} \end{aligned}$$

d. $s(t) = s(0) + \int_0^t (12x^2 - 30x + 12) dx = 1 + (4x^3 - 15x^2 + 12x) \Big|_0^t = 4t^3 - 15t^2 + 12t + 1$.

With the antiderivative method, we would have

$$s(t) = \int v(t) dt = \int (12t^2 - 30t + 12) dt = 4t^3 - 15t^2 + 12t + C,$$

and because $s(0) = 1$, we would have $C = 1$. Therefore, $s(t) = 4t^3 - 15t^2 + 12t + 1$.

4 The displacement is $\int_0^{1.5} 20 \cos \pi t dt = \left(\frac{20}{\pi} \sin \pi t \right) \Big|_0^{1.5} = -\frac{20}{\pi}$.

5 The position $s(t)$ and the displacement are the same, because the projectile started on the ground at position 0.

$$s(t) = \int_0^t v(x) dx = \int_0^t (20 - 10x) dx = (20x - 5x^2) \Big|_0^t = 20t - 5t^2.$$

Note that the projectile is moving up for $0 < t < 2$ and down for $2 < t < 4$. Thus the distance traveled is equal to the position for $0 < t < 2$, but for $2 < t < 4$ the distance traveled is $20 + (20 - (20t - 5t^2)) = 40 - 20t + 5t^2$.

6 $a(t) = -5$, so $v(t) = -5t + C$, and because $v(0) = 80$, we have $v(t) = 80 - 5t$. The position function $s(t) = \int v(t) dt = \int (80 - 5t) dt = 80t - 5t^2/2 + D$, and because $s(0) = 0$ we have $D = 0$. Thus, $s(t) = 80t - 5t^2/2$.

7

a. $v(t) = \int a(t) dt = \int 2 \sin(\pi t/4) dt = -\frac{8}{\pi} \cos(\pi t/4) + C$, and because $v(0) = -\frac{8}{\pi}$, we have $C = 0$.

Thus, $v(t) = -\frac{8}{\pi} \cos(\pi t/4)$.

$s(t) = \int v(t) dt = \int -\frac{8}{\pi} \cos(\pi t/4) dt = -\frac{32}{\pi^2} \sin(\pi t/4) + D$, and because $s(0) = 0$ we have $D = 0$.

Thus, $s(t) = -\frac{32}{\pi^2} \sin(\pi t/4)$.

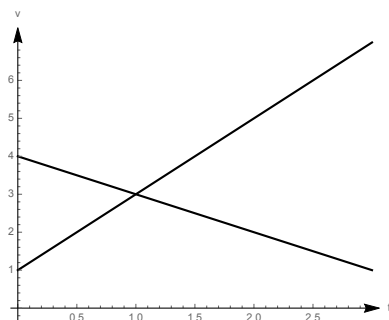
b. s is periodic with period 8, so we only consider $0 \leq t \leq 8$. There are critical numbers for s at $t = 2$ and $t = 6$. There is a maximum for s of $\frac{32}{\pi^2}$ at $t = 6$ and a minimum of $-\frac{32}{\pi^2}$ at $t = 2$.

c. The average velocity is $\frac{1}{8} \int_0^8 \frac{8}{\pi} \cos(\pi t/4) dt = -\frac{4}{\pi^2} (\sin(\pi t/4)) \Big|_0^8 = 0$.

The average position is $\frac{1}{8} \int_0^8 -\frac{32}{\pi^2} \sin(\pi t/4) dt = \frac{16}{\pi^3} (\cos(\pi t/4)) \Big|_0^8 = 0$.

8

a.



b. Anna runs $\int_0^1 (2t + 1) dt = (t^2 + t) \Big|_0^1 = 2$ miles, while Benny runs $\int_0^1 (4 - t) dt = (4t - t^2/2) \Big|_0^1 = (4 - 1/2) = 3.5$ miles. Note that the area under Anna's graph from 0 to 1 is smaller than the corresponding area under Benny's graph.

c. $s_A(t) = t^2 + t$, which is 6 for $t = 2$. $s_B(t) = 4t - t^2/2$ which is 6 for $t = 2$, so they both take exactly two hours to run 6 miles. At $t = 2$, both velocity function have the same area under them, namely 6.

9

a. For $0 \leq t \leq 8$ we have $R'(t) = 4t^{1/3}$, so $R(t) = 3t^{4/3} + C$, but $C = 0$, so $R(t) = 3t^{4/3}$.

b. Note that $R(8) = 48$, so for $t > 8$, $R(t) = 48 + \int_8^t 2 dx = 48 + 2(t - 8)$.

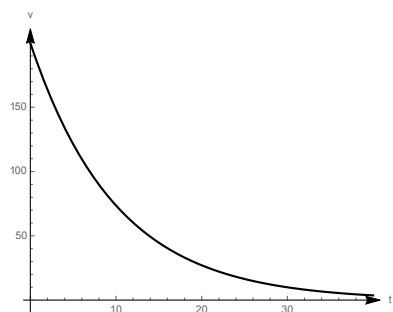
$$\text{We have } R(t) = \begin{cases} 3t^{4/3} & \text{if } 0 \leq t \leq 8; \\ 2t + 32 & \text{if } t > 8. \end{cases}$$

c. The fuel runs out when $150 = 48 + 2(t - 8)$, which occurs for $t = 59$.

10 Let $V'(t) = -\frac{15}{t+1}$. Then $V(t) = -15 \ln(t+1) + C$, and because $V(0) = 75$, we have $C = 75$. Thus, $V(t) = 75 - 15 \ln(t+1)$. This is 0 when $\ln(t+1) = 5$, which occurs when $t = e^5 - 1 \approx 147.413$ hours, so don't try to hold your breath while the tank is emptying.

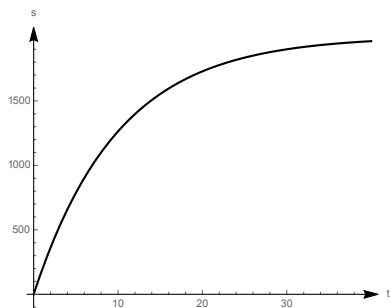
11

a.



b. The velocity is 50 when $200e^{-t/10} = 50$, which occurs when $e^{t/10} = 4$, so when $t = 10 \ln 4$.

c. .

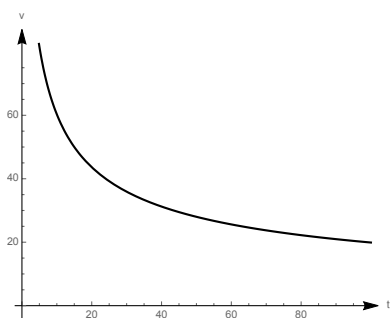


The position is given by $\int_0^t 200e^{-x/10} dx = -2000e^{-x/10} \Big|_0^t = 2000(1 - e^{-t/10})$.

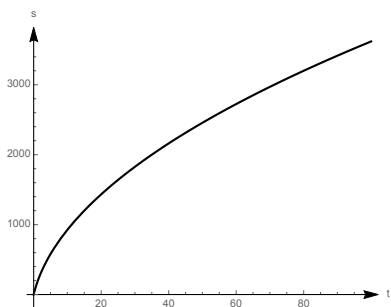
d. No. $\lim_{t \rightarrow \infty} s(t) = 2000 < 2500$.

12

a.



b.



The position is given by $\int_0^t 200/\sqrt{x+1} dx = (400\sqrt{x+1}) \Big|_0^t = 400(\sqrt{t+1} - 1)$.

c. Yes, the position is 2500 when $400\sqrt{t+1} = 2900$, which occurs when $t+1 = \left(\frac{29}{4}\right)^2$, so

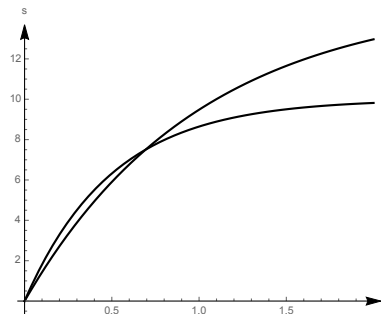
$$t = \left(\frac{29}{4}\right)^2 - 1 = 51.5625.$$

13

Tom's position function is given by

$$\int_0^t 20e^{-2x} dx = (-10e^{-2x}) \Big|_0^t = 10(1 - e^{-2t}).$$

a. Sue's position is given by $\int_0^t 15e^{-x} dx = (-15e^{-x}) \Big|_0^t = 15(1 - e^{-t}).$



b. We are looking for t so that $10(1 - e^{-2t}) = 15(1 - e^{-t})$, which occurs when $10(e^{-t})^2 - 15e^{-t} + 5 = 0$, or $2u^2 - 3u + 1 = 0$ where $u = e^{-t}$. This quadratic factors as $(2u - 1)(u - 1)$, so if $e^{-t} = 1$ or $e^{-t} = 1/2$, so $t = 0$ or $t = \ln 2$.

c. Sue takes the lead and doesn't relinquish it at $t = \ln 2$.

14 The area is given by

$$\int_0^1 ((x+1) - (2^x + x - 1)) dx = \int_0^1 (2 - 2^x) dx = \left(2x - \frac{2^x}{\ln 2}\right) \Big|_0^1 = \left(2 - \frac{2}{\ln 2}\right) - \left(0 - \frac{1}{\ln 2}\right) = 2 - \frac{1}{\ln 2}.$$

15 The area is given by

$$\int_0^{\pi/4} (\sin 2x - 1 - (-\cos 2x)) dx = \left(-\frac{\cos 2x}{2} - x + \frac{\sin 2x}{2}\right) \Big|_0^{\pi/4} = \left(0 - \frac{\pi}{4} + \frac{1}{2}\right) - \left(-\frac{1}{2} - 0 + 0\right) = 1 - \frac{\pi}{4}.$$

16 The area is given by

$$\int_0^{\pi/3} \cos y dy = \sin y \Big|_0^{\pi/3} = \frac{\sqrt{3}}{2}.$$

17 The area is given by

$$\int_0^1 (e^y - 1) dy = (e^y - y) \Big|_0^1 = (e - 1) - (1 - 0) = e - 2.$$

18 Because $\sin \frac{\pi}{4} = \cos \frac{\pi}{4}$, the curves intersect at $x = \frac{\pi}{2}$. By symmetry, the area is given by

$$2 \int_{\pi/2}^{5\pi/2} \left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) dx = 2 \left(-2 \cos \frac{x}{2} - 2 \sin \frac{x}{2}\right) \Big|_{\pi/2}^{5\pi/2} = -4 \left(\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right) = 8\sqrt{2}.$$

19 These two curves meet when $4x = x\sqrt{25 - x^2}$, so at $x = 0$ and when $4 = \sqrt{25 - x^2}$, which occurs for $x = 3$. On $[0, 3]$, we have $x\sqrt{25 - x^2} > 4x$ (for example check at $x = 1$), so the area is

$$A = \int_0^3 (x\sqrt{25 - x^2} - 4x) dx.$$

Note that $\int_0^3 -4x dx = -2x^2 \Big|_0^3 = -18$. For the first term, use the substitution $u = 25 - x^2$, so that $du = -2x dx$ and the integration bounds become $u = 25$ to $u = 16$. Then we have

$$A = -\frac{1}{2} \int_{25}^{16} u^{1/2} du - 18 = -\frac{1}{2} \left(\frac{2}{3} u^{3/2}\right) \Big|_{25}^{16} - 18 = -\frac{64}{3} + \frac{125}{3} - 18 = \frac{7}{3}.$$

20 The curves meet where $6 - 3x^2 = 6x - 3$, or $3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x + 3)(x - 1) = 0$, which occurs at $x = 1$ and $x = -3$. The area is given by

$$\int_{-3}^1 (6 - 3x^2 - (6x - 3)) dx = \int_{-3}^1 (-3x^2 - 6x + 9) dx = \left(-x^3 - 3x^2 + 9x \right) \Big|_{-3}^1 = (-1 - 3 + 9) - (-27 - 27 - 27) = 32.$$

21 The curves intersect for $x^2 = 2x^2 - 4x$, or $x^2 - 4x = x(x - 4) = 0$, so for $x = 0$ and $x = 4$. For $0 \leq x \leq 2$ the region is bounded above by x^2 and below by the x -axis, and for $2 \leq x \leq 4$ the region is bounded above by x^2 and below by $2x^2 - 4x$. The area is given by

$$\int_0^2 x^2 dx + \int_2^4 (x^2 - 2x^2 + 4x) dx = \left(x^3/3 \right) \Big|_0^2 + \left(-x^3/3 + 2x^2 \right) \Big|_2^4 = (8/3) + (-64/3 + 32) - (-8/3 + 8) = 8.$$

22 The area is given by

$$\int_0^1 (1 - \sqrt{x})^2 dx = \int_0^1 (1 - 2\sqrt{x} + x) dx = \left(x - 4x^{3/2}/3 + x^2/2 \right) \Big|_0^1 = \frac{1}{6}.$$

23 Note that we can write $1 - \left| \frac{x}{2} - 1 \right|$ as $\begin{cases} \frac{x}{2} & \text{if } x \leq 2; \\ 2 - \frac{x}{2} & \text{if } x \geq 2. \end{cases}$

For $0 \leq x \leq 2$, the region is bounded above by $\frac{x}{2}$ and below by $\frac{x}{6}$. For $2 \leq x \leq 3$, the region is bounded above by $2 - \frac{x}{2}$ and below by $\frac{x}{6}$. The area is then given by

$$\int_0^2 (x/2 - x/6) dx + \int_2^3 (2 - x/2 - x/6) dx = \left(x^2/6 \right) \Big|_0^2 + \left(2x - x^2/3 \right) \Big|_2^3 = \frac{2}{3} + (6 - 3) - (4 - 4/3) = 1.$$

24 $\int_0^1 (x^{1/p} - x^p) dx$ is evaluated as

$$\left(\frac{p}{p+1} x^{(p+1)/p} - \frac{1}{p+1} x^{p+1} \right) \Big|_0^1 = \frac{p}{p+1} - \frac{1}{p+1} = \frac{p-1}{p+1}.$$

For $p = 100$ the area is $99/101$, and for $p = 1000$ the area is $999/1001$.

25 The curve and the line intersect where $2x^2 - 6x + 5 = 1$, or $2x^2 - 6x + 4 = 2(x^2 - 3x + 2) = 2(x - 1)(x - 2) = 0$, which is for $x = 1$ and $x = 2$. The area is given by

$$\begin{aligned} \int_1^2 (1 - (2x^2 - 6x + 5)) dx &= \int_1^2 (-4 + 6x - 2x^2) dx = \left(-4x + 3x^2 - \frac{2x^3}{3} \right) \Big|_1^2 \\ &= -8 + 12 - \frac{16}{3} - \left(-4 + 3 - \frac{2}{3} \right) = \frac{1}{3}. \end{aligned}$$

26 Substituting x for y^2 in the equation $x = (2 - y^2)^2$, we obtain $x = (2 - x)^2$, which can be written as $x = 4 - 4x + x^2$ or $x^2 - 5x + 4 = (x - 4)(x - 1) = 0$, so the curves intersect at $x = 4$ and $x = 1$. The points of intersection are $(1, \pm 1)$ and $(4, \pm 2)$. It will be easier to integrate with respect to y . Exploiting symmetry, we can integrate with respect to y from 0 to 1 and then again from 1 to 2 and double our result.

Note that between $y = 1$ and $y = 2$ we have $y^2 > (2 - y^2)^2$, while between $y = 0$ and $y = 1$ the reverse

is true. Thus the area is given by

$$\begin{aligned}
 A &= 2 \int_0^1 ((2 - y^2)^2 - y^2) dy + 2 \int_1^2 (y^2 - (2 - y^2)^2) dy \\
 &= 2 \int_0^1 (y^4 - 5y^2 + 4) dy + 2 \int_1^2 (-y^4 + 5y^2 - 4) dy \\
 &= 2 \left(\frac{y^5}{5} - \frac{5y^3}{3} + 4y \right) \Big|_0^1 + 2 \left(-\frac{y^5}{5} + \frac{5y^3}{3} - 4y \right) \Big|_1^2 \\
 &= 2 \left(\frac{38}{15} \right) + 2 \left(-\frac{16}{15} + \frac{38}{15} \right) = \frac{120}{15} = 8.
 \end{aligned}$$

27 The area of R_1 is

$$\int_0^1 (3 - x - 2\sqrt{x}) dx = \left(3x - \frac{x^2}{2} - \frac{4x^{3/2}}{3} \right) \Big|_0^1 = 3 - \frac{1}{2} - \frac{4}{3} = \frac{7}{6}.$$

The area of R_2 can be computed by subtracting the area of R_1 from the area of a triangle with base 3 and height 3, so it is

$$\frac{9}{2} - \frac{7}{6} = \frac{20}{6} = \frac{10}{3}.$$

The area of R_3 is the difference of $\int_1^3 2\sqrt{x} dx$ and the area of a triangle with base 2 and height 2, so it is

$$\int_1^3 2\sqrt{x} dx - 2 = \left(\frac{4x^{3/2}}{3} \right) \Big|_1^3 - 2 = \frac{12\sqrt{3}}{3} - \frac{4}{3} - 2 = 4\sqrt{3} - \frac{10}{3}.$$

28
$$V = \pi \int_0^1 ((3 - x)^2 - 4x) dx = \pi \int_0^1 (x^2 - 10x + 9) dx = \pi \left(\frac{x^3}{3} - 5x^2 + 9x \right) \Big|_0^1 = \pi \left(\frac{1}{3} - 5 + 9 \right) = \frac{13\pi}{3}.$$

29

$$\begin{aligned}
 V &= 2\pi \int_0^1 x(3 - x - 2\sqrt{x}) dx = 2\pi \int_0^1 (3x - x^2 - 2x^{3/2}) dx \\
 &= 2\pi \left(\frac{3x^2}{2} - \frac{x^3}{3} - \frac{4x^{5/2}}{5} \right) \Big|_0^1 = 2\pi \left(\frac{3}{2} - \frac{1}{3} - \frac{4}{5} \right) = \frac{11\pi}{15}.
 \end{aligned}$$

30

$$\begin{aligned}
 V &= \pi \int_0^2 \left((3 - y)^2 - \frac{y^4}{16} \right) dy = \pi \int_0^2 \left(9 - 6y + y^2 - \frac{y^4}{16} \right) dy \\
 &= \pi \left(9y - 3y^2 + \frac{y^3}{3} - \frac{y^5}{80} \right) \Big|_0^2 = \pi \left(18 - 12 + \frac{8}{3} - \frac{32}{80} \right) = \frac{124\pi}{15}.
 \end{aligned}$$

31

$$\begin{aligned}
 V &= 2\pi \int_0^2 y \left(3 - y - \frac{y^2}{4} \right) dy = 2\pi \left(3y - y^2 - \frac{y^3}{4} \right) dy \\
 &= 2\pi \left(\frac{3y^2}{2} - \frac{y^3}{3} - \frac{y^4}{16} \right) \Big|_0^2 = 2\pi \left(6 - \frac{8}{3} - 1 \right) = \frac{14\pi}{3}.
 \end{aligned}$$

32
$$V = \pi \int_0^2 (3 - y)^2 dy + \pi \int_2^{2\sqrt{3}} \left(3 - \frac{y^2}{4} \right)^2 dy.$$

$$33 \quad V = 2\pi \int_1^3 (3-x)(2\sqrt{x}-3+x) \, dx.$$

34

$$a. \quad A = \int_0^2 ((4+y) - (y^2+2)) \, dy.$$

$$b. \quad V = 2\pi \int_0^2 y[(4+y) - (y^2+2)] \, dy.$$

$$c. \quad V = \pi \int_0^2 [(4+y)^2 - (y^2+2)^2] \, dy.$$

$$d. \quad V = \int_0^2 A(y) \, dy = \frac{\pi}{2} \int_0^2 \left(\frac{(4+y) - (y^2+2)}{2} \right)^2 \, dy.$$

$$35 \quad A = \int_0^1 1 \, dx + \int_1^4 (2-\sqrt{x}) \, dx = 1 + \left(2x - \frac{2x^{3/2}}{3} \right) \Big|_1^4 = 1 + \left(8 - \frac{16}{3} - \left(2 - \frac{2}{3} \right) \right) = \frac{7}{3}.$$

$$36 \quad V = 2\pi \int_0^1 y(2-y)^2 \, dy = 2\pi \int_0^1 (4y-4y^2+y^3) \, dy = 2\pi \left(2y^2 - \frac{4y^3}{3} + \frac{y^4}{4} \right) \Big|_0^1 = 2\pi \left(2 - \frac{4}{3} + \frac{1}{4} \right) = \frac{11\pi}{6}.$$

$$37 \quad V = \pi \int_0^1 (2-y)^4 \, dy = \pi \left(-\frac{(2-y)^5}{5} \right) \Big|_0^1 = \pi \left(-\frac{1}{5} + \frac{32}{5} \right) = \frac{31\pi}{5}.$$

38 For the portion generated over $0 \leq x \leq 1$, we have

$$S_1 = 2\pi \int_0^1 1\sqrt{1+0^2} \, dx = 2\pi.$$

For the portion over $1 \leq x \leq 4$, we have

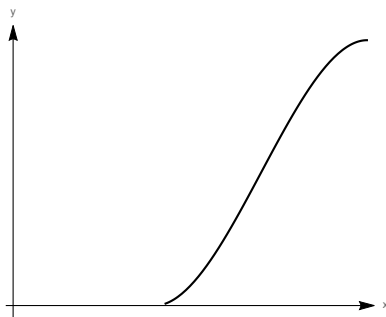
$$S_2 = 2\pi \int_1^4 (2-\sqrt{x})\sqrt{1+\frac{1}{4x}} \, dx \approx 8.96177.$$

This calculation was done using a computer algebra system. The surface area for the whole surface is approximately $S_1 + S_2 = 2\pi + 8.96177 \approx 15.24$.

39 The area of region R_1 is $\int_0^{\pi/3} \sec^2 x \, dx = \tan x \Big|_0^{\pi/3} = \sqrt{3}$. The area of region R_2 can be calculated by subtracting the area of R_1 from the area of a rectangle with base $\frac{\pi}{3}$ and height 4. Thus the area of R_2 is $\frac{4\pi}{3} - \sqrt{3}$.

40

a.



b. $R(x) = \int_a^x (f(t) - g(t)) dt$, and $R'(x) = f(x) - g(x)$.

41 $A(a) = \int_0^{\sqrt[3]{a}} (\sqrt{x/a} - x^2/a) dx = \left(\frac{2}{3} \frac{x^{3/2}}{\sqrt{a}} - \frac{x^3}{3a} \right) \Big|_0^{\sqrt[3]{a}} = \frac{1}{3}.$

42

a. $V = \pi \int_0^4 (\sqrt{4-y})^2 dy = \pi \int_0^4 (4-y) dy = \pi (4y - y^2/2) \Big|_0^4 = \pi(16 - 8 - (0 - 0)) = 8\pi.$

b. $V = 2\pi \int_0^2 x(4-x^2) dx = 2\pi \int_0^2 (4x - x^3) dx = 2\pi (2x^2 - x^4/4) \Big|_0^2 = 2\pi(8 - 4 - (0 - 0)) = 8\pi.$

43 From $x = 0$ to $x = 1$ the area function is $A(x) = (\sqrt{x})^2 = x$, while from $x = 1$ to $x = 2$ it is $A(x) = (2-x)^2$.

$$\int_0^1 x dx + \int_1^2 (2-x)^2 dx = \frac{1}{2}x^2 \Big|_0^1 - \frac{1}{3}(2-x)^3 \Big|_1^2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

44 The region is the same as in the previous exercise. From $x = 0$ to $x = 1$ the slices are semicircles with diameter $\sqrt{x}$, so their area is $\frac{\pi(\sqrt{x})^2}{8} = \frac{\pi}{8}x$. From $x = 1$ to $x = 2$, the slices are semicircles with diameter $2-x$, so their area is $\frac{\pi}{8}(2-x)^2$. Thus by the general slicing method, the volume is

$$\frac{\pi}{8} \left(\int_0^1 x dx + \int_1^2 (2-x)^2 dx \right) = \frac{\pi}{8} \left(\frac{1}{2}x^2 \Big|_0^1 - \frac{1}{3}(2-x)^3 \Big|_1^2 \right) = \frac{\pi}{8} \left(\frac{1}{2} + \frac{1}{3} \right) = \frac{5\pi}{48}.$$

45 Because the cross sections are parallel to the x -axis, solve for x to get $x = y^2$ and $x = 2 - y$. Then from $y = 0$ to $y = 1$ the area function is $A(y) = (y^2)^2 = y^4$ and from $y = 1$ to $y = 2$ it is $A(y) = (2-y)^2$. The by the general slicing method, the volume is

$$\int_0^1 y^4 dy + \int_1^2 (2-y)^2 dy = \frac{y^5}{5} \Big|_0^1 - \frac{1}{3}(2-y)^3 \Big|_1^2 = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}.$$

46

$$\begin{aligned} V &= \pi \int_0^2 ((-x^2 + 2x + 2)^2 - (2x^2 - 4x + 2)^2) dx = \pi \int_0^2 (24x - 24x^2 + 12x^3 - 3x^4) dx \\ &= \pi (12x^2 - 8x^3 + 3x^4 - 3x^5/5) \Big|_0^2 = \frac{64\pi}{5}. \end{aligned}$$

47

a. $V = 2\pi \int_0^1 x((1+\sqrt{x}) - (1-\sqrt{x})) dx = 4\pi \int_0^1 x^{3/2} dx = 4\pi \cdot \frac{2x^{5/2}}{5} \Big|_0^1 = \frac{8\pi}{5}.$

b. $V = \pi \int_0^2 (1^2 - ((1-y)^2)^2) dy = \pi \int_0^2 (1 - (y^4 - 4y^3 + 6y^2 - 4y + 1)) dy = \pi \int_0^2 (-y^4 + 4y^3 - 6y^2 + 4y) dy =$
 $\pi \left(-\frac{y^5}{5} + y^4 - 2y^3 + 2y^2 \right) \Big|_0^2 = \frac{8\pi}{5}.$

48 $V = \pi \int_0^{(\ln 2)/2} ((2e^{-x})^2 - (e^x)^2) dx = \pi \left(-2e^{-2x} - \frac{1}{2}e^{2x} \right) \Big|_0^{(\ln 2)/2} = \frac{\pi}{2}.$

49 It appears to be easiest to use shells, because then we won't have to split the integral up, and also the functions will integrate more easily. Solving for y we have $y = e^{x^2}$ and $y = e^{2-x^2}$. The height of each shell is then $e^{2-x^2} - e^{x^2}$ and the volume is

$$2\pi \int_0^1 x(e^{2-x^2} - e^{x^2}) dx = 2\pi \left(-\frac{1}{2}e^{2-x^2} - \frac{1}{2}e^{x^2} \right) \Big|_0^1 = \pi(-e - e + e^2 + 1) = \pi(e^2 - 2e + 1) = \pi(e - 1)^2.$$

50 Using washers, we have

$$\pi \int_0^{\pi/3} (4 - \sec^2 x) dx = \pi (4x - \tan x) \Big|_0^{\pi/3} = \pi \left(4\pi/3 - \sqrt{3} - (0 - 0) \right) = 4\pi^2/3 - \pi\sqrt{3}.$$

51 We use the shell method. $V = 2\pi \int_0^{\sqrt{3}/2} \frac{x}{\sqrt{1-x^2}} dx$. Let $u = 1 - x^2$ so that $du = -2x dx$. Substituting gives $V = 2\pi \int_1^{1/4} \frac{-1}{2\sqrt{u}} du = \pi \int_{1/4}^1 u^{-1/2} du = 2\pi \sqrt{u} \Big|_{1/4}^1 = 2\pi \left(1 - \frac{1}{2} \right) = \pi$.

52 Use the shell method. Then each shell has height $4 - x^2$ and the radius of the shell at x is $x - (-2) = x + 2$. The volume is then

$$\begin{aligned} V &= 2\pi \int_{-2}^2 (x+2)(4-x^2) dx = 2\pi \int_{-2}^2 (-x^3 - 2x^2 + 4x + 8) dx \\ &= 2\pi \left(-\frac{x^4}{4} - \frac{2}{3}x^3 + 2x^2 + 8x \right) \Big|_{-2}^2 = 2\pi \left(-\frac{32}{3} + 32 \right) = \frac{128\pi}{3}. \end{aligned}$$

53 We use the disk method. The radius of each disk is $4 - (x - 2)^2 = 4x - x^2$, so the volume is

$$\begin{aligned} V &= \pi \int_0^4 (4x - x^2)^2 dx = \pi \int_0^4 (x^4 - 8x^3 + 16x^2) dx \\ &= \pi \left(\frac{x^5}{5} - 2x^4 + \frac{16}{3}x^3 \right) \Big|_0^4 = \pi \left(\frac{1024}{5} - 512 + \frac{1024}{3} \right) = \frac{512\pi}{15}. \end{aligned}$$

54 The curves intersect when $6x = x^2 + 5$, or when $x^2 - 6x + 5 = (x - 1)(x - 5) = 0$, thus at $x = 1$ and $x = 5$. To revolve about the line $y = -1$, we use the washer method; the outer radius of each washer is $6x - (-1) = 6x + 1$ and the inner radius is $x^2 + 5 - (-1) = x^2 + 6$. The volume is thus

$$\begin{aligned} V &= \pi \int_1^5 ((6x+1)^2 - (x^2+6)^2) dx = \pi \int_1^5 (-x^4 + 24x^2 + 12x - 35) dx \\ &= \pi \left(-\frac{x^5}{5} + 8x^3 + 6x^2 - 35x \right) \Big|_1^5 = \pi \left(-625 + 1000 + 150 - 175 + \frac{1}{5} - 8 - 6 + 35 \right) \\ &= \pi \left(371 + \frac{1}{5} \right) = \frac{1856\pi}{5}. \end{aligned}$$

To revolve about the line $x = -1$, we use the shell method. The height of each shell is $6x - (x^2 + 5) = 6x - x^2 - 5$, and the radius at x is $x - (-1) = x + 1$. The volume is thus

$$\begin{aligned} V &= 2\pi \int_1^5 (x+1)(6x-x^2-5) dx = 2\pi \int_1^5 (-x^3 + 5x^2 + x - 5) dx = 2\pi \left(-\frac{x^4}{4} + \frac{5}{3}x^2 + \frac{1}{2}x^2 - 5x \right) \Big|_1^5 \\ &= 2\pi \left(-\frac{625}{4} + \frac{625}{3} + \frac{25}{2} - 25 + \frac{1}{4} - \frac{5}{3} - \frac{1}{2} + 5 \right) = \frac{256\pi}{3}. \end{aligned}$$

55 The two non-horizontal lines intersect at $(2, 4)$. To revolve about $y = -2$, we use the shell method. First solve for x to obtain $x = \frac{y}{2}$ and $x = 6 - y$; then the height of each shell is $6 - y - \frac{y}{2} = 6 - \frac{3}{2}y$, and the radius at y is $y - (-2) = y + 2$. The volume is

$$2\pi \int_0^4 (y+2) \left(6 - \frac{3}{2}y\right) dy = 2\pi \int_0^4 \left(12 + 3y - \frac{3}{2}y^2\right) dy = 2\pi \left(12y + \frac{3}{2}y^2 - \frac{1}{2}y^3\right) \Big|_0^4 = 80\pi.$$

To revolve about $x = -2$, we use the washer method. With the equations above, the outer radius of each washer is $6 - y - (-2) = 8 - y$ and the inner radius is $\frac{y}{2} - (-2) = 2 + \frac{y}{2}$. The volume is

$$\begin{aligned} V &= \pi \int_0^4 \left((8-y)^2 - \left(2 + \frac{y}{2}\right)^2\right) dy = \pi \int_0^4 \left(\frac{3}{4}y^2 - 18y + 60\right) dy \\ &= \pi \left(\frac{1}{4}y^3 - 9y^2 + 60y\right) \Big|_0^4 = \pi(16 - 144 + 240) = 112\pi. \end{aligned}$$

56

a. $V_x = \pi \int_1^2 \frac{1}{x^2} dx = \pi \left(-1/x\right) \Big|_1^2 = \frac{\pi}{2}.$

$$V_y = 2\pi \int_1^2 x \cdot x^{-1} dx = 2\pi, \text{ so } V_y > V_x.$$

b. $V_x = \pi \int_1^4 \frac{1}{x^6} dx = \pi \left(-x^{-5}/5\right) \Big|_1^4 = \frac{\pi}{5} \left(1 - \frac{1}{1024}\right).$

$$V_y = 2\pi \int_1^4 x \cdot x^{-3} dx = 2\pi \left(-1/x\right) \Big|_1^4 = \frac{3\pi}{2}, \text{ so } V_y > V_x.$$

c. $V_x = \pi \int_1^a \frac{1}{x^{2p}} dx = \begin{cases} \frac{\pi}{1-2p}(-1 + a^{1-2p}) & \text{if } p \neq 1/2, \\ \pi \ln a & \text{if } p = 1/2. \end{cases}$

d. $V_y = 2\pi \int_1^a \frac{1}{x^{p-1}} dx = \begin{cases} \frac{2\pi(a^{2-p} - 1)}{2-p} & \text{if } p \neq 2, \\ 2\pi \ln a & \text{if } p = 2. \end{cases}$

e. Using part d, let $h = 2 - p$, and note that V_y is continuous. So $\lim_{h \rightarrow 0} V_y = \lim_{p \rightarrow 2} \frac{2\pi(a^{2-p} - 1)}{2-p} = \lim_{h \rightarrow 0} \frac{2\pi(a^h - 1)}{h} = 2\pi \ln a$, so $\lim_{h \rightarrow 0} \frac{(a^h - 1)}{h} = \ln a$. A similar calculation can be done with the result of part c.

f. No, $V_y > V_x$ for all values of a and p , because $\frac{1}{x^{2p}} < \frac{1}{x^{p-1}}$ for $x > 1$ and $p > 0$.

57 We use disks when revolving about the x -axis. The volume is

$$\pi \int_0^1 c^2 x^2 (1-x)^2 dx = \pi c^2 \int_0^1 (x^2 - 2x^3 + x^4) dx = \pi c^2 \left(\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right) \Big|_0^1 = \pi c^2 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi c^2}{30}.$$

We use shells when revolving about the y -axis. The volume is

$$2\pi \int_0^1 x \cdot cx(1-x) dx = 2\pi c \int_0^1 (x^2 - x^3) dx = 2\pi c \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right) \Big|_0^1 = \frac{\pi c}{6}.$$

These are equal when $\frac{\pi c^2}{30} = \frac{\pi c}{6}$, which for $c \neq 0$ occurs when $c = 5$.

$$58 \quad L = \int_{-2}^2 \sqrt{1+4} \, dx = \sqrt{5}(2 - (-2)) = 4\sqrt{5}.$$

$$59 \quad L = \int_{\sqrt{2}}^{\sqrt{5}} \sqrt{1 + \frac{1}{x^2 - 1}} \, dx = \int_{\sqrt{2}}^{\sqrt{5}} \frac{x}{\sqrt{x^2 - 1}} \, dx. \text{ Let } u = x^2 - 1 \text{ so that } du = 2x \, dx. \text{ Substituting gives}$$

$$\frac{1}{2} \int_1^4 u^{-1/2} \, du = (\sqrt{u}) \Big|_1^4 = 2 - 1 = 1.$$

$$60 \quad \text{Note that } y' = x^2/2 - 1/(2x^2), \text{ so } 1 + y'^2 = (x^2/2 + 1/(2x^2))^2. \quad L = \int_1^2 \sqrt{1 + y'^2} \, dy = \int_1^2 (x^2/2 + 1/(2x^2)) \, dx = (x^3/6 - 1/(2x)) \Big|_1^2 = \frac{17}{12}.$$

$$61 \quad \text{Note that } y' = 1/(2\sqrt{x}) - \sqrt{x}/2, \text{ so } 1 + y'^2 = (1/(2\sqrt{x}) + \sqrt{x}/2)^2. \quad L = \int_1^3 \sqrt{1 + y'^2} \, dy = \int_1^3 (1/(2\sqrt{x}) + \sqrt{x}/2) \, dx = (\sqrt{x} + x^{3/2}/3) \Big|_1^3 = 2\sqrt{3} - 4/3.$$

$$62 \quad f'(x) = \pi \cos \pi x. \text{ Therefore the arc length is } L = \int_0^1 \sqrt{1 + \pi^2 \cos^2 \pi x} \, dx \approx 2.305.$$

$$63 \quad y' = 2(x+1), \text{ so } \sqrt{1 + f'(x)^2} = \sqrt{1 + (2x+2)^2} = \sqrt{4x^2 + 8x + 5}.$$

The arc length is $\int_2^4 \sqrt{4x^2 + 8x + 5} \, dx \approx 16.127.$

$$64 \quad y' = x^2 + x, \text{ so } \sqrt{1 + f'(x)^2} = \sqrt{1 + (x^2 + x)^2} = \sqrt{x^4 + 2x^3 + x^2 + 1}. \text{ So the arc length is } L = \int_0^2 \sqrt{x^4 + 2x^3 + x^2 + 1} \, dx \approx 5.351.$$

$$65 \quad \text{Note that } y' = 1/x, \text{ so } 1 + y'^2 = \frac{x^2 + 1}{x^2}, \text{ and } \sqrt{1 + y'^2} = \frac{\sqrt{x^2 + 1}}{x}.$$

So

$$L = \int_1^b \frac{\sqrt{x^2 + 1}}{x} \, dx = \left(\sqrt{x^2 + 1} - \ln \left(\frac{1 + \sqrt{x^2 + 1}}{x} \right) \right) \Big|_1^b = \sqrt{b^2 + 1} - \sqrt{2} + \ln \left(\frac{(\sqrt{b^2 + 1} - 1)(1 + \sqrt{2})}{b} \right).$$

Using a computer algebra system, we see that this has value 2 for $b \approx 2.715$.

66

$$\text{a. } S = 2\pi \int_0^2 (x^3/3) \sqrt{1 + x^4} \, dx. \text{ Let } u = 1 + x^4, \text{ so that } du = 4x^3 \, dx. \text{ Substituting gives } \frac{\pi}{6} \int_1^{17} u^{1/2} \, du =$$

$$\frac{\pi}{9} (u^{3/2}) \Big|_1^{17} = \frac{\pi}{9} (17^{3/2} - 1).$$

$$\text{b. } V = 2\pi \int_0^2 x(x^3/3) \, dx = 2\pi \int_0^2 x^4/3 \, dx = 2\pi (x^5/15) \Big|_0^2 = 2\pi(32/15 - 0) = 64\pi/15.$$

$$\text{c. } V = \pi \int_0^2 x^6/9 \, dx = \pi (x^7/63) \Big|_0^2 = \pi(128/63) = 128\pi/63.$$

67

$$\begin{aligned} \text{a. } S &= 2\pi \int_0^3 \sqrt{3x-x^2} \sqrt{1 + \frac{(3-2x)^2}{4(3x-x^2)}} dx = 2\pi \int_0^3 \sqrt{3x-x^2} \sqrt{\frac{12x-4x^2+9-12x+4x^2}{4(3x-x^2)}} dx = \\ &= \pi \int_0^3 3 dx = 3\pi(3-0) = 9\pi. \end{aligned}$$

$$\text{b. } V = \pi \int_0^3 (3x-x^2) dx = \pi \left(3x^2/2 - x^3/3 \right) \Big|_0^3 = \pi (27/2 - 9) = 9\pi/2.$$

68 Consider the line $y = x/2$ over the interval $[0, 8]$. If we revolve this around the x -axis, we generate a cone with radius 4 and height 8. The surface of this cone is $2\pi \int_0^8 (x/2) \sqrt{1+1/4} dx = \sqrt{5}\pi/2 (x^2/2) \Big|_0^8 = 16\sqrt{5}\pi$.

69

$$\begin{aligned} \text{a. } S &= 2\pi \int_1^2 \left(\frac{x^4}{2} + \frac{1}{16x^2} \right) \sqrt{1 + (2x^3 - (1/8x^3))^2} dx = \\ &= 2\pi \int_1^2 \left(\frac{x^4}{2} + \frac{1}{16x^2} \right) \sqrt{1 + (4x^6 - 1/2 + (64/x^6))} dx = 2\pi \int_1^2 \left(\frac{x^4}{2} + \frac{1}{16x^2} \right) \sqrt{4x^6 + 1/2 + 64/x^6} dx = \\ &= 2\pi \int_1^2 \left(\frac{x^4}{2} + \frac{1}{16x^2} \right) \sqrt{(2x^3 + (1/8x^3))^2} dx = 2\pi \int_1^2 \left(\frac{x^4}{2} + \frac{1}{16x^2} \right) \left(2x^3 + \frac{1}{8x^3} \right) dx = 2\pi \int_1^2 (x^7 + \\ &= x/16 + x/8 + x^{-5}/128) dx = 2\pi \int_1^2 (x^7 + 3x/16 + x^{-5}/128) dx = 2\pi (x^8/8 + 3x^2/32 - x^{-4}/512) \Big|_1^2 = \\ &= 2\pi((32 + 3/8 - 1/8192) - (1/8 + 3/32 - 1/512)) = \frac{263439\pi}{4096}. \end{aligned}$$

$$\begin{aligned} \text{b. } &\text{Using the fact that } \sqrt{1 + f'(x)^2} = 2x^3 + 1/(8x^3) \text{ which was discovered during the previous calculation,} \\ &\text{we have } L = \int_1^2 (2x^3 + x^{-3}/8) dx = (x^4/2 - x^{-2}/16) \Big|_1^2 = (8 - 1/64) - (1/2 - 1/16) = \frac{483}{64}. \end{aligned}$$

$$\begin{aligned} \text{c. } V &= 2\pi \int_1^2 (x^5/2 + x^{-1}/16) dx = 2\pi (x^6/12 + (1/16) \ln x) \Big|_1^2 = 2\pi(16/3 + (\ln 2)/16 - (1/12 + 0)) = \\ &= \frac{21\pi}{2} + \frac{\pi \ln 2}{8}. \end{aligned}$$

d.

$$\begin{aligned} V &= \pi \int_1^2 (x^4/2 + x^{-2}/16)^2 dx = \pi \int_1^2 (x^8/4 + x^2/16 + x^{-4}/256) dx = \pi (x^9/36 + x^3/48 - x^{-3}/768) \Big|_1^2 \\ &= \pi (512/36 + 8/48 - 1/6144 - (1/36 + 1/48 - 1/768)) = \frac{264341\pi}{18432}. \end{aligned}$$

$$\text{70 } m = \int_0^9 (3 + 2\sqrt{x}) dx = \left(3x + 4x^{3/2}/3 \right) \Big|_0^9 = 63 \text{ gm.}$$

$$\text{71 } m = \int_0^3 150e^{-x/3} dx = \left(-450e^{-x/3} \right) \Big|_0^3 = 450(1 - e^{-1}) \text{ gm.}$$

$$\text{72 } m = \int_0^2 dx + \int_2^4 2 dx + \int_4^6 4 dx = 2 + 4 + 8 = 14.$$

73

$$\text{a. } \text{Because } 50 = \int_0^{.2} kx dx = \frac{kx^2}{2} \Big|_0^{.2} = \frac{k}{50}, \text{ we must have } k = 2500.$$

$$W = \int_{0.2}^{0.7} 2500x dx = 1250x^2 \Big|_{0.2}^{0.7} = 562.5 \text{ J.}$$

b. $f(0.2) = 0.2k = 50$, so $k = 250$, and thus

$$W = \int_{0.2}^{0.7} 250x \, dx = 125x^2 \Big|_{0.2}^{0.7} = 56.25 \text{ J.}$$

74 The force function is given by $f(y) = 9.8 \left(7 - \frac{y}{4}\right)$. So the work required is

$$W = 9.8 \int_0^8 \left(7 - \frac{y}{4}\right) dy = 9.8 \left(7y - \frac{y^2}{8}\right) \Big|_0^8 = 9.8(56 - 8) = 9.8(48) = 470.4 \text{ J.}$$

75

- a. Note that the density of the chain is 2 kg/m.

$$W = \int_0^{10} 2(9.8)(10 - y) dy = 19.6 \left(10y - \frac{y^2}{2} \right) \Big|_0^{10} = 19.6(100 - 50) = 980 \text{ J.}$$

- b. The lower 6 meters of the chain are lifted 4 meters. The work required for this is $6(2)(9.8)(4) = 470.4 \text{ J}$
The work required to lift the upper four meters of the chain is

$$\int_6^{10} 2(9.8)(10 - y) dy = 19.6 \left(10y - \frac{y^2}{2} \right) \Big|_6^{10} = 19.6(100 - 50 - (60 - 18)) = 156.8.$$

So the total work required is $470.4 + 156.8 = 627.2 \text{ J}$.

76 The density of the chain is 3 kg/m. The work to lift the mass 5 meters is $4(9.8)(5) = 196 \text{ J}$. The work to lift the chain is $W = \int_0^5 3(9.8)(5 - y) dy = 29.4 \left(5y - \frac{y^2}{2} \right) \Big|_0^5 = 29.4(25 - 12.5) = 29.4(12.5) = 367.5 \text{ J}$. Therefore the total work is $196 + 367.5 = 563.5 \text{ J}$.

77

a. $W = \int_0^6 1000(9.8)(8)(6 - y) dy = 78,400 \left(6y - \frac{y^2}{2} \right) \Big|_0^6 = 78,400(36 - 18) = 1,411,200 \text{ J.}$

b. $W = \int_0^2 1000(9.8)(8)(7 - y) dy = 78,400 \left(7y - \frac{y^2}{2} \right) \Big|_0^2 = 78,400(14 - 2) = 940,800 \text{ J.}$

78 $W = \int_0^6 \pi 1000(9.8) \cdot 16(6 - y) dy = 16\pi(9800) \left(6y - \frac{y^2}{2} \right) \Big|_0^6 = 288\pi(9800) \approx 8,866,830 \text{ J.}$

79

- a.

$$\begin{aligned} W &= \int_0^6 1000(9.8) \frac{4}{9} \pi y^2 (6 - y) dy = \frac{39200}{9} \pi \int_0^6 (6y^2 - y^3) dy \\ &= \frac{39200}{9} \pi \left(2y^3 - \frac{y^4}{4} \right) \Big|_0^6 = \frac{39200}{9} \pi (432 - 324) \approx 1,477,805 \text{ J.} \end{aligned}$$

- b. The work to pump out the top 3 feet is

$$\int_3^6 \frac{39200}{9} \pi (6y^2 - y^3) dy = \frac{39200}{9} \pi \left(2y^3 - \frac{y^4}{4} \right) \Big|_3^6 = \frac{39200}{9} \pi (432 - 324 - (54 - 20.25)) \approx 1,015,592 \text{ J.}$$

Therefore, the work to pump out the bottom 3 feet is about $1,477,805 - 1,015,592 = 461,814 \text{ J}$.

80

$$\begin{aligned} W &= \int_{-2}^0 9.8(1000)(4 - y^2)\pi(2 - y) dy = 9800\pi \int_{-2}^0 (8 - 4y - 2y^2 + y^3) dy \\ &= 9800\pi \left(8y - 2y^2 - \frac{2y^3}{3} + \frac{y^4}{4} \right) \Big|_{-2}^0 = 9800\pi \left(16 + 8 - \frac{16}{3} - 4 \right) = 9800\pi \cdot \frac{44}{3} \approx 451,552 \text{ J.} \end{aligned}$$

81 $W = \int_0^9 1000(9.8)\pi y(10 - y) dy = 9800\pi \int_0^9 (10y - y^2) dy = 9800\pi \left(5y^2 - \frac{y^3}{3} \right) \Big|_0^9 = 9800\pi(405 - 243) \approx 4,987,592 \text{ J.}$

$$82 \quad F = \int_0^1 1000(9.8)(3-y) dy = 9800 \left(3y - \frac{y^2}{2} \right) \Big|_0^1 = 9800(2.5) = 24,500 \text{ N.}$$

$$83 \quad F = \int_0^1 1000(9.8)(3-y)\frac{y}{2} dy = 4900 \int_0^1 (3y-y^2) dy = 4900 \left(\frac{3y^2}{2} - \frac{y^3}{3} \right) \Big|_0^1 = 4900 \left(\frac{3}{2} - \frac{1}{3} \right) \approx 5716.7 \text{ N.}$$

$$84 \quad F = \int_{-2}^2 1000(9.8)(4-y)2\sqrt{4-y^2} dy = 19600 \int_{-2}^2 4\sqrt{4-y^2} dy - 19600 \int_{-2}^2 y\sqrt{4-y^2} dy. \text{ The first term}$$

in this sum is $19,600 \cdot 4$ times the area of the upper half of a circle of radius 2 so it is $19600 \cdot 4 \cdot 2\pi = 156,800\pi$. The second term is zero because it is the integral of an odd function over a symmetric interval. Therefore $F = 156,800\pi \approx 492,602 \text{ N}$.

85 Orient the semicircle so that the center is at the point $(0, 20)$. Then the force is given by

$$\int_0^{20} \rho g(20-y)(2)\sqrt{40y-y^2} dy.$$

. Let $u = 40y - y^2$ so that $du = (40 - 2y) dy$. Then we have

$$\rho g \int_0^{400} u^{1/2} du = \rho g \left(\frac{2}{3} u^{3/2} \right) \Big|_0^{400} \approx 5.2 \times 10^7.$$

86 The equation of L_1 is $y = (2ap + b)(x - p) + f(p)$, which can be written $y = (2ap + b)x - ap^2 + c$. The equation of L_2 is $y = (2aq + b)(x - q) + f(q)$, which can be written $y = (2aq + b)x - aq^2 + c$. If we set these equal to each other and solve to find the point of intersection, we find $s = \frac{p+q}{2}$. Thus, the area

$$\begin{aligned} R_1 &= \int_p^s f(x) - ((2ap + b)x - ap^2 + c) dx = \int_p^s ax^2 - 2apx + ap^2 dx \\ &= \left(ax^3/3 - apx^2 + ap^2x \right) \Big|_p^s = \frac{as^3}{3} - aps^2 + ap^2s - \frac{ap^3}{3} = \frac{a}{24}(q-p)^3. \end{aligned}$$

Similarly, the area

$$\begin{aligned} R_2 &= \int_s^q f(x) - ((2aq + b)x - aq^2 + c) dx = \int_s^q ax^2 - 2aqx + aq^2 dx \\ &= \left(ax^3/3 - aqx^2 + aq^2x \right) \Big|_s^q = \frac{aq^3}{3} - \frac{as^3}{3} + aqs^2 - aq^2s = \frac{aq^3}{3} - \frac{as}{3}(s^2 - 3qs + 3q^2) = \frac{a}{24}(q-p)^3. \end{aligned}$$

Chapter 7

Logarithmic, Exponential, and Hyperbolic Functions

7.1 Logarithmic and Exponential Functions Revisited

7.1.1 The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

7.1.2 This represents the area under the function $\frac{1}{t}$ and above the t axis between the vertical line $t = 1$ and the vertical line $t = x$.

$$\mathbf{7.1.3} \quad \int 4^x dx = \int e^{x \ln 4} dx = \frac{1}{\ln 4} e^{x \ln 4} + C = \frac{4^x}{\ln 4} + C.$$

7.1.4 The inverse is $y = e^x$, whose domain is $(-\infty, \infty)$ and whose range is $(0, \infty)$.

$$\mathbf{7.1.5} \quad 3^x = e^{x \ln 3}, \quad x^\pi = e^{\pi \ln x}, \quad x^{\sin x} = e^{\sin x \cdot \ln x}.$$

$$\mathbf{7.1.6} \quad \frac{d}{dx} 3^x = \frac{d}{dx} e^{x \ln 3} = e^{x \ln 3} \cdot \ln 3 = \ln 3 \cdot 3^x.$$

$$\mathbf{7.1.7} \quad \frac{d}{dx} (x \ln x^3) = x \cdot \frac{1}{x^3} \cdot 3x^2 + \ln x^3 = 3 + \ln x^3 = 3 + 3 \ln x = 3(1 + \ln x).$$

$$\mathbf{7.1.8} \quad \frac{d}{dx} (\ln(\ln(x))) = \frac{1}{\ln x} \cdot \frac{1}{x}.$$

$$\mathbf{7.1.9} \quad \frac{d}{dx} \sin(\ln x) = \cos(\ln x) \cdot \frac{1}{x} = \frac{\cos(\ln x)}{x}.$$

$$\mathbf{7.1.10} \quad \frac{d}{dx} \ln(\cos^2 x) = \frac{1}{\cos^2 x} \cdot (2 \cos x)(-\sin x) = -\frac{2 \sin x}{\cos x} = -2 \tan x.$$

$$\mathbf{7.1.11} \quad \frac{d}{dx} ((\ln 2x)^{-5}) = -5(\ln 2x)^{-6} (1/x) = -\frac{5}{x \ln^6 2x}.$$

$$\mathbf{7.1.12} \quad \frac{d}{dx} (\ln^3(3x^2 + 2)) = 3 \ln^2(3x^2 + 2) \left(\frac{1}{3x^2 + 2} \right) (6x) = \frac{18x \ln^2(3x^2 + 2)}{3x^2 + 2}.$$

$$\mathbf{7.1.13} \quad 2x^{4x} = e^{4x \ln 2x}, \text{ so}$$

$$\frac{d}{dx} 2x^{4x} = \frac{d}{dx} e^{4x \ln 2x} = e^{4x \ln(2x)} \left(4x \cdot \frac{1}{x} + \ln 2x \cdot 4 \right) = (2x)^{4x} (4 + 4 \ln 2x) = 4^{2x+1} x^{4x} (1 + \ln 2x).$$

$$7.1.14 \quad \frac{d}{dx} x^\pi = \pi x^{\pi-1}.$$

$$7.1.15 \quad \frac{d}{dx} 2^{(x^2)} = 2^{(x^2)} \cdot \ln 2 \cdot (2x) = 2^{(x^2+1)} x \ln 2.$$

$$7.1.16 \quad \frac{d}{dt} (\sin t)^{\sqrt{t}} = \frac{d}{dt} e^{\sqrt{t} \ln(\sin t)} = e^{\sqrt{t} \ln(\sin t)} \cdot \left(\sqrt{t} \cot t + \ln(\sin t) \cdot \frac{1}{2\sqrt{t}} \right), \text{ which can be written as}$$

$$(\sin t)^{\sqrt{t}} \left(\sqrt{t} \cot t + \ln(\sin t) \cdot \frac{1}{2\sqrt{t}} \right) = (\sin t)^{\sqrt{t}} \left(\frac{2t \cot t + \ln(\sin t)}{2\sqrt{t}} \right).$$

$$7.1.17 \quad \frac{d}{dx} e^{2x \ln(x+1)} = e^{2x \ln(x+1)} \cdot \left(\frac{2x}{x+1} + 2 \ln(x+1) \right), \text{ which can be written as}$$

$$(x+1)^{2x} \left(\frac{2x}{x+1} + 2 \ln(x+1) \right).$$

$$7.1.18 \quad \frac{d}{dx} e^{-(\ln x)^2} = e^{-(\ln x)^2} \cdot \left(-\frac{2 \ln x}{x} \right), \text{ which can be written as } -\frac{x^{-\ln x} \cdot 2 \ln x}{x}.$$

$$7.1.19 \quad \frac{d}{dy} e^{\sin y \ln y} = e^{\sin y \ln y} \left(\cos y \ln y + \frac{\sin y}{y} \right), \text{ which can be written as } y^{\sin y} \left(\cos y \ln y + \frac{\sin y}{y} \right).$$

$$7.1.20 \quad \frac{d}{dt} e^{(\ln t)/t} = e^{(\ln t)/t} \left(\frac{t(1/t) - \ln t}{t^2} \right), \text{ which can be written as } t^{1/t} \left(\frac{1 - \ln t}{t^2} \right).$$

$$7.1.21 \quad \frac{d}{dx} e^{-10x^2} = e^{-10x^2} \cdot -20x = -20xe^{-10x^2}.$$

$$7.1.22 \quad \frac{d}{dx} (x^e + e^x) = ex^{e-1} + e^x.$$

$$7.1.23 \quad \frac{d}{dx} x^{2x} = \frac{d}{dx} e^{2x \ln x} = e^{2x \ln x} (2 + 2 \ln x) = 2x^{2x} (1 + \ln x).$$

$$7.1.24 \quad \frac{d}{dx} x^{\tan x} = \frac{d}{dx} e^{\tan x \ln x} = e^{\tan x \ln x} ((\tan x)/x + \sec^2 x \ln x) = x^{\tan x} \left(\frac{\tan x}{x} + \sec^2 x \ln x \right).$$

$$7.1.25 \quad \frac{d}{dx} (1/x)^x = \frac{d}{dx} e^{x \ln(1/x)} = \frac{d}{dx} e^{-x \ln x} = e^{-x \ln x} (-1 - \ln x) = \left(\frac{1}{x} \right)^x (-1 - \ln x).$$

$$7.1.26 \quad \frac{d}{dx} (x^{x^{10}}) = \frac{d}{dx} e^{(x^{10} \ln x)} = e^{(x^{10} \ln x)} (x^9 + 10x^9 \ln x) = x^{(x^{10})} (x^9) (1 + 10 \ln x).$$

7.1.27

$$\begin{aligned} \frac{d}{dx} (1 + (4/x))^x &= \frac{d}{dx} e^{x \ln(1 + (4/x))} = e^{x \ln(1 + (4/x))} \left(\ln(1 + (4/x)) + \frac{x}{1 + (4/x)} \cdot -\frac{4}{x^2} \right) \\ &= (1 + (4/x))^x (\ln(1 + (4/x)) - \frac{4}{x + 4}) \end{aligned}$$

7.1.28

$$\begin{aligned} \frac{d}{dx} \cos(x^{2 \sin x}) &= \frac{d}{dx} \cos(e^{2 \sin x \ln x}) = -\sin(x^{2 \sin x}) \cdot e^{2 \sin x \ln x} \cdot \left(\frac{2 \sin x}{x} + 2 \cos x \ln x \right) \\ &= -\sin(x^{2 \sin x}) \cdot x^{2 \sin x} \cdot \left(\frac{2 \sin x}{x} + 2 \cos x \ln x \right). \end{aligned}$$

$$7.1.29 \quad \int_0^3 \frac{2x-1}{x+1} dx = \int_0^3 \left(2 - \frac{3}{x+1} \right) dx = (2x - 3 \ln(x+1)) \Big|_0^3 = 6(1 - \ln 2).$$

7.1.30 Let $u = 4x^3 + 7$. Then $du = 12x^2 dx$. Substituting gives

$$\frac{1}{12} \int \frac{1}{u} du = \frac{1}{12} \ln |u| + C = \frac{1}{12} \ln |4x^3 + 7| + C.$$

7.1.31 Let $u = \ln x$ so that $du = \frac{1}{x} dx$. Then

$$\int_e^{e^2} \frac{dx}{x \ln^3 x} = \int_1^2 u^{-3} du = \frac{-1}{2} (u^{-2}) \Big|_1^2 = \frac{-1}{2} (1/4 - 1) = \frac{3}{8}.$$

7.1.32 Let $u = 1 + \cos x$ so that $du = -\sin x dx$. Then

$$\int_0^{\pi/2} \frac{\sin x}{1 + \cos x} dx = \int_1^2 \frac{1}{u} du = (\ln u) \Big|_1^2 = \ln 2.$$

7.1.33 Let $u = 4 + e^{2x}$ so that $du = 2e^{2x} dx$. Substituting gives

$$\frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln(4 + e^{2x}) + C.$$

7.1.34 Let $u = \ln(\ln x)$ so that $du = \frac{1}{x \ln x} dx$. Substituting yields

$$\int \frac{1}{u} du = \ln |u| + C = \ln |\ln(\ln x)| + C.$$

7.1.35 Let $u = \ln(\ln x)$ so that $du = \frac{1}{x \ln x} dx$. Substituting yields

$$\int_{\ln 2}^{\ln 3} \frac{1}{u^2} du = \left(\frac{-1}{u} \right) \Big|_{\ln 2}^{\ln 3} = \frac{1}{\ln 2} - \frac{1}{\ln 3}.$$

7.1.36 Let $u = \ln(y^2 + 1)$ so that $du = \frac{2y}{y^2 + 1} dy$. Substituting yields

$$\frac{1}{2} \int_0^{\ln 2} u^4 du = \frac{1}{2} (u^5/5) \Big|_0^{\ln 2} = (\ln 2)^5/10.$$

7.1.37 Let $u = -x^2/2$ so that $du = -x dx$. Substituting yields

$$-4 \int_0^{-2} e^u du = 4 \int_{-2}^0 e^u du = 4 (e^u) \Big|_{-2}^0 = 4(1 - e^{-2}) = 4 - \frac{4}{e^2}.$$

7.1.38 Let $u = \sin x$ so that $du = \cos x dx$. Then

$$\int \frac{e^{\sin x}}{\sec x} dx = \int e^{\sin x} \cos x dx = \int e^u du = e^u + C = e^{\sin x} + C.$$

7.1.39 Let $u = \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}} dx$. Then

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C.$$

7.1.40 Let $u = e^{z/2} + 1$, so that $du = \frac{1}{2}e^{z/2} dz$.

Then

$$\int_{-2}^2 \frac{e^{z/2}}{e^{z/2} + 1} dz = \int_{(1/e)+1}^{e+1} \frac{2}{u} du = (2 \ln u) \Big|_{(1/e)+1}^{e+1} = 2 \ln \left(\frac{e+1}{e^{-1}+1} \right) = 2 \ln(e) = 2.$$

7.1.41 Let $u = e^x - e^{-x}$ so that $du = e^x + e^{-x} dx$. Then we have

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int \frac{1}{u} du = \ln |u| + C = \ln |e^x - e^{-x}| + C.$$

7.1.42 Note that the denominator of the integrand is $(e^x - e^{-x})^2$. Let $u = e^x - e^{-x}$ so that $du = (e^x + e^{-x}) dx$. Substituting gives

$$\int_{3/2}^{8/3} u^{-2} du = (-1/u) \Big|_{3/2}^{8/3} = -3/8 + 2/3 = \frac{-9+16}{24} = \frac{7}{24}.$$

7.1.43
$$\int_{-1}^1 10^x dx = \left(\frac{10^x}{\ln 10} \right) \Big|_{-1}^1 = \frac{10 - 10^{-1}}{\ln 10} = \frac{99}{10 \ln 10}.$$

7.1.44 Let $u = \sin x$ so that $du = \cos x dx$. Then

$$\int_0^{\pi/2} 4^{\sin x} \cos x dx = \int_0^1 4^u du = \left(\frac{4^u}{\ln 4} \right) \Big|_0^1 = \frac{3}{\ln 4}.$$

7.1.45
$$\int_1^2 (1 + \ln x)x^x dx = (x^x) \Big|_1^2 = 4 - 1 = 3.$$

7.1.46 Let $u = 1/p$, so that $du = -\frac{1}{p^2} dp$. Then

$$\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2} dp = \int_2^3 10^u du = \left(\frac{10^u}{\ln 10} \right) \Big|_2^3 = \frac{900}{\ln 10}.$$

7.1.47 Let $u = x^3 + 8$ so that $du = 3x^2 dx$. Substituting gives

$$\frac{1}{3} \int 6^u du = \frac{6^u}{3 \ln 6} + C = \frac{6^{x^3+8}}{3 \ln 6} + C.$$

7.1.48 Let $u = \cot x$ so that $du = -\csc^2 x dx = \frac{-1}{\sin^2 x} dx$. Substituting yields

$$-\int 4^u du = \frac{-4^u}{\ln 4} + C = \frac{-4^{\cot x}}{\ln 4} + C.$$

7.1.49 Let $u = 3x^2 + 1$ so that $du = 6x dx$. Substituting yields

$$\frac{1}{6} \int e^u du = \frac{1}{6} e^u + C = \frac{1}{6} e^{3x^2+1} + C.$$

7.1.50
$$\int 7^{2x} dx = \int e^{2x \ln 7} dx = e^{2x \ln 7} \cdot \frac{1}{2 \ln 7} + C = \frac{7^{2x}}{2 \ln 7} + C.$$

7.1.51
$$\int 3^{-2x} dx = \int e^{-2x \ln 3} dx = e^{-2x \ln 3} \cdot \frac{1}{-2 \ln 3} + C = -\frac{3^{-2x}}{2 \ln 3} + C = \frac{-1}{9x \ln 9} + C.$$

7.1.52
$$\int_0^5 5^{5x} dx = \left(\frac{5^{5x}}{5 \ln 5} \right) \Big|_0^5 = \frac{5^{25} - 1}{5 \ln 5}.$$

7.1.53 Let $u = x^3$, so that $du = 3x^2 dx$. Then we have

$$\int \frac{1}{3} 10^u du = \frac{10^u}{3 \ln 10} + C = \frac{10^{x^3}}{3 \ln 10} + C.$$

7.1.54 Let $u = \sin x$ so that $du = \cos x dx$. Then we have

$$\int_0^0 2^u du = \left(\frac{2^u}{\ln 2} \right) \Big|_0^0 = 0.$$

7.1.55 Let $u = \ln x$ so that $du = \frac{1}{x} dx$. Then we have

$$\int_0^{\ln(2)+1} 3^u du = \left(\frac{3^u}{\ln 3} \right) \Big|_0^{\ln(2)+1} = \frac{3 \cdot 3^{\ln 2} - 1}{\ln 3}.$$

7.1.56 Let $u = \ln x$ so that $du = \frac{1}{x} dx$. Then we have

$$\int \frac{1}{4} \sin u du = -\frac{1}{4} \cos u + C = -\frac{1}{4} \cos(\ln x) + C.$$

7.1.57 Let $u = \ln x$ so that $du = \frac{1}{x} dx$. Then we have

$$\int_0^2 u^5 du = \left(\frac{u^6}{6} \right) \Big|_0^2 = \frac{32}{3}.$$

7.1.58 Let $u = \ln x$ so that $du = \frac{1}{x} dx$. Then we have

$$\int (u^2 + 2u - 1) du = u^3/3 + u^2 - u + C = (\ln x)^3/3 + (\ln x)^2 - \ln x + C.$$

7.1.59 Let $u = e^{3x} + e^{-3x}$ so that $du = 3(e^{3x} - e^{-3x}) dx$. Substituting gives

$$\frac{1}{3} \int_2^{65/8} \frac{1}{u} du = \frac{1}{3} (\ln u) \Big|_2^{65/8} = \frac{1}{3} (\ln(65/8) - \ln 2) = \frac{1}{3} \ln(65/16).$$

7.1.60 Let $u = e^{2x} + 1$. Then $du = 2e^{2x} dx$. Substituting yields

$$\frac{1}{2} \int u^{-2} du = \frac{1}{2} \left(-\frac{1}{u} \right) = -\frac{1}{2(e^{2x} + 1)}.$$

7.1.61 Let $u = 5 + \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}$. Substituting yields

$$\int 2e^u du = 2e^u + C = 2e^{5+\sqrt{x}} + C.$$

7.1.62 Note that $4^{2x} = (4^2)^x = 16^x$. So we are computing $\int_0^1 1 dx = 1$.

7.1.63

h	$(1+2h)^{1/h}$	h	$(1+2h)^{1/h}$
10^{-1}	6.1917	-10^{-1}	9.3132
10^{-2}	7.2446	-10^{-2}	7.5404
10^{-3}	7.3743	-10^{-3}	7.4039
10^{-4}	7.3876	-10^{-4}	7.3905
10^{-5}	7.3889	-10^{-5}	7.3892
10^{-6}	7.3890	-10^{-6}	7.3891

Let $y = (1+2h)^{1/h}$. Then $\ln y = \frac{\ln(1+2h)}{h}$.
 $\lim_{h \rightarrow 0} \ln y = \lim_{h \rightarrow 0} \frac{\ln(1+2h)}{h} = \lim_{h \rightarrow 0} \frac{2}{1+2h} = 2$, so
the limit of y as $h \rightarrow 0$ is e^2 .

7.1.64

h	$(1+3h)^{2/h}$	h	$(1+3h)^{2/h}$
10^{-1}	190.0496	-10^{-1}	1253.25
10^{-2}	369.3558	-10^{-2}	442.2350
10^{-3}	399.8214	-10^{-3}	407.0834
10^{-4}	403.0659	-10^{-4}	403.7921
10^{-5}	403.3925	-10^{-5}	403.4651
10^{-6}	403.4252	-10^{-6}	403.4324
10^{-7}	403.4284	-10^{-7}	403.4288
10^{-8}	403.4288	-10^{-8}	403.4388
10^{-9}	403.4290	-10^{-9}	403.4290

Let $y = (1+3h)^{2/h}$. Then $\ln y = \frac{2 \ln(1+3h)}{h}$.
 $\lim_{h \rightarrow 0} \ln y = \lim_{h \rightarrow 0} \frac{2 \ln(1+3h)}{h} = \lim_{h \rightarrow 0} \frac{6}{1+3h} = 6$, so
the limit of y as $h \rightarrow 0$ is e^6 .

7.1.65

x	$\frac{2^x - 1}{x}$	x	$\frac{2^x - 1}{x}$
10^{-1}	.71773	-10^{-1}	.66967
10^{-2}	.69556	-10^{-2}	.69075
10^{-3}	.69339	-10^{-3}	.69291
10^{-4}	.69317	-10^{-4}	.69312
10^{-5}	.69315	-10^{-5}	.69314
10^{-6}	.69315	-10^{-6}	.69315

$$\lim_{x \rightarrow 0} \frac{2^x - 1}{x} = \lim_{x \rightarrow 0} \frac{2^x \ln 2}{1} = \ln 2.$$

7.1.66

x	$\frac{\ln(1+x)}{x}$	x	$\frac{\ln(1+x)}{x}$
10^{-1}	.9531	-10^{-1}	1.0536
10^{-2}	.9950	-10^{-2}	1.0050
10^{-3}	.9995	-10^{-3}	1.0005
10^{-4}	1.0000	-10^{-4}	1.0001
10^{-5}	1.0000	-10^{-5}	1.0000
10^{-6}	1.0000	-10^{-6}	1.0000

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1/(1+x)}{1} = 1.$$

7.1.67

- True. This follows because $e^{\ln xy} = xy = e^{\ln x} e^{\ln y} = e^{\ln x + \ln y}$, and because the exponential function is one-to-one.
- False. Zero is not in the domain of the natural logarithm function.
- False. For example, $\ln(1+1) = \ln 2 \neq \ln(1) + \ln(1) = 0$.
- False. $e^{2 \ln x} = e^{\ln x^2} = x^2 \neq 2^x$.
- False. $\int_0^e \frac{1}{x} dx = \lim_{b \rightarrow 0^+} \int_b^e \frac{1}{x} dx = \lim_{b \rightarrow 0^+} (\ln x) \Big|_b^e = 1 - \lim_{b \rightarrow 0^+} \ln b$, which does not exist.

7.1.68 $\ln(x/y) = \int_1^{x/y} \frac{1}{t} dt = \int_1^x \frac{1}{t} dt + \int_x^{x/y} \frac{1}{t} dt$. For the second integral, let $u = x/t$ so that $du = -\frac{1}{t^2} dx$.

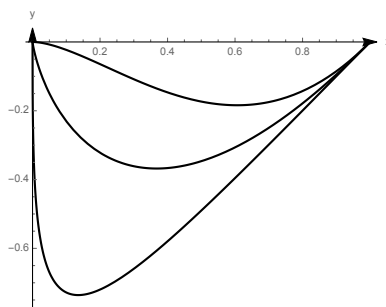
Then we have $\ln(x) + \int_1^{1/y} \frac{1}{u} du = \ln(x) + \ln(1/y)$. All that remains is to prove that $\ln(1/y) = -\ln(y)$.

In the text it was proved that $\ln xy = \ln x + \ln y$, so if we apply this rule to $0 = \ln(1) = \ln(y(1/y)) = \ln(y) + \ln(1/y)$, we see that $\ln(1/y) = -\ln(y)$.

7.1.69 The average value is given by $\frac{1}{p-1} \int_1^p \frac{1}{x} dx = \frac{1}{p-1} (\ln x) \Big|_1^p = \frac{\ln p}{p-1}$. Note that

$$\lim_{p \rightarrow \infty} \frac{\ln p}{p-1} = \lim_{p \rightarrow \infty} \frac{1/p}{1} = 0.$$

7.1.70 Each has limit 0 as $x \rightarrow 0^+$ (by L'Hôpital's rule), and each has value 0 at $x = 1$. Each is negative on $(0, 1)$. Note that $\frac{d}{dx} x^p \ln x = x^{p-1} + px^{p-1} \ln x = x^{p-1}(1 + p \ln x)$, which is 0 on $(0, 1)$ for $x = e^{-1/p}$. This critical point leads to a minimum in each case, but the location of the minimums increases as p increases, and the value of the minimum increases as well.



7.1.71

a. No. Let

$$h(a) = \int_{1-a}^{1+a} \frac{1-x}{x} dx = \int_{1-a}^1 \frac{1-x}{x} dx + \int_1^{1+a} \frac{1-x}{x} dx = -\int_1^{1-a} \frac{1-x}{x} dx + \int_1^{1+a} \frac{1-x}{x} dx.$$

Then

$$h'(a) = \frac{1-(1-a)}{1-a} + \frac{1-(1+a)}{1+a} = \frac{a}{1-a} - \frac{a}{1+a} = \frac{a+a^2+(-a)+a^2}{1-a^2} = \frac{2a^2}{1-a^2} > 0.$$

Also, $h(a) = 0$. Because h is increasing on $(0, 1)$, there are no other numbers on that interval where h is zero.

b. No. Let

$$g(a) = \int_{1/a}^a f(x) dx = \int_{1/a}^{1/2} f(x) dx + \int_{1/2}^a f(x) dx = -\int_{1/2}^{1/a} f(x) dx + \int_{1/2}^a f(x) dx.$$

Then

$$g'(a) = -f(1/a) \cdot -\frac{1}{a^2} + f(a) = \frac{1-(1/a)}{a} + \frac{1-a}{a} = \frac{2-(a+1/a)}{a} < 0,$$

because $a + 1/a > 2$ for $a > 1$. Because g has value 0 at $a = 1$ and is decreasing on $(1, \infty)$ it is never 0 on that interval.

7.1.72 For $x > 0$, $\frac{d}{dx} \ln x = \frac{1}{x}$. For $x < 0$, $\frac{d}{dx} \ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$. So $\frac{d}{dx} \ln|x| = \frac{1}{x}$.

7.1.73

- a. Let $\exp 0 = z$. Then $\ln \exp 0 = \ln z$, so $\ln z = 0$. Then because $\ln z = \ln 1$, we have $z = 1$, so $\exp 0 = 1$.
- b. $\ln \left(\frac{\exp x}{\exp y} \right) = \ln(\exp x) - \ln(\exp y) = x - y$. Thus, $\exp(x - y) = \frac{\exp x}{\exp y}$.
- c. $\ln((\exp x)^p) = p \ln(\exp x) = px$, so $\exp px = (\exp x)^p$.

7.1.74

- a. The region under $1/x$ and above the x -axis between $x = 1$ and $x = 2$ has area $\int_1^2 1/x \, dx = (\ln x) \Big|_1^2 = \ln 2$.
- b. Because the region bounded by the x -axis and the lines $y = 1/2$ and $x = 1$ and $x = 2$ has area $1/2$, and because this region is encompassed by the region under $y = 1/x$ and above the x -axis between $x = 1$ and $x = 2$, we must have $\ln 2 > 1/2$.
- c. Let $n > 0$. Because of the result of part b), we must have $n \ln 2 > n/2$, so $\ln 2^n > n/2$. Also, $-n \ln 2 < -n/2$, so $\ln(2^{-n}) < -n/2$.
- d. Because $n/2 \rightarrow \infty$ as $n \rightarrow \infty$, and because $2^n > 2/n$, we must have $\ln(2^n) \rightarrow \infty$ as $n \rightarrow \infty$. So $\ln x \rightarrow \infty$ as $x \rightarrow \infty$.
- Also because $\ln(2^{-n}) \rightarrow -\infty$ as $n \rightarrow \infty$ (because $\ln(2^{-n}) < -n/2$), we must have that $\ln x \rightarrow -\infty$ as $x \rightarrow 0^+$.

7.1.75 Because $1/x$ is decreasing, we know that the left Riemann sum is an overestimate. Using $n = 2$ subintervals, we have $\int_1^2 1/x \, dx = \ln 2 < \frac{1}{1} \cdot \frac{1}{2} + \frac{1}{3/2} \cdot \frac{1}{2} = \frac{5}{6} < 1$.

Because $1/x$ is decreasing, we know that the right Riemann sum is an underestimate. We will use $n = 8$ subintervals. We have $\int_1^3 1/x \, dx = \ln 3 > 2 \left(\frac{1}{9} + \frac{1}{11} + \cdots + \frac{1}{21} \right) > 1$.

7.1.76 With y fixed, we have $\frac{d}{dx} \ln(xy) = \frac{1}{xy} \cdot y = \frac{1}{x} = \frac{d}{dx} \ln(x)$.

Thus, $\ln(xy) = \ln(x) + C$. If we let $x = 1$, we see that $\ln(y) = 0 + C$, so $C = \ln(y)$, and $\ln(xy) = \ln(x) + \ln(y)$.

7.1.77 Because $1/x$ is decreasing, the left Riemann sum is an overestimate to the integral. Over the interval $[1, n+1]$ with n subdivisions, we must have $\int_1^{n+1} \frac{1}{x} \, dx = \ln(n+1) < \frac{n+1-1}{n} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) = \sum_{i=1}^n \frac{1}{i}$. Because $\ln(n+1) \rightarrow \infty$ as $n \rightarrow \infty$, it must follow that the harmonic partial sum $\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) \rightarrow \infty$ as well.

7.1.78 Note that the area of the triangle described is $\frac{1}{2}xy$, so if $xy \geq \frac{1}{2}$, then $\frac{1}{2}xy \geq \frac{1}{4}$, which is not what we are seeking. However, if $xy < 1/2$, then the triangle formed will give us what we want.

Note that for any choice of x with $0 < x < 1/2$ and any choice of possible y , we have $xy < 1/2$ because $0 < y < 1$. Thus because there is a probability of $\frac{1}{2}$ of choosing $0 < x < 1/2$, the probability we seek is at least $1/2$. In addition, for $1/2 < x < 1$, if $y < \frac{1}{2x}$, then $xy < \frac{1}{2}$. So we should add $\int_{1/2}^1 \frac{1}{2x} \, dx = \left(\frac{1}{2} \ln x \right) \Big|_{1/2}^1 = (\ln 2)/2$. Thus the probability we seek is $\frac{1}{2}(1 + \ln 2)$.

7.2 Exponential Models

7.2.1 Exponential growth occurs for a constant relative growth rate.

7.2.2 The initial value, and the relative growth rate.

7.2.3 It is the time it takes the population to double in size.

7.2.4 It is the time it takes for the population to become half its size.

7.2.5 For exponential growth modeled by $y = y_0 e^{kt}$, the doubling time T_2 is $T_2 = \frac{\ln 2}{k}$, where $k > 0$ is the growth constant.

7.2.6 For exponential decay modeled by $y = y_0 e^{kt}$, the half-life $T_{1/2}$ is $T_{1/2} = \frac{\ln(1/2)}{k}$, where $k < 0$ is the decay constant.

7.2.7 The doubling time is $\frac{\ln 2}{k}$, so we solve $\frac{\ln 2}{k} = 20$, or $k = \frac{\ln 2}{20} \approx 0.03466$.

7.2.8 $\frac{\ln(1/2)}{k} = \frac{\ln(1/2)}{-0.05} \approx 13.9$ years.

7.2.9 Compound interest and world population growth.

7.2.10 Radioactive decay and depreciation of durable goods.

7.2.11 We have $y(t) = y_0 e^{kt}$ and $y(1) = y_0 e^k = 1.11y_0$. Therefore $e^k = 1.11$, and $k = \ln 1.11 \approx 0.10436$.

7.2.12 We have $y(t) = y_0 e^{kt}$ and $y(1) = y_0 e^k = 0.955y_0$. Therefore $e^k = 0.955$, and $k = \ln 0.955 \approx -0.046$.

7.2.13 For $f(t)$, $\frac{df}{dt} = 10.5$, so the absolute growth rate is constant. For $g(t)$, $\frac{dg}{dt} = 100e^{t/10} \cdot \frac{1}{10} = 10e^{t/10}$, so the growth rate is not constant but the relative growth rate $\frac{1}{g(t)}g'(t) = \frac{1}{10}$ is constant.

7.2.14 For $f(t)$, $\frac{df}{dt} = 400$, so the absolute growth rate is constant. For $g(t)$, $\frac{dg}{dt} = 400 \cdot 2^{t/20} \cdot \frac{\ln 2}{20}$, so the growth rate is not constant but the relative growth rate $\frac{1}{g(t)}g'(t) = \frac{\ln 2}{20}$ is constant.

7.2.15

- The growth is modeled by $p(t) = 90,000e^{kt}$, with $t = 0$ corresponding to 2016, and time measured in years. Because $(1.024)(90,000) = 90,000e^k$, we have $k = \ln(1.024) \approx 0.02373$.
- $120,000 = 90,000e^{t \ln 1.024}$, so $\frac{4}{3} = e^{t \ln 1.024}$, and $\ln \frac{4}{3} = t \ln 1.024$, so $t = \frac{\ln(4/3)}{\ln 1.024} \approx 12.1$ years. The population will reach 120,000 in 2028.

7.2.16

- The growth is modeled by $y(t) = 2.115e^{kt}$ where $t = 0$ corresponds to 2015 and time is measured in years. Because the growth rate is 1.5%, $k = \ln(1.015) \approx 0.01489$.
- In 2025 we would have $t = 10$. We have $y(10) = 2.115e^{10 \ln 1.015} \approx 2.455$ million people.

7.2.17

- The growth is modeled by $y(t) = 50000e^{kt}$. When $t = 10$, we have $p(10) = 50000e^{10k} = 55000$, so $k = \frac{\ln 1.1}{10} \approx 0.00953$.

b. In 20 years, the population should be about $50000e^{2\ln 1.1} = 50000(1.1)^2 = 60,500$.

7.2.18

a. Our model is given by $y(t) = 1500e^{kt}$, and $y(1) = 1500 \cdot 1.031 = 1500e^k$, so $k = \ln(1.031) \approx 0.03053$.

b. We seek t so that $2500 = 1500e^{\ln(1.031)t}$. Thus $\ln(5/3) = \ln(1.031)t$, so $t = \frac{\ln(5/3)}{\ln(1.031)} \approx 16.7$ years.

7.2.19

a. The price of a cart of groceries is modeled by $y(t) = 100e^{kt}$ where $t = 0$ corresponds to 2010, and t is measured in years and $k = \ln(1.016) \approx 0.01587$.

b. The price of the groceries in 2025 when $t = 15$ should be about $y(15) = 100e^{15\ln(1.016)} \approx \126.88 .

7.2.20

a. Because the doubling time is 6 weeks, $k = \frac{\ln 2}{6}$. The number of cells is modeled by $y(t) = 8e^{(\ln 2/6)t}$ where t is measured in weeks.

b. The population will reach 1500 cells when $\ln(1500/8) = \frac{\ln 2}{6}t$, so when $t = \frac{6\ln(1500/8)}{\ln 2} \approx 45.3$ weeks.

7.2.21 $1200 = 1000e^{5k}$, so $k = \frac{\ln 1.2}{5}$, and therefore $y(t) = 1000e^{t(\ln 1.2)/5}$. In one year, the account grows by a factor of $e^{(\ln 1.2)/5} \approx 1.0371$, so the APY is about 3.71%.

7.2.22 Because $73 = 64e^{33k}$, we have $\frac{73}{64} = e^{33k}$, so $33k = \ln\left(\frac{73}{64}\right)$, and $k = \frac{\ln(73/64)}{33} \approx 0.004$. We are looking for t so that $64e^{0.004t} = 100$. We have $e^{0.004t} = \frac{100}{64}$, so $t = \frac{\ln(100/64)}{0.004} \approx 111.572$ days, so about 112 days.

7.2.23

a. The population is modeled by $P(t) = 334.4e^{t\ln(1.0079)}$. The doubling time is $\frac{\ln 2}{\ln 1.0079} \approx 88.1$ years. The population in 2050 will be about $P(30) = 334.4e^{30\ln(1.0079)} \approx 423.44$ million.

b. If the growth rate is 0.7%, the doubling time is $\frac{\ln 2}{\ln 1.007} \approx 99.4$ years and the population in 2050 will be about $334.4e^{30\ln(1.007)} \approx 412.2$ million.

7.2.24

a. $P(t)$ has the form $P(t) = 2000e^{kt}$, and because $P(1) = 1.013 \cdot 2000$, we have $k = \ln(1.013)$. Thus $P(t) = 2000e^{t\ln(1.013)}$.

b. This is given by $\int_0^4 P(t) dt = \frac{2000}{\ln(1.013)} e^{t\ln(1.013)} \Big|_0^4 \approx 8210.3$.

c. This is given by $\int_0^t P(s) ds = \frac{2000}{\ln(1.013)} e^{s\ln(1.013)} \Big|_0^t = -154844 + 154844e^{t\ln(1.013)}$.

7.2.25 The growth is modeled by $p(t) = 25.1e^{kt}$. When $t = 6$, we have $p(6) = 25.1e^{6k} = 26.47$, so $k = \frac{\ln(26.47/25.1)}{6} \approx 0.00886$. Thus $p(15) \approx 25.1e^{0.00886(15)} \approx 28.67$ million people.

7.2.26

- a. Note that $k = \ln(1.015)$. The amount consumed over the year would be

$$\int_0^1 1.2e^{\ln(1.015)t} dt = \left(\frac{1.2}{\ln(1.015)} e^{\ln(1.015)t} \right) \Big|_0^1 \approx 1.20898$$

million barrels of oil.

b. $\int_0^t 1.2e^{\ln(1.015)x} dx = \left(\frac{1.2}{\ln(1.015)} e^{\ln(1.015)x} \right) \Big|_0^t = \frac{1.2}{\ln(1.015)} (e^{\ln(1.015)t} - 1)$ million barrels.

c. This will occur when $\frac{1.2}{\ln(1.015)} (e^{\ln(1.015)t} - 1) = 10$, or $e^{\ln(1.015)t} = \frac{10 \ln(1.015)}{1.2} + 1$, or

$$t = \ln \left(\frac{10 \ln(1.015)}{1.2} + 1 \right) / (\ln(1.015)) \approx 7.85551 \text{ years.}$$

7.2.27 The homicide rate is modeled by $H(t) = 800e^{kt}$, where $t = 0$ corresponds to 2018, and time is measured in years, and $k = \ln 0.97$. The rate will reach 600 when $\frac{600}{800} = e^{(\ln 0.97)t}$, or $t = \frac{\ln(6/8)}{\ln 0.97} \approx 9.44$ years. So it should achieve this rate in 2027.

7.2.28 The amount of drug in the body is modeled by $y(t) = y_0 e^{kt}$, where y_0 is the initial dose and time is measured in hours and $k = \ln(0.85)$. The amount in the body should be 10 percent of the initial dose when $0.1 = e^{\ln(0.85)t}$, or $t = \frac{\ln(0.1)}{\ln(0.85)} \approx 14.1681$ hours.

7.2.29 The amount of Valium in the bloodstream is modeled by $a(t) = 20e^{kt}$. If the half-life is 36 hours, then $36 = \frac{\ln 2}{k}$, so $k = \frac{\ln 1/2}{36}$. So $a(12) = 20e^{-12 \ln(2)/36} = 20e^{-\ln(2)/3} \approx 15.87$ mg.

The concentration of Valium will reach 2 mg when $0.1 = e^{-\ln(2)t/36}$, which is when $t = \frac{-36 \ln(0.1)}{\ln 2} \approx 119.59$ hours.

7.2.30 The population is modeled by $p(t) = 1.2e^{\ln(.995)t}$. After 50 years, the population should be about $p(50) = 1.2e^{\ln(.995) \cdot 50} \approx 0.933975$ billion people (or about 934 million people). This rate is not sufficient to meet the goal of 700 million people.

7.2.31 The population is modeled by $p(t) = 1.853e^{kt}$. Because $p(6) = 1.853e^{6k} = 1.831$, we have $e^{6k} = \frac{1.831}{1.853}$, so $k = \frac{\ln(1.831/1.853)}{6}$. The population in 2025 is predicated to be $1.853e^{\frac{1}{6} \ln(1.831/1.853) 15} \approx 1.798$ million. This may not be an appropriate long-term model because the downturn in the population of West Virginia may be temporary.

7.2.32

- a. The value after t years is given by $V(t) = 2.5e^{kt}$. Because $V(1) = 2.5 \cdot 0.932$, we have $k = \ln(0.932)$. Thus, after 10 years, we have $V(10) = 2.5e^{10 \ln(0.932)} \approx 1.2$ million dollars.

- b. The value is 10 percent of the original when $0.10 = e^{t \ln(0.932)}$, which occurs when $t = \frac{\ln(0.10)}{\ln(0.932)} \approx 32.7$ years.

7.2.33 Let $y = 1000e^{kx}$ model the pressure at x feet above sea level. We know that $\frac{1}{3} = e^{30000k}$, so $k = \frac{\ln(1/3)}{30000}$. We want to know for what x does $\frac{1}{2} = e^{kx}$, so we are seeking $x = \frac{\ln 2}{k} = \frac{30000 \ln(2)}{\ln 3} \approx 18,928$ feet above sea level.

The pressure is 1/100th of the sea-level pressure when $x = \frac{30000 \ln(100)}{\ln 3} \approx 125,754$ feet.

7.2.34

- a. The decay is modeled by $a(t) = a_0 e^{kt}$ with $k = \frac{\ln(0.5)}{5730}$. We seek t so that $0.77a_0 = a_0 e^{kt}$, so $t = \frac{\ln(0.77)}{k} = \frac{\ln(0.77)(5730)}{\ln(0.5)} \approx 2160.6$. So the cloth was painted about 2160.6 year ago.
- b. In a similar manner to part (a) we have $t = \frac{\ln(0.062)(5730)}{\ln(0.5)} \approx 22,986.4$, so the wood was cut about 23,000 year ago.

7.2.35 The amount of U-238 in the rock is modeled by $a(t) = a_0 e^{kt}$ with $k = \frac{\ln(0.5)}{4.5}$, where time is measured in billions of years. We seek t so that $a_0 = a_0 e^{kt}$, so $t = \frac{\ln(0.85)}{k} = \frac{\ln(0.85)(4.5)}{\ln(0.5)} \approx 1.055$. So the cloth was painted about 1.055 billion years ago.

7.2.36

- a. After t days there would be $y = 100e^{(-t \ln 2)/8}$ millicuries after t days.
- b. We seek t so that $10 = 100e^{(-t \ln 2)/8}$, so $t = \frac{-8 \ln(0.1)}{\ln 2} \approx 26.58$ days.
- c. We seek t so that $10 = 105e^{(-t \ln 2)/8}$, so $t = \frac{-8 \ln(2/21)}{\ln 2} \approx 27.1385$ days.

7.2.37 The amount of caffeine is modeled by $y(t) = e^{kt}$, and $0.8 = e^{2k}$, so $k = \frac{\ln 0.8}{2}$. The half-life is $\frac{\ln(0.5)}{k} = \frac{2 \ln(0.5)}{\ln(0.8)} \approx 6.2$ hours.

7.2.38

- a. Because the half-life is $\frac{\ln(0.5)}{k} = 5.7$, we have $k = \frac{\ln(0.5)}{5.7}$. The amount of caffeine in the bloodstream after 1 hour is $90e^{\ln(0.5)/5.7} \approx 79.7$ mg.
- b. Right before the individual drinks the second cup of coffee (which is two hours after drinking the first cup) the amount of caffeine in the bloodstream is $90e^{2 \ln(0.5)/5.7}$. As soon as he chugs the second cup, he has this amount plus 90 in his bloodstream, so $90e^{2 \ln(0.5)/5.7} + 90$. Therefore the amount of caffeine t hours after the second cup of coffee is $y(t) = (90e^{2 \ln(0.5)/5.7} + 90)e^{t \ln(0.5)/5.7}$, and $y(1)$ is

$$y(1) = (90e^{2 \ln(0.5)/5.7} + 90)e^{\ln(0.5)/5.7} \approx 142.2 \text{ mg.}$$

7.2.39 Let $y(t)$ equal the cost of the light t years after 2018. Then $y(t) = 4e^{kt}$. Ten years from now, it is predicted that the light will cost 1/10 of its current cost, which means that $y(10) = 4/10 = 0.4$. Replace y with 0.4 and t with 10 in the equation $y(t) = 4e^{kt}$, we have $0.4 = 4e^{10k}$, or $e^{10k} = 0.1$. Solving for k we find that $k = (\ln 0.1)/10 \approx -0.2303$. This means that $y(t) = 4e^{-0.2302t}$ and the cost of the light in 2021 is predicted to be $y(3) = 4e^{-0.2302(3)} \approx \2.01 .

7.2.40 Let y_0 equal the current light output of an LED package. In the equation $y(t) = y_0 e^{kt}$, if we replace $y(t)$ with $20y_0$ and t with 10, we have $20y_0 = y_0 e^{10k}$, or $20 = e^{10k}$. Solving for k , we find that $k = \frac{\ln 20}{10} \approx 0.2996$. So three years from now, $y(3) \approx 2.46y_0$. So three years from now, an LED package will produce almost two and a half times as much light output as the current LED package.

7.2.41 The initial volume is $V_0 = \frac{4\pi}{3} \left(\frac{5}{10000} \right)^3 = \frac{\pi}{6 \times 10^9}$ cubic cm. $k = \frac{\ln 2}{35}$. Suppose $0.5 = V_0 e^{kt} = \frac{\pi}{6 \times 10^9} e^{(t \ln 2)/35}$. Then $\frac{t \ln 2}{35} = \ln \left(\frac{3 \times 10^9}{\pi} \right)$, and $t = \frac{35}{\ln 2} \ln \left(\frac{3 \times 10^9}{\pi} \right) \approx 1044$ days.

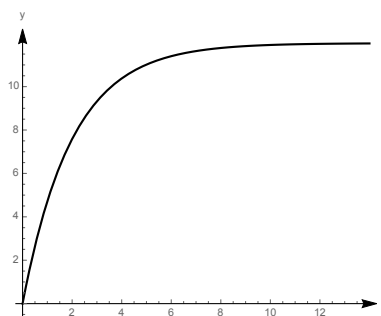
7.2.42 The time required to increase p -fold occurs when $py_0 = y_0e^{kt}$, or when $t = \frac{\ln p}{k}$. Thus, the tripling time is $\frac{\ln 3}{k}$.

7.2.43

- False. If that was the correct formula, then $y(1) = y_0e^{.06} = (1.06184)y_0 \neq (1.06)y_0$.
- False. If it increases by ten percent per year, then after 3 years it increases by a factor of $(1.1)^3 = 1.331$ which corresponds to 33.1 percent.
- True. The relative decay rate is constant, so the decay is exponential.
- True. This follows because the doubling time is related to k by the equation $T_2 = \frac{\ln 2}{k}$.
- True. This time would be the constant $\frac{\ln 10}{k}$.

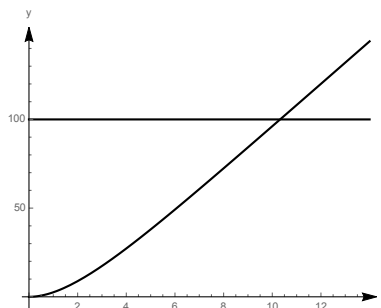
7.2.44

- The runner's velocity approaches but does not attain 12 meters per second.

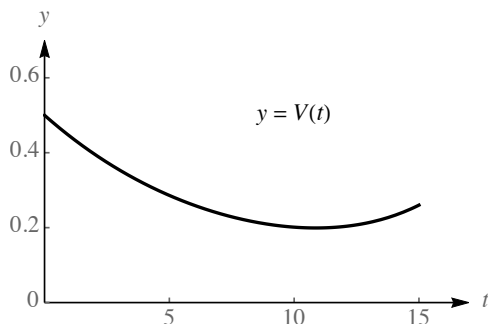


- The position function is $s(t) = \int_0^t 12(1 - e^{-x/2}) dx = \left(12(x + 2(e^{-x/2})) \right) \Big|_0^t = 12t + 24e^{-t/2} - 24$.

- $s = 100$ when $12t + 24e^{-t/2} - 24 = 100$, which occurs when $t \approx 10.325$ seconds.

**7.2.45**

- $V_1(t) = 0.99(0.5)e^{kt}$, where $k = \frac{\ln(0.5)}{5.7} \approx -0.1216$. So $V_1(t) = 0.495e^{-0.1216t}$.
- $V_2(t) = 0.01(0.5)e^{kt}$, where $k = \frac{\ln 2}{2.9} \approx 0.239$. So $V_2(t) = 0.005e^{0.239t}$.
- $V(t) = V_1(t) + V_2(t) = 0.495e^{-0.1216t} + 0.005e^{0.239t}$.
- The tumor initially shrinks significantly in size but eventually starts growing again.



- d. Solving the equation $V'(t) = 0$ for t , we find that $t \approx 10.9$ days. So it's best to treat the mouse again just before the end of the 10th day after the first treatment.

7.2.46

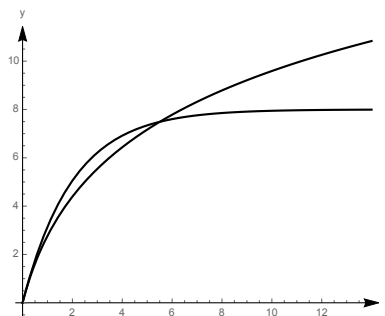
- a. We are seeking t so that $500,000e^{\ln(1.03)t} = 300,000e^{\ln(1.05)t}$. This occurs when $\frac{5}{3} = e^{(\ln(1.05) - \ln(1.03))t}$, so $\ln(5/3) = (\ln(1.05) - \ln(1.03))t$, so $t = \frac{\ln(5/3)}{(\ln(1.05) - \ln(1.03))} \approx 26.5621$ years.
- b. We are seeking y_0 and p so that $500,000e^{\ln(1.03)10} = y_0e^{\ln(1+p)10}$. This occurs when

$$y_0 = 500,000e^{(10(\ln(1.03) - \ln(1+p)))} = 500,000 \left(\frac{1.03}{1+p} \right)^{10}.$$

7.2.47

- a. After 5 hours, Abe has run $\int_0^5 \frac{4}{t+1} dt = (4 \ln(t+1))|_0^5 = 4 \ln 6 \approx 7.17$ miles. After 5 hours, Bob has run $\int_0^5 4e^{-t/2} dt = (8e^{-t/2})|_5^0 = 8(1 - e^{-5/2}) \approx 7.34$. So after 5 hours, Bob is ahead.
- After 10 hours, Abe has run $\int_0^{10} \frac{4}{t+1} dt = (4 \ln(t+1))|_0^{10} = 4 \ln 11 \approx 9.59$ miles. After 10 hours, Bob has run $\int_0^{10} 4e^{-t/2} dt = (8e^{-t/2})|_{10}^0 = 8(1 - e^{-5}) \approx 7.95$. So after 10 hours, Abe is ahead.

- b. Bob's distance function is given by $8(1 - e^{-t/2})$ and is bounded above by 8. Abe's distance function is given by $4 \ln(t+1)$ and is unbounded.



7.2.48 $T_2 = \frac{\ln 2}{k} = \frac{\ln 2}{\ln(1 + .0p)} \approx \frac{0.693}{\ln(1 + .0p)} \approx \frac{70}{p}$. The approximation is best when $\ln(1 + .0p) \approx .0p$, which occurs when p is small.

7.2.49 $(1 + 0.008)^{12} - 1 \approx 10.034\%$, which is more than $12 \cdot 0.008 \approx 9.6\%$.

7.2.50

- a. If $\frac{dv}{dt} = -kv$, then $\int \frac{1}{v} \frac{dv}{dt} dt = \int -k dt$, so $\ln(v(t)) = -kt + C$, so $v(t) = v_0 e^{-kt} = 10e^{-kt}$.
- b. $s(t) = \int 10e^{-kt} dt = \frac{10}{-k} e^{-kt} + C = \frac{10}{-k} e^{-kt} + \frac{10}{k}$. (Because $s(0) = 0$.)
- c. $dv/dt = \frac{dv}{ds} \frac{ds}{dt}$, so $-kv = \frac{dv}{ds} \cdot v$, so $\frac{dv}{ds} = -k$. Thus $v = -ks + C$ for some constant C , and because $v(0) = 10$, we have $v = 10 - ks$.

7.2.51 As in the previous problem, $s(t) = \frac{v_0}{k}(1 - e^{-kt})$. Consider the equations $s(t_2) - s(t_1) = 1200 = \frac{v_0}{k}(e^{-kt_1} - e^{-kt_2})$ and $v(t_2) - v(t_1) = -100 = v_0(e^{-kt_2} - e^{-kt_1})$. Dividing these two equations yields $k = \frac{1}{12}$.

Now $v(t_1) = 1000 = v_0 e^{-t_1/12}$ and $v(t_2) = 900 = v_0 e^{-t_2/12}$, so $\frac{1000}{900} = e^{(-1/12)(t_1 - t_2)}$, so $t_1 - t_2 = -12 \ln(10/9) \approx -1.2643$ seconds. The deceleration takes about 1.2643 seconds to occur.

7.2.52 $R_T = \frac{y(t+T) - y(t)}{y(t)} = \frac{y_0 e^{k(t+T)} - y_0 e^{kt}}{y_0 e^{kt}} = \frac{y_0 e^{kt}(e^{kT} - 1)}{y_0 e^{kt}} = e^{kT} - 1$. Hence, R_T is constant for all t .

7.2.53 If $y_0 e^{kt} = y_0(1+r)^t$, then $(e^k)^t = (1+r)^t$, so $e^k = 1+r$, and $k = \ln(1+r)$. If $y_0 e^{kt} = y_0 2^{t/T_2}$, then $(e^k)^t = (2^{1/T_2})^t$, so $e^k = 2^{1/T_2}$, and $k = \frac{\ln 2}{T_2}$, or $T_2 = \frac{\ln 2}{k}$. Also, we have $T_2 = \frac{\ln 2}{k} = \frac{\ln 2}{\ln(1+r)}$, so $T_2 \ln(1+r) = \ln 2$. Then $(1+r)^{T_2} = 2$, so $r = 2^{1/T_2} - 1$.

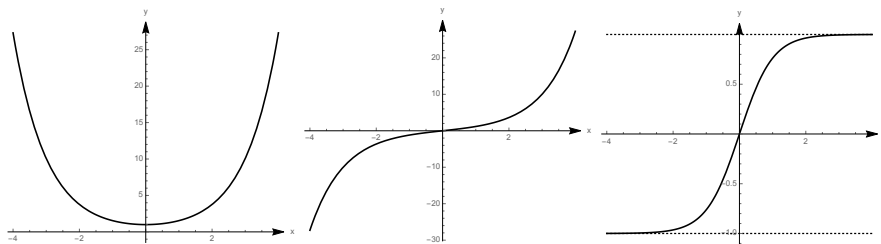
7.2.54 If $y(p) = \sqrt{y(m)y(n)}$, then $y_0 e^{kp} = \sqrt{y_0^2 e^{mk} e^{nk}}$, so $e^{kp} = e^{(m+n)k/2}$, so $p = \frac{m+n}{2}$.

7.2.55 The time required to double occurs when $2y_0 = y_0 e^{kt}$, or when $t = \frac{\ln 2}{k}$. Therefore, the doubling time is $\frac{\ln 2}{k}$ which is constant as a function of t .

7.3 Hyperbolic Functions

7.3.1 $\cosh x = \frac{e^x + e^{-x}}{2}$; $\sinh x = \frac{e^x - e^{-x}}{2}$.

7.3.2 From left to right, $y = \cosh x$ (an even function), $y = \sinh x$ (an odd function), and $y = \tanh x$ (an odd function). Only $y = \tanh x$ has asymptotes: they are $y = 1$ and $y = -1$.



7.3.3 $\cosh^2 x - \sinh^2 x = 1$.

7.3.4 They have a very similar basic form, but the plus and minus signs appear in spots that aren't completely analogous.

7.3.5 $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$.

7.3.6 The domain is $\{x : 0 < x \leq 1\}$. $\operatorname{sech}^{-1} x = \cosh^{-1}(1/x)$.

7.3.7 Evaluate $\sinh^{-1}(1/5)$, which has the same value.

7.3.8 This is valid on the domain of the inverse hyperbolic tangent, which is the interval $(-1, 1)$.

7.3.9 $\int \frac{dx}{16-x^2} = \frac{1}{4} \coth^{-1}(x/4) + C$ for $|x| > 4$; in this case, the values in the interval of integration satisfy $|x| > 4$.

7.3.10 As a increases, the curves become wider and less steep. The value of the minimum at $x = 0$ increases.

$$\mathbf{7.3.11} \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

$$\mathbf{7.3.12} \quad \tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{\frac{e^{-x} - e^x}{2}}{\frac{e^{-x} + e^x}{2}} = \frac{-\sinh x}{\cosh x} = -\tanh x.$$

7.3.13

$$\begin{aligned} \cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} = \cosh(2x). \end{aligned}$$

$$\mathbf{7.3.14} \quad 2 \sinh \ln \sec x = 2 \left(\frac{e^{\ln \sec x} - e^{-\ln \sec x}}{2} \right) = \sec x - \cos x = \frac{1 - \cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x} = \sin x \tan x.$$

$$\mathbf{7.3.15} \quad \cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{2e^x}{2} = e^x.$$

$$\mathbf{7.3.16} \quad \coth^2 x - 1 = \frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} = \operatorname{csch}^2 x.$$

$$\begin{aligned} \mathbf{7.3.17} \quad \frac{1 + \cosh 2x}{2} &= \frac{1 + \cosh^2 x + \sinh^2 x}{2} = \frac{\cosh^2 x + (\sinh^2 x + 1)}{2} = \frac{\cosh^2 x + \cosh^2 x}{2} = \cosh^2 x. \\ \frac{\cosh 2x - 1}{2} &= \frac{\cosh^2 x + \sinh^2 x - 1}{2} = \frac{\sinh^2 x + (\cosh^2 x - 1)}{2} = \frac{\sinh^2 x + \sinh^2 x}{2} = \sinh^2 x. \end{aligned}$$

$$\mathbf{7.3.18} \quad \cosh(2x) = \cosh(x+x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x.$$

$$\mathbf{7.3.19} \quad \frac{d}{dx}(\coth x) = \frac{d}{dx} \frac{\cosh x}{\sinh x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x.$$

$$\mathbf{7.3.20} \quad \frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx} \frac{1}{\cosh x} = \frac{\cosh x \cdot 0 - \sinh x}{\cosh^2 x} = -\operatorname{sech} x \tanh x.$$

$$\mathbf{7.3.21} \quad \frac{d}{dx}(\operatorname{csch} x) = \frac{d}{dx} \frac{1}{\sinh x} = \frac{\sinh x \cdot 0 - \cosh x}{\sinh^2 x} = -\operatorname{csch} x \coth x.$$

$$\mathbf{7.3.22} \quad f'(x) = 4 \cosh 4x.$$

$$\mathbf{7.3.23} \quad f'(x) = 2(\cosh x)(\sinh x).$$

$$\mathbf{7.3.24} \quad f'(x) = -3(\sinh^2 4x)(\cosh 4x)(4) = -12(\sinh^2 4x)(\cosh 4x).$$

$$\mathbf{7.3.25} \quad f'(x) = 2(\tanh x)(\operatorname{sech}^2 x).$$

$$7.3.26 \quad f'(x) = \frac{1}{2\sqrt{\coth 3x}} (-\operatorname{csch}^2(3x))(3) = \frac{-3\operatorname{csch}^2(3x)}{2\sqrt{\coth(3x)}}.$$

$$7.3.27 \quad f'(x) = \frac{1}{\operatorname{sech} 2x} (-\operatorname{sech} 2x \tanh 2x)(2) = -2 \tanh 2x.$$

$$7.3.28 \quad f'(x) = \tanh x + x \operatorname{sech}^2 x.$$

$$7.3.29 \quad f'(x) = 2x \cosh^2 3x + x^2(2 \cosh(3x) \sinh(3x))(3) = 2x \cosh(3x)(\cosh(3x) + 3x \sinh(3x)).$$

$$7.3.30 \quad f'(x) = \frac{\operatorname{csch} x + x \operatorname{csch} x \coth x}{\operatorname{csch}^2 x} = \frac{1}{\operatorname{csch} x} + \frac{x \coth x}{\operatorname{csch} x} = \sinh x + x \cosh x.$$

$$7.3.31 \quad f'(x) = \frac{1}{\sqrt{16x^2 - 1}} \cdot 4 = \frac{4}{\sqrt{16x^2 - 1}}.$$

$$7.3.32 \quad f'(t) = \frac{2}{1-t} \left(\frac{1}{2\sqrt{t}} \right) = \frac{1}{\sqrt{t}(1-t)}.$$

$$7.3.33 \quad f'(v) = \frac{1}{\sqrt{v^4 + 1}} \cdot (2v) = \frac{2v}{\sqrt{v^4 + 1}}.$$

$$7.3.34 \quad f'(x) = \frac{-1}{|2x|\sqrt{1+4/x^2}} \left(\frac{-2}{x^2} \right) = \frac{1}{|x|\sqrt{1+4/x^2}} = \frac{1}{\sqrt{x^2 + 4}}.$$

$$7.3.35 \quad f'(x) = \sinh^{-1}(x) + \frac{x}{\sqrt{x^2 + 1}} - \frac{x}{\sqrt{x^2 + 1}} = \sinh^{-1}(x).$$

$$7.3.36 \quad f'(u) = \frac{1}{\sqrt{\tan^2 u + 1}} \cdot \sec^2 u = \frac{1}{\sqrt{\sec^2 u}} \cdot \sec^2 u = \frac{\sec^2 u}{|\sec u|} = |\sec u|.$$

$$7.3.37 \quad \int \cosh 2x \, dx = \frac{1}{2} \sinh 2x + C.$$

7.3.38 Let $u = \tanh x$ so that $du = \operatorname{sech}^2 x \, dx$. Then we have $\int \operatorname{sech}^2 x \tanh x \, dx = \int u \, du = u^2/2 + C = \frac{\tanh^2 x}{2} + C$. Note: It is also possible to let $u = \operatorname{sech} x$ and obtain a different-looking but equivalent answer.

7.3.39 Let $u = 1 + \cosh x$ so that $du = \sinh x \, dx$. Substituting gives $\int \frac{1}{u} \, du = \ln |u| + C = \ln |1 + \cosh x| + C$.

7.3.40 Let $u = \coth x$ so that $du = -\operatorname{csch}^2 x \, dx$. Substituting gives $\int -u^2 \, du = -u^3/3 + C = -(\coth x)^3/3 + C$.

7.3.41 Recall that $\tanh^2 x = 1 - \operatorname{sech}^2 x$. So we have $\int (1 - \operatorname{sech}^2 x) \, dx = x - \tanh x + C$.

7.3.42 Recall that $\sinh^2 x = \frac{\cosh 2x - 1}{2}$. Thus we have $\frac{1}{2} \int (\cosh 2x - 1) \, dx = \frac{1}{2} \left(\frac{\sinh 2x}{2} - x \right) + C = \frac{\sinh 2x}{4} - \frac{x}{2} + C$.

7.3.43 Let $u = \cosh 3x$ so that $du = 3 \sinh 3x \, dx$. Substituting gives $\frac{1}{3} \int_1^{\cosh 3} u^3 \, du = \left(u^4/12 \right) \Big|_1^{\cosh 3} = \cosh^4 3/12 - 1/12 \approx 856.034$.

7.3.44 Let $u = \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}} dx$. Substituting yields $2 \int_0^2 \operatorname{sech}^2 u \, du = 2 (\tanh u) \Big|_0^2 = 2(\tanh(2) - 0) = 2 \tanh(2) \approx 1.93$.

7.3.45 $\int_0^{\ln 2} \tanh x \, dx = \int_0^{\ln 2} \frac{\sinh x}{\cosh x} dx$. Let $u = \cosh x$ so that $du = \sinh x \, dx$. Substituting gives $\int_1^{5/4} \frac{1}{u} du = (\ln u) \Big|_1^{5/4} = \ln(5/4)$.

7.3.46 We use the formula in Theorem 6.9: $\int \operatorname{csch} y \, dy = \ln |\tanh(y/2)| + C$. We have $\int_{\ln 2}^{\ln 3} \operatorname{csch} y \, dy = (\ln |\tanh(y/2)|) \Big|_{\ln 2}^{\ln 3} = \ln(\tanh((\ln 3)/2)) - \ln(\tanh((\ln 2)/2)) \approx 0.405$.

7.3.47 We have $\frac{1}{8} \int \frac{dx}{1 - (x/\sqrt{8})^2} dx$. Let $u = x/\sqrt{8}$ so that $du = \frac{dx}{\sqrt{8}}$. Substituting gives $\frac{1}{\sqrt{8}} \int \frac{du}{1 - u^2} = \frac{1}{\sqrt{8}} \coth^{-1}(u) + C = \frac{1}{\sqrt{8}} \coth^{-1}(x/\sqrt{8}) + C$.

7.3.48 $\frac{1}{4} \int \frac{dx}{\sqrt{(x/4)^2 - 1}}$. Let $u = x/4$ so that $du = dx/4$. Substituting gives $\int \frac{du}{\sqrt{u^2 - 1}} = \cosh^{-1}(u) + C = \cosh^{-1}(x/4) + C$.

7.3.49 We have $\frac{1}{36} \int \frac{e^x}{1 - (e^x/6)^2} dx$.
Let $u = e^x/6$ so that $du = e^x/6 \, dx$. Substituting gives

$$\frac{1}{6} \int \frac{du}{1 - u^2} = \frac{1}{6} \tanh^{-1}(u) + C = \frac{1}{6} \tanh^{-1}(e^x/6) + C.$$

7.3.50 This can be computed using part 5 of Theorem 7.9. We have

$$\int \frac{1}{x\sqrt{4^2 + x^2}} dx = -\frac{1}{4} \operatorname{csch}^{-1} \left(\frac{|x|}{4} \right) + C.$$

7.3.51 We have $\frac{1}{2} \int \frac{dx}{x\sqrt{1 - (x^4/2)^2}}$. Let $u = x^4/2$ so that $du = 2x^3 \, dx$. Substituting gives

$$\frac{1}{2} \int \frac{du}{4u\sqrt{1 - u^2}} = -\frac{1}{8} \operatorname{sech}^{-1} u + C = -\frac{1}{8} \operatorname{sech}^{-1}(x^4/2) + C.$$

7.3.52 Let $u = x^2$ so that $du = 2x \, dx$. Substituting gives

$$\int \frac{du}{2u\sqrt{1 + u^2}} = -\frac{1}{2} \operatorname{csch}^{-1}|u| + C = -\frac{1}{2} \operatorname{csch}^{-1}x^2 + C.$$

7.3.53 $\int \frac{\cosh z}{\sinh^2 z} dz = \int \coth z \operatorname{csch} z \, dz = -\operatorname{csch} z + C$.

7.3.54 Let $u = \sin \theta$ so that $du = \cos \theta \, d\theta$. Substituting gives

$$\int \frac{du}{3^2 - u^2} = \frac{1}{3} \tanh^{-1}(u/3) + C = \frac{1}{3} \tanh^{-1}((\sin \theta)/3) + C.$$

7.3.55 Let $u = \sinh^{-1} x$ so that $du = \frac{dx}{\sqrt{x^2 + 1}}$. Substituting gives

$$\int_{\sinh^{-1}(5/12)}^{\sinh^{-1}(3/4)} u \, du = (u^2/2) \Big|_{\sinh^{-1}(5/12)}^{\sinh^{-1}(3/4)} = \frac{1}{2} \left((\sinh^{-1}(3/4))^2 - (\sinh^{-1}(5/12))^2 \right) \approx 0.158.$$

7.3.56 Consider $\int_{25}^{225} \frac{dx}{\sqrt{x}\sqrt{x^2+25}}$, and let $u = \sqrt{x}$, so that $du = \frac{dx}{2\sqrt{x}}$. Substituting gives

$$2 \int_5^{15} \frac{du}{\sqrt{u^2+5^2}} = 2 \left(\sinh^{-1}(u/5) \right) \Big|_5^{15} = 2(\sinh^{-1}(3) - \sinh^{-1}(1)) = 2(\ln(3 + \sqrt{10}) - \ln(1 + \sqrt{2})) \approx 1.874.$$

7.3.57

a. Note that $\sinh \ln x = \frac{e^{\ln x} - e^{-\ln x}}{2} = \frac{x - 1/x}{2} = x/2 - 1/(2x)$. So $\int \frac{\sinh \ln x}{x} dx = \int (1/2 - x^{-2}/2) dx = x/2 + x^{-1}/2 + C = \frac{x^2 + 1}{2x} + C$.

b. Let $u = \ln x$, so that $du = \frac{1}{x} dx$. Substituting gives $\int \sinh u du = \cosh u + C = \cosh \ln x + C = \frac{e^{\ln x} + e^{-\ln x}}{2} + C = \frac{x + 1/x}{2} + C = \frac{x^2 + 1}{2x} + C$.

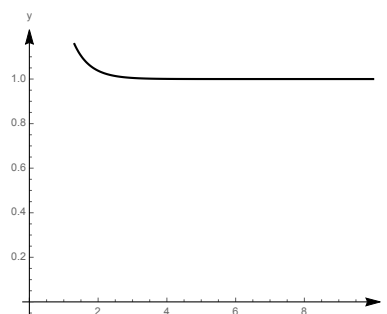
7.3.58

a. Note that $\cosh \ln x = \frac{e^{\ln x} + e^{-\ln x}}{2} = \frac{x + 1/x}{2} = \frac{x^2 + 1}{2x}$. Thus $\operatorname{sech} \ln x = \frac{1}{\cosh \ln x} = \frac{2x}{x^2 + 1}$. We have $\int_1^{\sqrt{3}} \frac{\sinh \ln x}{x} dx = \int_1^{\sqrt{3}} \frac{2}{x^2 + 1} dx = (2 \tan^{-1} x) \Big|_1^{\sqrt{3}} = 2(\pi/3 - \pi/4) = \pi/6$.

b. Let $u = \ln x$, so that $du = \frac{1}{x} dx$. Substituting gives $\int_0^{\ln \sqrt{3}} \operatorname{sech} u du = (\tan^{-1} |\sinh(u)|) \Big|_0^{\ln \sqrt{3}} = \tan^{-1} \left(\frac{\sqrt{3} - (1/\sqrt{3})}{2} \right) - 0 = \tan^{-1} (1/\sqrt{3}) = \pi/6$.

7.3.59

a. The values of $\coth x$ on the interval $[5, 10]$ are very close to 1. So the area under the curve is very close to that of a 5×1 rectangle.

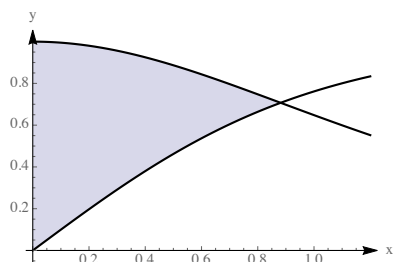


b. $\int_5^{10} \coth x dx = \int_5^{10} \frac{\cosh x}{\sinh x} dx = (\ln |\sinh x|) \Big|_5^{10} = \ln \sinh 10 - \ln \sinh 5 \approx 5.0000454$. The absolute value of the error is about 0.0000454.

7.3.60 Let $u = \sinh x$ so that $du = \cosh x dx$. Then $\int \coth x dx = \int \frac{\cosh x}{\sinh x} dx = \int \frac{1}{u} du = \ln |u| + C = \ln |\sinh x| + C$.

7.3.61

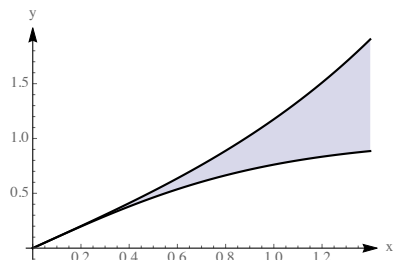
a. The curves intersect when $\frac{1}{\cosh x} = \frac{\sinh x}{\cosh x}$, or when $\sinh x = 1$, so at $x = \sinh^{-1}(1) = \ln(1 + \sqrt{2})$.



b. We need to compute $A = \int_0^{\ln(1+\sqrt{2})} \text{sech } x - \tanh x \, dx$. Recall that $\int \text{sech } x \, dx = \tan^{-1} |\sinh x| + C$ and $\int \tanh x \, dx = \ln \cosh x + C$. Thus, $A = (\tan^{-1} \sinh x - \ln \cosh x) \Big|_0^{\ln(1+\sqrt{2})} = \tan^{-1} \sinh(\ln(1+\sqrt{2})) - \ln \cosh(\ln(1+\sqrt{2})) - (0 - 0) \approx 0.44$.

7.3.62

a. The curves intersect when $\sinh x = \frac{\sinh x}{\cosh x}$, or when $\sinh x = 0$, (or $\cosh x = 1$), so at $x = 0$.



b. We need to compute $\int_0^{\ln 3} \sinh x - \tanh x \, dx = (\cosh x - \ln \cosh x) \Big|_0^{\ln 3} = \cosh(\ln 3) - \ln \cosh(\ln 3) - 1 = 5/3 - \ln(5/3) - 1 = 2/3 - \ln(5/3) \approx 0.156$.

7.3.63 Let $u = \ln x$ so that $du = \frac{dx}{x}$. Substituting gives

$$\int_0^2 \frac{du}{\sqrt{u^2 + 1}} = (\sinh^{-1} u) \Big|_0^2 = \sinh^{-1}(2) - 0 = \ln(2 + \sqrt{5}).$$

$$7.3.64 \int_5^{3\sqrt{5}} \frac{dx}{\sqrt{x^2 - 3^2}} = (\cosh^{-1}(x/3)) \Big|_5^{3\sqrt{5}} = \cosh^{-1}(\sqrt{5}) - \cosh^{-1}(5/3) = \ln(\sqrt{5} + 2) - \ln(5/3 + 4/3) = \ln\left(\frac{\sqrt{5} + 2}{3}\right).$$

$$7.3.65 \int_{-2}^2 \frac{dt}{t^2 - 3^2} = -\frac{1}{3} (\tanh^{-1}(x/3)) \Big|_{-2}^2 = -\frac{1}{3} (\tanh^{-1}(2/3) - \tanh^{-1}(-2/3)) = -\frac{1}{3} \left(\frac{1}{2} \ln((5/3)/(1/3)) - \frac{1}{2} \ln((1/3)/(5/3)) \right) = -\frac{1}{3} \ln 5 \approx -0.54.$$

$$7.3.66 \text{ Let } u = 2t \text{ so that } du = 2 \, dt. \text{ Substituting gives } \int_{1/3}^{1/2} \frac{du}{u\sqrt{1-u^2}} = (-\text{sech}^{-1} u) \Big|_{1/3}^{1/2} = \text{sech}^{-1}(1/3) - \text{sech}^{-1}(1/2) = \cosh^{-1}(3) - \cosh^{-1}(2) = \ln(3 + \sqrt{8}) - \ln(2 + \sqrt{3}) = \ln((3 + \sqrt{8})/(2 + \sqrt{3})) \approx 0.45.$$

$$7.3.67 \text{ Let } u = x^{1/3} \text{ so that } du = \frac{1}{3} x^{-2/3} dx. \text{ Substituting gives } \int_{1/2}^1 \frac{3u^2 du}{u^3 \sqrt{1+u^2}} = 3 \int_{1/2}^1 \frac{du}{u\sqrt{1+u^2}} = -3 (\text{csch}^{-1} x) \Big|_{1/2}^1 = 3(\text{csch}^{-1}(1/2) - \text{csch}^{-1}(1)) = 3(\sinh^{-1}(2) - \sinh^{-1}(1)) = 3(\ln(2 + \sqrt{5}) - \ln(1 + \sqrt{2})) \approx 1.69.$$

7.3.68 Let $u = \sinh x$ so that $du = \cosh x \, dx$. Substituting gives $\int_{12/5}^{40/9} \frac{du}{4-u^2} = \frac{1}{2} (\coth^{-1}(u/2)) \Big|_{12/5}^{40/9} = \frac{1}{2} (\coth^{-1}(20/9) - \coth^{-1}(6/5)) = \frac{1}{2} (\tanh^{-1}(9/20) - \tanh^{-1}(5/6)) = \frac{1}{4} (\ln((29/20)/(11/20)) - \ln((11/6)/(1/6))) = \frac{1}{4} (\ln(29/121)) \approx -0.36$.

7.3.69 By symmetry, we compute $I = 2 \int_0^{\cosh^{-1}(17/15)} (17/15 - \cosh x) \, dx$ and divide by $2 \cosh^{-1}(17/15)$. The value of I is $2 \left((17/15)x - \sinh x \right) \Big|_0^{\cosh^{-1}(17/15)} = 2 \left((17/15) \cosh^{-1}(17/15) - \sinh(\cosh^{-1}(17/15)) \right) = (34/15) \cosh^{-1}(17/15) - 16/15$. When we divide by $2 \cosh^{-1}(17/15)$ we obtain $\frac{17}{15} - \frac{8}{15 \cosh^{-1}(17/15)} \approx 0.09$.

7.3.70 $L = \int_0^a \sqrt{1 + \sinh^2 x} \, dx = \int_0^a \cosh x \, dx = (\sinh x) \Big|_0^a = \sinh a$.

7.3.71

- Let a be the value that produces a sag of 10 feet. Note that $f(0) + \text{sag} = f(50)$, so $a + \text{sag} = a \cosh(50/a)$, and dividing by a yields $1 + \frac{\text{sag}}{a} = \cosh(50/a)$, which can be written as $\cosh(50/a) - 1 = 10/a$.
- If $t = 10/a$, we have $\cosh(5t) - 1 = t$. Solving for t yields $t \approx 0.08$.

- If $0.08 = 10/a$, then $a = 10/0.08 = 125$. The length of the power line is $2 \int_0^{50} \sqrt{1 + \sinh^2(x/125)} \, dx = 2 \int_0^{50} \cosh(x/125) \, dx = 250 (\sinh(x/125)) \Big|_0^{50} = 250 \sinh(50/125) = 250 \sinh(2/5) \approx 102.7$ feet.

7.3.72 We have a right triangle with hypotenuse 50.5 and side adjacent to θ of length 50, so $\cos \theta = 50/50.5$, so $\theta = \cos^{-1}(50/50.5) \approx 0.14$ radians.

7.3.73 We are seeking λ so that $7 = \sqrt{\frac{9.8\lambda}{2\pi} \tanh\left(\frac{20\pi}{\lambda}\right)}$. A computer algebra system finds the approximate root to be 32.81 meters.

7.3.74

- If $\lambda = 50$ and $d = 20$ then $v = \sqrt{\frac{(9.8)(50)}{2\pi} \tanh\left(\frac{40\pi}{50}\right)} \approx 8.77$ meters per second.
- If $4.5 = \sqrt{\frac{(9.8)(15)}{2\pi} \tanh\left(\frac{2\pi d}{15}\right)}$, then $d \approx 3.14$ meters.

7.3.75

- $f'(x) = \text{sech}^2(x)$, so $f'(0) = 1$. The point of tangency is $(0, \tanh 0) = (0, 0)$, so the linearization is $L(x) = 0 + 1(x - 0) = x$.
- When $d/\lambda < 0.05$, $2\pi d/\lambda$ is small. Because $\tanh x \approx x$ for small values of x , $\tanh(2\pi d/\lambda) \approx 2\pi d/\lambda$. Thus,

$$v = \sqrt{\frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi d}{\lambda}\right)} \approx \sqrt{\frac{g\lambda}{2\pi} \cdot \frac{2\pi d}{\lambda}} = \sqrt{gd}.$$

- $v = \sqrt{gd}$ is a function of depth only. As $d \rightarrow 0$, $v \rightarrow 0$ as well.

7.3.76 With $\lambda = 150$ km which is 150,000 meters, we still have $\frac{d}{\lambda} = \frac{4000}{150000} \approx 0.027$, which is less than 0.05. The conclusions from the previous solution (part b) still apply.

7.3.77

- a. False. $\sinh(\ln 3)$ is a constant, so its derivative is zero.
- b. False. $\frac{d}{dx} \cosh x = \sinh x$, not $-\sinh x$.
- c. True. Note that $-\ln(\sqrt{2}-1) = \ln\left(\frac{1}{\sqrt{2}-1}\right) = \ln\left(\frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \ln(\sqrt{2}+1)$.
- d. False. $\int_0^1 \frac{dx}{2^2 - x^2} = \frac{1}{2} \tanh^{-1}(x/2) \Big|_0^1 = \frac{1}{2} \tanh^{-1}\left(\frac{1}{2}\right) \neq \frac{1}{2} \left(\coth^{-1} \frac{1}{2} - \coth^{-1} 0\right)$.

7.3.78

- a. $\coth(4) \approx 1.00067$
- b. Does not exist
- c. $\operatorname{csch}^{-1}(5) \approx 0.19869$
- d. $\operatorname{csch} \Big|_{1/2}^2 \approx -1.64331$
- e. $\ln |\tanh(x/2)| \Big|_1^{10} \approx 0.771846$
- f. $\tan^{-1}(\sinh x) \Big|_{-3}^3 \approx 2.94261$
- g. $\frac{1}{4} \coth^{-1}(x/4) \Big|_{20}^{36} \approx -0.0227902$

7.3.79

- a. $\cosh(0) = 1$
- b. $\tanh(0) = 0$
- c. Does not exist
- d. $\operatorname{sech}(0) = 1$
- e. $\coth(\ln 5) = \frac{5 + 1/5}{5 - 1/5} = \frac{13}{12}$
- f. $\sinh(\ln 9) = \frac{9 - 1/9}{2} = \frac{40}{9}$
- g. $\left(\frac{e + (1/e)}{2}\right)^2 = \left(\frac{e^2 + 1}{2e}\right)^2$
- h. Does not exist
- i. $\ln\left(17/8 + \sqrt{\frac{17^2 - 8^2}{8^2}}\right) = \ln 4$
- j. $\sinh^{-1}\left(\frac{e^1 - e^{-1}}{2}\right) = 1$

7.3.80 $f'(x) = \cosh x > 0$ for all x , so f is increasing on $(-\infty, \infty)$. $f''(x) = \sinh x$, which is zero only for $x = 0$. Note that f'' is positive on $(0, \infty)$ and negative on $(-\infty, 0)$, so f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. There are no extrema, but there is an inflection point at $(0, 0)$.

7.3.81 $f'(x) = 2 \sinh x \cosh x \cosh x + \sinh^2 x \sinh x = \sinh x(2 \cosh^2 x + \sinh^2 x)$. Note that $\sinh x = 0$ for $x = 0$, while $2 \cosh^2 x + \sinh^2 x > 0$ for all x . So the only critical point is $x = 0$.

7.3.82

- a. $f'(x) = \frac{x \sinh x - \cosh x}{x^2}$. This is zero when $x \sinh x = \cosh x$, or $x = \coth x$.
- b. Using a computer algebra system, we find that $x = \pm 1.2$ are critical points of f .

$$7.3.83 \quad f'(x) = 2 \tanh x \operatorname{sech}^2 x = \frac{2 \sinh x}{\cosh^3 x}.$$

$$f''(x) = \frac{2 \cosh^4 x - 2(\sinh x)(3 \cosh^2 x)(\sinh x)}{\cosh^6 x} = \frac{2 \cosh^2 x - 6 \sinh^2 x}{\cosh^4 x}.$$

This is zero for $\tanh^2 x = 1/3$, or $x = \tanh^{-1}(\pm\sqrt{1/3}) = \pm \tanh^{-1}(\sqrt{1/3})$.

7.3.84 $f'(x) = -\operatorname{sech} x \tanh x$. $f''(x) = (\operatorname{sech} x \tanh x) \tanh x - \operatorname{sech} x \operatorname{sech}^2 x = \operatorname{sech} x (\tanh^2 x - \operatorname{sech}^2 x)$. Because $\operatorname{sech} x > 0$ for all x , this quantity is zero when $\tanh x = \pm \operatorname{sech} x$, which occurs for $\sinh x = \pm 1$, or $x = \sinh^{-1}(\pm 1)$. The potential inflection points occur at $\ln(\sqrt{2} + 1)$ and $\ln(\sqrt{2} - 1)$. A check of the sign of f'' on the relevant intervals confirms that there are inflection points at these x -values.

7.3.85 This area would be the area under $\operatorname{sech} x$ between 0 and 1, minus the area of a quarter circle. Thus we have $\int_0^1 \operatorname{sech} x \, dx - \frac{\pi}{4} = (\tan^{-1} |\sinh x|) \Big|_0^1 - \frac{\pi}{4} = \tan^{-1}(\sinh(1)) - \frac{\pi}{4} \approx 0.08$.

$$7.3.86 \quad V = \pi \int_0^1 (\operatorname{sech}^2 x - (1 - x^2)) \, dx = \pi \left(\tanh x - x + x^3/3 \right) \Big|_0^1 = \tanh(1) - 1 + 1/3 \approx 0.095 \text{ cubic units.}$$

7.3.87 Applying L'Hôpital's rule twice brings you back to the initial limit. If we write $\frac{\sinh x}{\cosh x} = \frac{(\sinh x)e^x}{(\cosh x)e^x} = \frac{e^{2x} - 1}{e^{2x} + 1}$, then applying L'Hôpital's rule once shows that the limit is 1.

$$7.3.88 \quad \lim_{x \rightarrow \infty} \frac{1 - \coth x}{1 - \tanh x} = \lim_{x \rightarrow \infty} \frac{\operatorname{csch}^2 x}{-\operatorname{sech}^2 x} = - \lim_{x \rightarrow \infty} \frac{\cosh^2 x}{\sinh^2 x} = -1.$$

$$7.3.89 \quad \lim_{x \rightarrow 0} \frac{\tanh^{-1} x}{\tan(\pi x/2)} = \lim_{x \rightarrow 0} \frac{\frac{1}{1-x^2}}{(\pi/2) \sec^2(\pi x/2)} = \frac{2}{\pi} \lim_{x \rightarrow 0} \frac{\cos^2(\pi x/2)}{1 - x^2} = \frac{2}{\pi}.$$

7.3.90

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\tanh^{-1} x}{\tan(\pi x/2)} &= \lim_{x \rightarrow 1} \frac{\frac{1}{1-x^2}}{(\pi/2) \sec^2(\pi x/2)} = \frac{2}{\pi} \lim_{x \rightarrow 1} \frac{\cos^2(\pi x/2)}{1 - x^2} \\ &= \frac{2}{\pi} \lim_{x \rightarrow 1} \frac{2 \cos(\pi x/2)(-\sin(\pi x/2))(\pi/2)}{-2x} = \frac{2}{\pi} \cdot \frac{0}{-2} = 0. \end{aligned}$$

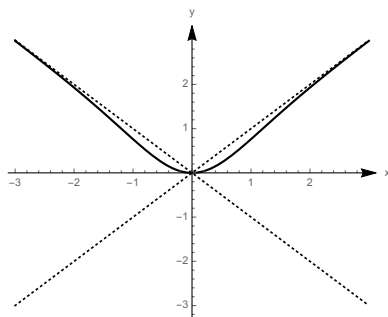
7.3.91 Let $y = (\tanh x)^x$. Then $\ln y = x \ln(\tanh x) = \frac{\ln \tanh x}{1/x}$. Note that $\lim_{x \rightarrow 0} \frac{\ln \tanh x}{1/x} =$

$$\lim_{x \rightarrow 0} \frac{1/(\sinh x \cosh x)}{-1/x^2} = \lim_{x \rightarrow 0} \frac{-x^2}{\sinh x \cosh x} = \lim_{x \rightarrow 0} \frac{-2x}{\sinh^2 x + \cosh^2 x} = 0.$$

Thus, $\lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} (\tanh x)^x = e^0 = 1$.

7.3.92

a. Note that $y = -x$ is also a slant asymptote, by symmetry.



b. Because $\lim_{x \rightarrow \infty} \tanh x = 1$, we expect $x \tanh x$ to behave like $x \cdot 1$ as x gets large.

c. Note that $x \tanh x - x = x \left(\frac{e^{2x} - 1}{e^{2x} + 1} - 1 \right) = x \left(\frac{e^{2x} - 1 - (e^{2x} + 1)}{e^{2x} + 1} \right) = \frac{-2x}{e^{2x} + 1}$. So $\lim_{x \rightarrow \infty} x \tanh x - x = \lim_{x \rightarrow \infty} \frac{-2x}{e^{2x} + 1} = \lim_{x \rightarrow \infty} \frac{-2}{2e^{2x}} = 0$.

7.3.93 Let A be the area under $3 - \cosh x$ between $\cosh^{-1}(-3)$ and $\cosh^{-1}(3)$. Then the volume of the kiln will be $6A$. By symmetry, $A = 2 \int_0^{\cosh^{-1}(3)} (3 - \cosh x) dx = 2 (3x - \sinh x) \Big|_0^{\cosh^{-1}(3)} = 6 \cosh^{-1}(3) - 2 \sinh(\cosh^{-1}(3)) \approx 4.92$. So the volume of the kiln is about $6 \cdot 4.92 \approx 29.5$

7.3.94 Note that f has odd symmetry, so nonzero critical points should come in positive, negative pairs. $f'(x) = \operatorname{sech} x - x \operatorname{sech} x \tanh x = \operatorname{sech} x (1 - x \tanh x)$. Because $\operatorname{sech} x > 0$ for all x , any roots must come from solutions to the equation $1 - x \tanh x = 0$. Let $g(x) = 1 - x \tanh x$. Applying Newton's method with the recursion $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{1 - x_n \tanh x_n}{-\tanh x_n - x_n \operatorname{sech}^2 x_n}$ starting with $x_0 = 1$ yields $x_1 = 1.20177$, $x_2 = 1.9968$, $x_3 = 1.9968$ and so on, so the local extreme values for f occur at ± 1.9968 .

7.3.95

a. $d(10) = \frac{75}{0.2} \ln \left(\cosh \left(\sqrt{\frac{(0.2)(9.8)}{75}} \cdot 10 \right) \right) \approx 360.8$.

b. We need to solve $100 = \frac{75}{0.2} \ln \left(\cosh \left(\sqrt{\frac{(0.2)(9.8)}{75}} \cdot t \right) \right)$ for t . We have $t = \cosh^{-1} \left(e^{\frac{100 \cdot 0.2}{75}} \right) \cdot \sqrt{\frac{75}{(0.2)(9.8)}} \approx 6.97$ seconds. The first 200 feet take $t = \cosh^{-1} \left(e^{\frac{200 \cdot 0.2}{75}} \right) \cdot \sqrt{\frac{75}{(0.2)(9.8)}} \approx 6.97$ seconds, which means that the second 100 feet took about $6.97 - 4.72 = 2.25$ seconds. The average velocity over the first 100 feet is $\frac{100}{4.72} \approx 21.2$ meters per second, while over the second 100 feet is $\frac{100}{2.25} \approx 44.5$ meters per second.

7.3.96

a. $d'(t) = \frac{m}{k} \frac{1}{\cosh \left(\sqrt{\frac{kg}{m}} t \right)} \sinh \left(\sqrt{\frac{kg}{m}} t \right) \sqrt{\frac{kg}{m}} = \sqrt{\frac{mg}{k}} \tanh \left(\sqrt{\frac{kg}{m}} t \right)$.

b. $d'(10) = \sqrt{\frac{(75)(9.8)}{0.2}} \tanh \left(\sqrt{\frac{(75)(9.8)}{0.2}} (10) \right) \approx 56.02$ meters per second.

c. We seek t so that $d'(t) = \sqrt{\frac{mg}{k}} \tanh \left(\sqrt{\frac{kg}{m}} t \right) = 45$. This can be written as $60.6218 \tanh(0.161658t) = 45$, so $t \approx \tanh^{-1} \left(\frac{45}{60.618} \right) \cdot \frac{1}{0.161658} \approx 5.911$ seconds.

7.3.97

a. $\sqrt{\frac{mg}{k}} \lim_{t \rightarrow \infty} \tanh \left(\sqrt{\frac{kg}{m}} t \right) = \sqrt{\frac{mg}{k}} \cdot 1 = \sqrt{\frac{mg}{k}}$.

b. $\sqrt{\frac{(75)(9.8)}{0.2}} \approx 60.6$ meters per second.

c. We seek t so that $v(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}t\right) = 0.95\sqrt{\frac{mg}{k}}$.

So $\tanh\left(\sqrt{\frac{kg}{m}}t\right) = 0.95$, and $t = \tanh^{-1}(0.95)\sqrt{\frac{m}{kg}}$.

d. It takes $\tanh^{-1}(0.95)\sqrt{\frac{75}{(0.2)(9.8)}} \approx 11.33$ seconds to achieve 95 percent of terminal velocity. It takes $d(11.33) \approx 436.5$ feet to achieve this. So the cliff should be at least $436.5 + 300 = 736.5$ feet high.

7.3.98

a. $a(t) = \frac{d}{dt}\sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}t\right) = \sqrt{\frac{mg}{k}} \operatorname{sech}^2\left(\sqrt{\frac{kg}{m}}t\right) \sqrt{\frac{kg}{m}} = g \operatorname{sech}^2\left(\sqrt{\frac{kg}{m}}t\right)$.

b. $\lim_{t \rightarrow \infty} g \operatorname{sech}^2\left(\sqrt{\frac{kg}{m}}t\right) = g \left(\lim_{t \rightarrow \infty} \operatorname{sech}\left(\sqrt{\frac{kg}{m}}t\right)\right)^2 = g(0)^2 = 0$. Once terminal velocity is achieved, the velocity is constant, so the acceleration is zero.

7.3.99 For $y = A \sinh kx$, we have $y' = kA \cosh kx$ and $y'' = k^2 A \sinh kx$. So $y'' - k^2 y = k^2 A \sinh kx - k^2 A \sinh kx = 0$.

For $y = B \cosh kx$, we have $y' = kB \sinh kx$ and $y'' = k^2 B \cosh kx$. So $y'' - k^2 y = k^2 B \cosh kx - k^2 B \cosh kx = 0$.

7.3.100 Using symmetry,

$$\begin{aligned} A &= 4\pi \int_0^{\ln 2} \cosh x \sqrt{1 + \sinh^2 x} \, dx = 4\pi \int_0^{\ln 2} \cosh^2 x \, dx = 4\pi \int_0^{\ln 2} \frac{1 + \cosh 2x}{2} \, dx \\ &= 2\pi \left(x + (1/2) \sinh 2x\right) \Big|_0^{\ln 2} = 2\pi(\ln 2 + 15/16) = \pi(\ln 4 + 15/8) \approx 10.25. \end{aligned}$$

7.3.101

$$\begin{aligned} \sinh(\cosh^{-1}(x)) &= \sinh(\ln(x + \sqrt{x^2 - 1})) = \frac{e^{\ln(x + \sqrt{x^2 - 1})} - \frac{1}{e^{\ln(x + \sqrt{x^2 - 1})}}}{2} \\ &= \frac{1}{2} \left(x + \sqrt{x^2 - 1} - \frac{1}{x + \sqrt{x^2 - 1}}\right). \end{aligned}$$

Multiplying the last term by $\frac{x - \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}}$ gives

$$\frac{1}{2} \left((x + \sqrt{x^2 - 1}) - (x - \sqrt{x^2 - 1})\right) = \sqrt{x^2 - 1}.$$

7.3.102

$$\begin{aligned} \cosh(\sinh^{-1}(x)) &= \cosh(\ln(x + \sqrt{x^2 + 1})) = \frac{e^{\ln(x + \sqrt{x^2 + 1})} + \frac{1}{e^{\ln(x + \sqrt{x^2 + 1})}}}{2} \\ &= \frac{1}{2} \left(x + \sqrt{x^2 + 1} + \frac{1}{x + \sqrt{x^2 + 1}}\right). \end{aligned}$$

Multiplying the last term by $\frac{x - \sqrt{x^2 + 1}}{x - \sqrt{x^2 + 1}}$ gives

$$\frac{1}{2} \left((x + \sqrt{x^2 + 1}) - (x - \sqrt{x^2 + 1})\right) = \sqrt{x^2 + 1}.$$

7.3.103 $\cosh(x+y) = \frac{e^{x+y} + e^{-(x+y)}}{2} = \frac{2e^{x+y} + 2e^{-(x+y)}}{4}$. This can be written as

$$\begin{aligned} & \frac{1}{4}(e^{x+y} + e^{x-y} + e^{y-x} + e^{-(x+y)} + e^{x+y} - e^{x-y} - e^{y-x} + e^{-(x+y)}) \\ &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ &= \cosh x \cosh y + \sinh x \sinh y \end{aligned}$$

7.3.104 $\sinh(x+y) = \frac{e^{x+y} - e^{-(x+y)}}{2} = \frac{2e^{x+y} - 2e^{-(x+y)}}{4}$. This can be written as

$$\begin{aligned} & \frac{1}{4}(e^{x+y} + e^{x-y} - e^{y-x} - e^{-(x+y)} + e^{x+y} - e^{x-y} + e^{y-x} - e^{-(x+y)}) \\ &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ &= \sinh x \cosh y + \cosh x \sinh y \end{aligned}$$

7.3.105 First suppose $x \geq 0$. Then $\cosh^{-1}(\cosh x) = \ln(\cosh x + \sqrt{1 + \cosh^2 x}) = \ln(\cosh x + \sqrt{\sinh^2 x}) = \ln(\cosh x + \sinh x) = \ln e^x = x$.

Now suppose $x < 0$. Then $\cosh^{-1}(\cosh x) = \ln(\cosh x + \sqrt{1 + \cosh^2 x}) = \ln(\cosh x + \sqrt{\sinh^2 x}) = \ln(\cosh x - \sinh x) = \ln e^{-x} = -x$.

Thus $\cosh^{-1}(\cosh x) = |x|$.

7.3.106

a. Let $y = \cosh^{-1}(x)$. Then $x = \cosh y$, so $1 = \sinh y \cdot \frac{dy}{dx}$. Thus $\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sinh(\cosh^{-1}(x))} = \frac{1}{\sqrt{x^2 - 1}}$.

b. $\frac{d}{dx} \ln(x + \sqrt{x^2 + 1}) = \frac{1}{x + \sqrt{x^2 + 1}} \left(1 + \frac{2x}{2\sqrt{x^2 + 1}}\right) = \frac{1}{x + \sqrt{x^2 + 1}} \left(\frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}\right) = \frac{1}{\sqrt{x^2 + 1}}$.

7.3.107

a. $\int \operatorname{sech} x \, dx = \int \frac{\cosh x}{1 + \sinh^2 x} \, dx$. Let $u = \sinh x$ so that $du = \cosh x \, dx$. Then we have $\int \frac{1}{1 + u^2} \, du = \tan^{-1}(u) + C = \tan^{-1}(\sinh x) + C$.

b. Note that $\operatorname{sech} x = \frac{\operatorname{sech}^2 x}{\sqrt{\operatorname{sech}^2 x}} = \frac{\operatorname{sech}^2 x}{\sqrt{1 - \tanh^2 x}}$. So $\int \operatorname{sech} x \, dx = \int \frac{\operatorname{sech}^2 x}{\sqrt{1 - \tanh^2 x}} \, dx$. Let $u = \tanh x$ so that $du = \operatorname{sech}^2 x \, dx$. Substituting gives $\int \frac{1}{\sqrt{1 - u^2}} \, du = \sin^{-1}(u) + C = \sin^{-1}(\tanh x) + C$.

c. $\frac{d}{dx}(2 \tan^{-1}(e^x) + C) = 2 \left(\frac{1}{1 + e^{2x}}\right) e^x + 0 = \frac{2e^x}{1 + e^{2x}} \cdot \frac{e^{-x}}{e^{-x}} = \frac{2}{e^{-x} + e^x} = \frac{1}{\cosh x} = \operatorname{sech} x$.

7.3.108

a. Consider $I = \int \operatorname{csch} x \, dx$ and let $u = x/2$, so that $du = dx/2$. Substituting gives $I = 2 \int \operatorname{csch} 2u \, du = \int \frac{2 \, du}{\sinh 2u}$.

b. Because $\sinh 2x = 2 \sinh x \cosh x$, we have $I = \int \frac{2 \, du}{2 \sinh u \cosh u} = \int \frac{1}{\sinh u \cosh u} \cdot \frac{\frac{1}{\cosh^2 u}}{\frac{1}{\cosh^2 u}} \, du = \int \frac{\operatorname{sech}^2 u}{\tanh u} \, du$.

c. Let $w = \tanh u$ so that $dw = \operatorname{sech}^2 u \, du$. Substituting gives $I = \int \frac{1}{w} dw = \ln |w| + C = \ln |\tanh u| + C = \ln |\tanh(x/2)| + C$.

7.3.109 $L = \int_{\ln 2}^{\ln 8} \sqrt{1 + \operatorname{csch}^2 x} \, dx = \int_{\ln 2}^{\ln 8} \sqrt{\coth^2 x} \, dx = \int_{\ln 2}^{\ln 8} \coth x \, dx = \int_{\ln 2}^{\ln 8} \frac{\cosh x}{\sinh x} \, dx$. Let $u = \sinh x$ so that $du = \cosh x \, dx$. Substituting gives $\int_{3/4}^{63/16} \frac{1}{u} \, du = (\ln u) \Big|_{3/4}^{63/16} = \ln(63/16) - \ln(3/4) = \ln(21/4) \approx 1.66$.

7.3.110 Let $x = \tanh y$ so that $x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \cdot \frac{e^y}{e^y} = \frac{e^{2y} - 1}{e^{2y} + 1}$. Then $xe^{2y} + x = e^{2y} - 1$, so $x + 1 = e^{2y} - xe^{2y}$, and $x + 1 = e^{2y}(1 - x)$. Dividing through by $1 - x$ gives $\frac{x + 1}{1 - x} = e^{2y}$, so $\ln\left(\frac{x + 1}{1 - x}\right) = 2y$, and $y = \frac{1}{2} \ln\left(\frac{x + 1}{1 - x}\right)$.

7.3.111 Suppose $r > 0$ and $0 < x < 1$. If $u = x^r$, then $du = rx^{r-1} \, dx$. Substituting gives $\int \frac{du}{ru\sqrt{1-u^2}} = -\frac{1}{r}(\operatorname{sech}^{-1}(u)) + C = -\frac{1}{r}(\operatorname{sech}^{-1}(x^r)) + C$.

7.3.112

a. Consider the triangle with vertices $(0, 0)$, $(x, 0)$ and (x, y) . Twice the area of the shaded region is twice the area of the triangle less twice the area under the curve $y = \sqrt{z^2 - 1}$ between $z = 1$ and $z = x$. The area of the triangle is $\frac{1}{2}xy$ and the area under the curve is given by $\int_1^x \sqrt{z^2 - 1} \, dz$. Thus, twice the area is given by

$$t = 2 \left(\frac{1}{2}xy - \int_1^x \sqrt{z^2 - 1} \, dz \right) = x\sqrt{x^2 - 1} - 2 \int_1^x \sqrt{z^2 - 1} \, dz.$$

b. Given the formula, we have $\int_1^x \sqrt{z^2 - 1} \, dz = \left(\frac{z}{2} \sqrt{z^2 - 1} - \frac{1}{2} \ln |z + \sqrt{z^2 - 1}| \right) \Big|_1^x = \frac{x}{2} \sqrt{x^2 - 1} - \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| - (0 - 0) = \frac{x}{2} \sqrt{x^2 - 1} - \frac{1}{2} \ln(x + \sqrt{x^2 - 1})$. The absolute values can be dropped because $x > 1$ so $x + \sqrt{x^2 - 1} > 0$.

Now using the previous part of this problem, we have

$$t = x\sqrt{x^2 - 1} - 2 \left(\frac{x}{2} \sqrt{x^2 - 1} - \frac{1}{2} \ln(x + \sqrt{x^2 - 1}) \right) = \ln(x + \sqrt{x^2 - 1}).$$

c. Because $t = \ln(x + \sqrt{x^2 - 1})$, we have $e^t = x + \sqrt{x^2 - 1}$, so $e^{-t} = \frac{1}{x + \sqrt{x^2 - 1}} = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{x - \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = x - \sqrt{x^2 - 1}$. Thus $e^t + e^{-t} = 2x$, so $x = \frac{e^t + e^{-t}}{2}$.

d. We have $y = \sqrt{x^2 - 1} = \sqrt{\frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{4}{4}} = \sqrt{\frac{e^{2t} - 2 + e^{-2t}}{4}} = \sqrt{\left(\frac{e^t - e^{-t}}{2}\right)^2} = \frac{e^t - e^{-t}}{2}$.

Chapter Seven Review

1

- a. False. For example, when t goes from 2 to 3, the value of y goes from 3 to 4, which is not doubling.
- b. False. It grows from A to $A \cdot e^1 \approx 1.10517A$, so the growth is about 10.517%.
- c. False. For example, if $x = e$ and $y = 1$, this equation would imply that $\ln(e \cdot 1) = (\ln e)(\ln 1)$, which is not true because $1 \neq 0$.

d. True. $\sinh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{2} = \frac{x - 1/x}{2} \cdot \frac{x}{x} = \frac{x^2 - 1}{2x}.$

2 Let $u = 4e^x + 6$ so that $du = 4e^x dx$. Then we have $\frac{1}{4} \int \frac{1}{u} du = \frac{1}{4} \ln |u| + C = \frac{1}{4} \ln(4e^x + 6) + C.$

3 Let $u = \ln x$ so that $du = \frac{1}{x} dx$. Then we have $\int_2^8 \frac{1}{u} du = (\ln u) \Big|_2^8 = \ln 4.$

4 Let $u = \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}} dx$. Then we have $2 \int_1^2 10^u du = 2 (10^u / (\ln 10)) \Big|_1^2 = \frac{2}{\ln 10} \cdot 90 = \frac{180}{\ln 10}.$

5 Let $u = x^2 + 8x + 25$ so that $du = (2x + 8) dx = 2(x + 4) dx$. Then we have $\frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln(x^2 + 8x + 25) + C.$

6 $\int_{\ln 2}^{\ln 3} \frac{\cosh s}{\sinh s} ds = (\ln(\sinh s)) \Big|_{\ln 2}^{\ln 3} = \ln(4/3) - \ln(3/4) = \ln(16/9).$

7 $\int \frac{dx}{\sqrt{x^2 - 9}} = \cosh^{-1}(x/3) + C = \ln(x + \sqrt{x^2 - 9}) + C.$

8 Let $u = e^x$ so that $du = e^x dx$. Then we have $\int \frac{du}{\sqrt{u^2 + 4}} = \sinh^{-1}(u/2) + C = \sinh^{-1}(e^x/2) + C.$

9 Let $u = x^3$ so that $du = 3x^2 dx$. Substituting gives

$$\frac{1}{3} \int_0^1 \frac{du}{3^2 - u^2} = \frac{1}{9} (\tanh^{-1}(u/3)) \Big|_0^1 = \frac{1}{9} \tanh^{-1}(1/3) \approx 0.0385.$$

10 We can write $f(x)$ as $f(x) = \ln 3 + 2 \ln(\sin 4x)$, so $f'(x) = 2 \cdot \frac{1}{\sin 4x} \cdot (\cos 4x) \cdot 4 = \frac{8 \cos 4x}{\sin 4x} = 8 \cot 4x.$

11 We can write $g(x)$ as $g(x) = e^{(3x^2+1) \ln x}$, so

$$g'(x) = e^{(3x^2+1) \ln x} \left(6x \ln x + (3x^2 + 1) \frac{1}{x} \right) = x^{3x^2+1} \left(6x \ln x + 3x + \frac{1}{x} \right).$$

12 $f'(x) = \frac{(1 + \sinh x) \cosh x - (\sinh x)(\cosh x)}{(1 + \sinh x)^2} = \frac{\cosh x}{(1 + \sinh x)^2}.$

13 $f'(t) = \sinh t \sinh t + \cosh t \cosh t = \sinh^2 t + \cosh^2 t = \cosh 2t.$

14 $g'(w) = \operatorname{sech}^2(4w^2 + w)(8w + 1) = (8w + 1)\operatorname{sech}^2(4w^2 + w).$

15 $f'(x) = 2 \cosh(3x - 1) \cdot \sinh(3x - 1) \cdot 3 = 6 \cosh(3x - 1) \sinh(3x - 1) = 3 \sinh(2(3x - 1)) = 3 \sinh(6x - 2).$

$$16 \quad h'(z) = \frac{\cosh z}{\sinh z} - \coth z + z \operatorname{csch}^2 z = z \operatorname{csch}^2 z.$$

$$17 \quad f'(x) = \frac{-\sin x}{1 - \cos^2 x} = -\frac{\sin x}{\sin^2 x} = -\frac{1}{\sin x} = -\csc x.$$

$$18 \quad g'(t) = \frac{1}{\sqrt{(\sqrt{t})^2 + 1}} \cdot \frac{1}{2\sqrt{t}} = \frac{1}{2\sqrt{t^2 + t}}.$$

$$19 \quad f'(x) = -\frac{1}{\frac{1}{x^2}\sqrt{1 - \frac{1}{x^4}}} \cdot \frac{-2}{x^3} = \frac{2}{x\sqrt{1 - \frac{1}{x^4}}} = \frac{2x}{\sqrt{x^4 - 1}}.$$

20

a. We can model the population size by $y(t) = 1.3e^{kt}$. Because $1.45 = 1.3e^{10k}$, we have $\frac{1.45}{1.3} = e^{10k}$, so $k = \frac{\ln(1.45/1.3)}{10} \approx 0.01092$. So $y(20) = 1.3e^{20(0.01092)} \approx 1.62$ million.

b. The doubling time is approximately $\frac{\ln 2}{0.01092} \approx 63.5$ years.

21 Because the half-life is 5.5, $k = \frac{\ln(1/2)}{5.5}$, and $y(t) = 75e^{\frac{\ln(1/2)}{5.5}t}$. We are seeking t so that $30 = 75e^{\frac{\ln(1/2)}{5.5}t}$, or $\frac{30}{75} = e^{\frac{\ln(1/2)}{5.5}t}$, so $\ln\left(\frac{30}{75}\right) = \frac{-\ln 2}{5.5}t$, so $t = \frac{5.5 \ln(30/75)}{-\ln 2} \approx 7.3$ hours.

22 $65 = 80e^{1.5k}$, so $1.5k = \ln \frac{65}{80}$, and $k = \frac{2}{3} \ln \frac{65}{80}$.

The amount of caffeine in his system t hours after his second cup of coffee is $y(t) = 145e^{\frac{2}{3}t \ln \frac{65}{80}}$. The amount of caffeine in his system 7 hours after his first cup of coffee is $y(5.5) = 145e^{\frac{2}{3} \cdot 5.5 \cdot \ln \frac{65}{80}} \approx 67.7$ mg.

23

a. Because the doubling time is 2, $k = \frac{\ln 2}{2}$, so $y(t) = 29,000e^{\frac{\ln 2}{2}t}$.

b. $y(21) = 29,000e^{\frac{21 \ln 2}{2}} \approx 41,996,486$ transistors. This closely approximates the actual number of transistors.

24 Because the half-life is 1500 years, we know that $k = \frac{\ln(1/2)}{1500}$. We are seeking t so that $0.7y_0 = y_0e^{kt}$, so $\ln(0.7) = kt$, and thus $t = \frac{\ln(0.7)}{k} = \frac{\ln(0.7) \cdot 1500}{-\ln 2} \approx 771.86$ years ago.

25 Growth is modeled by $p(t) = 150,000e^{kt}$ where $k = \ln(1.04)$. The population reaches 1,000,000 when $1,000,000 = 150,000e^{\ln(1.04)t}$, or when $\ln(20/3) = \ln(1.04)t$, so when $t = \frac{\ln(20/3)}{\ln(1.04)} \approx 48.37$ years.

26

a. The balance at time t years is given by $1500(1.054)^t$.

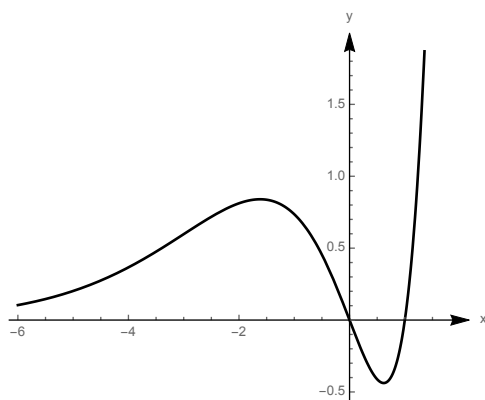
b. The balance doubles when $(1.054)^t = 2$, which occurs for $t = \frac{\ln 2}{\ln 1.054} \approx 13.18$ years.

c. The balance reaches \$5000 when $5000 = 1500(1.054)^t$, which occurs when $t = \frac{\ln(10/3)}{\ln(1.054)} \approx 22.89$ years.

27 The domain of f is the set of all real numbers. Note that $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = 0$.

$f'(x) = e^x(2x-1) + (x^2-x)e^x = e^x(x^2+x-1)$, which is 0 for $x = \frac{-1 \pm \sqrt{5}}{2}$. Note that $f'(x) > 0$ on the interval $(-\infty, (-1 - \sqrt{5})/2)$ and on $((-1 + \sqrt{5})/2, \infty)$, so f is increasing on those intervals, while $f'(x) < 0$ (and so f is decreasing) on $((-1 - \sqrt{5})/2, (-1 + \sqrt{5})/2)$. There is a local maximum at $(-1 - \sqrt{5})/2$ and a local minimum at $(-1 + \sqrt{5})/2$.

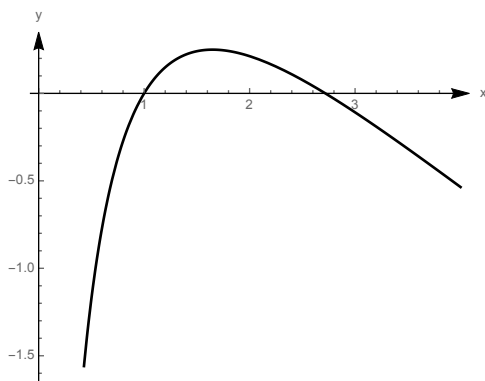
Note that $f''(x) = e^x(2x+1) + (x^2+x-1)e^x = e^x(x^2+3x)$, which is 0 for $x = 0$ and $x = -3$. $f''(x) > 0$ on $(-\infty, -3)$ and on $(0, \infty)$, so f is concave up on those intervals, while $f''(x) < 0$ on $(-3, 0)$, so f is concave down there. There are inflection points at $x = -3$ and at $x = 0$.



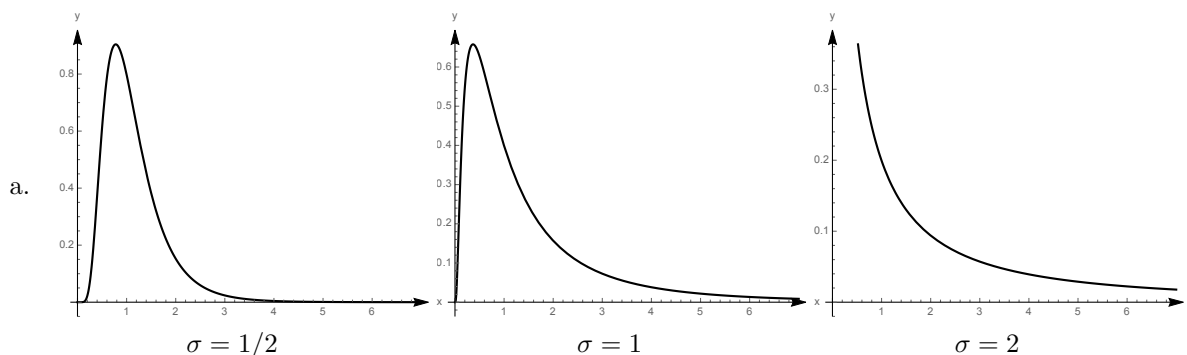
28 The domain of f is $(0, \infty)$, and has x -intercepts where $\ln x(1 - \ln x) = 0$, so at $x = 1$ and $x = e$. Note that $\lim_{x \rightarrow \infty} f(x) = -\infty$ and $\lim_{x \rightarrow 0^+} f(x) = -\infty$.

$f'(x) = \frac{1 - 2 \ln x}{x}$, which is 0 for $x = \sqrt{e}$. Note that $f'(x) > 0$ on the interval $(0, \sqrt{e})$, so f is increasing on that interval, while $f'(x) < 0$ (and so f is decreasing) on $(\sqrt{e}, \infty)$. There is a local maximum at $x = \sqrt{e}$.

Note that $f''(x) = \frac{-2 - (1 - 2 \ln x)}{x^2} = \frac{2 \ln x - 3}{x^2}$, which is 0 for $x = e^{3/2}$. Note that $f''(x) < 0$ on $(0, e^{3/2})$, so f is concave down on that interval, while $f''(x) > 0$ on $(e^{3/2}, \infty)$ so f is concave up there. The point at $x = e^{3/2}$ is an inflection point.



29



It appears that $\lim_{x \rightarrow 0} f(x) = 0$.

b. Let $x = e^y$, so that $y = \ln x$ and as $x \rightarrow 0$ we have $y \rightarrow -\infty$. Then $\lim_{x \rightarrow 0} f(x) = \lim_{y \rightarrow -\infty} \frac{e^{-y^2/2\sigma^2}}{\sigma\sqrt{2\pi}e^y} = \frac{1}{\sigma\sqrt{2\pi}} \lim_{y \rightarrow -\infty} \frac{1}{e^y \cdot e^{y^2/2\sigma^2}} = 0$.

c. Write f as $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \left(\frac{1}{xe^{(\ln^2 x)/2\sigma^2}} \right)$. Then

$$f'(x) = \frac{-1}{\sigma\sqrt{2\pi}} \left(\frac{1}{x^2 e^{(\ln^2 x)/\sigma^2}} \right) \left(e^{(\ln^2 x)/2\sigma^2} + xe^{(\ln^2 x)/2\sigma^2} \cdot \frac{1}{\sigma^2} \ln x \cdot \frac{1}{x} \right) =$$

$$\frac{-1}{\sigma\sqrt{2\pi}} \left(\frac{1}{x^2} \right) \left(1 + \frac{\ln x}{\sigma^2} \right).$$

This quantity is 0 only when $1 + \frac{\ln x}{\sigma^2} = 0$, which occurs for $x^* = e^{-\sigma^2}$, and this critical number yields a maximum.

d. $f(e^{-\sigma^2}) = \frac{1}{\sigma\sqrt{2\pi}} \left(\frac{1}{(e^{-\sigma^2} e^{(\ln^2(e^{-\sigma^2}))/2\sigma^2})} \right) = \frac{1}{\sigma\sqrt{2\pi}} \left(\frac{1}{e^{-\sigma^2} e^{\sigma^2/2}} \right) = \frac{e^{\sigma^2/2}}{\sigma\sqrt{2\pi}}.$

e. Let $g(\sigma) = \frac{e^{\sigma^2/2}}{\sigma\sqrt{2\pi}}$. Then $g'(\sigma) = \frac{e^{\frac{\sigma^2}{2}}(\sigma^2 - 1)}{\sqrt{2\pi}\sigma^2}$, so there is a critical point at $\sigma = 1$. Note that g is decreasing on $(0, 1)$ and increasing on $(1, \infty)$, so g has a minimum at $\sigma = 1$.

30 Note that the curves intersect when $\frac{8}{\cosh^2 x} = \cosh x$, which occurs when $\cosh^3 x = 8$ or $\cosh x = 2$,

which is $x = \cosh^{-1}(2)$. by symmetry, we are seeking $2 \int_0^{\cosh^{-1}(2)} (8 \operatorname{sech}^2 x - \cosh x) dx =$

$$2(8 \tanh x - \sinh x) \Big|_0^{\cosh^{-1}(2)} = 2(4\sqrt{3} - \sqrt{3}) = 6\sqrt{3}.$$

31 $f'(\ln 3) = \sinh(\ln 3) = 4/3$. So the linearization is $L(x) = 5/3 + (4/3)(x - \ln 3)$. Using this approximation, we have $\cosh(1) \approx L(1) = (5/3) + (4/3)(1 - \ln 3) \approx 1.535$.

32 Let $y = (\tanh x)^x$. Then we have $\ln y = x \ln(\tanh x)$. So

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln \tanh x}{1/x} = \lim_{x \rightarrow \infty} \frac{\operatorname{sech}^2 x / \tanh x}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{-x^2}{\sinh x \cosh x}$$

$$= \lim_{x \rightarrow \infty} \frac{-2x}{\sinh^2 x + \cosh^2 x} = \lim_{x \rightarrow \infty} \frac{-2x}{\cosh 2x} = \lim_{x \rightarrow \infty} \frac{-2}{2 \sinh 2x} = 0.$$

Because $\lim_{x \rightarrow \infty} \ln y = 0$, we have $\lim_{x \rightarrow \infty} y = e^0 = 1$.

33

a. Because $\frac{d}{dx} \cosh x = \sinh x$ and $\frac{d}{dx} \sinh x = \cosh x$, we see that every n th derivative of $\cosh x$ where n is even is equal to $\cosh x$. So $\frac{d^6}{dx^6} \cosh x = \cosh x$.

b. $\frac{d}{dx} x \operatorname{sech} x = \operatorname{sech} x - x \operatorname{sech} x \tanh x = \operatorname{sech} x(1 - x \tanh x)$.

34 Note that $y' = 1/x$, so $1 + y'^2 = \frac{x^2 + 1}{x^2}$, and $\sqrt{1 + y'^2} = \frac{\sqrt{x^2 + 1}}{x}$.

So

$$L = \int_1^b \frac{\sqrt{x^2 + 1}}{x} dx = \left(\sqrt{x^2 + 1} - \ln \left(\frac{1 + \sqrt{x^2 + 1}}{x} \right) \right) \Big|_1^b = \sqrt{b^2 + 1} - \sqrt{2} + \ln \left(\frac{(\sqrt{b^2 + 1} - 1)(1 + \sqrt{2})}{b} \right).$$

Using a computer algebra system, we see that this has value 2 for $b \approx 2.715$.

Chapter 8

Integration Techniques

8.1 Basic Approaches

8.1.1 Let $u = 4 - 7x$. Then $du = -7 dx$ and we obtain $-\frac{1}{7} \int u^{-6} du$.

8.1.2 $\int (\sec x + 1)^2 dx = \int (\sec^2 x + 2 \sec x + 1) dx = \tan x + 2 \ln |\sec x + \tan x| + x + C$.

8.1.3 $\sin^2 x = \frac{1 - \cos 2x}{2}$.

8.1.4 $\int \frac{4x^3 + x^2 + 4x + 2}{x^2 + 1} dx = \int \left(4x + 1 + \frac{1}{x^2 + 1} \right) dx = 2x^2 + x + \tan^{-1} x + C$.

8.1.5 Complete the square in the denominator to get $\int \frac{10}{(x-2)^2 + 1} dx$.

8.1.6

a. $\int \frac{2x+1}{x^2+1} dx = \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx$.

b. $\ln(x^2 + 1) + \tan^{-1} x + C$.

8.1.7 Let $u = 3 - 5x$ so that $du = -5 dx$. Substituting gives

$$-\frac{1}{5} \int u^{-4} du = \frac{1}{15} u^{-3} + C = \frac{1}{15(3-5x)^3} + C.$$

8.1.8 Let $u = 9x - 2$ so that $du = 9 dx$. Substituting gives

$$\frac{1}{9} \int u^{-3} du = \frac{-1}{18} u^{-2} + C = \frac{-1}{18(9x-2)^2} + C.$$

8.1.9 Let $u = 2x - \pi/4$ so that $du = 2 dx$. Substituting gives

$$\frac{1}{2} \int_{-\pi/4}^{\pi/2} \sin u du = \frac{1}{2} (-\cos u) \Big|_{-\pi/4}^{\pi/2} = \frac{1}{2} (0 + \sqrt{2}/2) = \frac{\sqrt{2}}{4}.$$

8.1.10 Let $u = 3 - 4x$ so that $du = -4 dx$. Substituting gives

$$-\frac{1}{4} \int e^u du = -\frac{1}{4} e^u + C = -\frac{1}{4} e^{3-4x} + C.$$

8.1.11 Let $u = \ln(2x)$ so that $du = \frac{dx}{x}$. Substituting gives

$$\int u \, du = u^2/2 + C = \frac{1}{2} \ln^2 2x + C.$$

8.1.12 Let $u = 4 - x$ so that $du = -dx$. Substituting gives

$$-\int_9^4 u^{-1/2} \, du = \int_4^9 u^{-1/2} \, du = 2\sqrt{u} \Big|_4^9 = 6 - 4 = 2.$$

8.1.13 Let $u = e^x + 1$ so that $du = e^x \, dx$. Substituting gives

$$\int \frac{1}{u} \, du = \ln |u| + C = \ln(e^x + 1) + C.$$

8.1.14 Let $u = x^2 + 1$ so that $du = 2x \, dx$. Note that $0^2 + 1 = 1$ and $1^2 + 1 = 2$. Substituting gives

$$\frac{1}{2} \int_1^2 3^u \, du = \frac{1}{2 \ln 3} 3^u \Big|_1^2 = \frac{1}{2 \ln 3} (9 - 3) = \frac{3}{\ln 3}.$$

8.1.15 Let $u = \ln x^2 = 2 \ln x$. Then $du = \frac{2}{x} \, dx$. Substituting gives

$$\frac{1}{2} \int_0^4 u^2 \, du = (u^3/6) \Big|_0^4 = 32/3.$$

8.1.16 Let $u = t^3$ so that $du = 3t^2 \, dt$. Note that $0^3 = 0$ and $1^3 = 1$. Substituting gives

$$\frac{1}{3} \int_0^1 \frac{du}{1+u^2} = \frac{1}{3} \tan^{-1} u \Big|_0^1 = \frac{1}{3} \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{12}.$$

8.1.17 Let $u = s - 1$, so that $du = ds$ and $s = u + 1$. Note that $1 - 1 = 0$ and $2 - 1 = 1$, so the new limits of integration are 0 and 1. Substituting gives

$$\int_0^1 u^9(u+1) \, du = \int_0^1 (u^{10} + u^9) \, du = \left(\frac{u^{11}}{11} + \frac{u^{10}}{10} \right) \Big|_0^1 = \frac{1}{11} + \frac{1}{10} = \frac{21}{110}.$$

8.1.18 Let $u = t - 3$ so that $du = dt$ and $t = u + 3$. Note that $3 - 3 = 0$ and $7 - 3 = 4$, so the new limits of integration are 0 and 4. Substituting gives

$$\int_0^4 (u-3)\sqrt{u} \, du = \int_0^4 (u^{3/2} - 3u^{1/2}) \, du = \left(\frac{2}{5} u^{5/2} - 2u^{3/2} \right) \Big|_0^4 = \frac{64}{5} - 16 = -\frac{16}{5}.$$

8.1.19 Let $u = \ln w - 1$. Then $du = \frac{dw}{w}$ and $\ln w = u + 1$. Substituting gives

$$\int u^7(u+1) \, du = \int (u^8 + u^7) \, du = \frac{u^9}{9} + \frac{u^8}{8} + C = \frac{(\ln w - 1)^9}{9} + \frac{(\ln w - 1)^8}{8} + C.$$

8.1.20 Let $u = 1 + e^x$ so that $du = e^x \, dx$ and $e^x = u - 1$. Substituting gives

$$\begin{aligned} \int u^9(2-u) \, du &= \int (2u^9 - u^{10}) \, du = \frac{u^{10}}{5} - \frac{u^{11}}{11} + C = \frac{(1+e^x)^{10}}{5} - \frac{(1+e^x)^{11}}{11} + C \\ &= (1+e^x)^{10} \left(\frac{11}{55} - \frac{5(1+e^x)}{55} \right) + C = \frac{(1+e^x)^{10}(6-5e^x)}{55} + C. \end{aligned}$$

8.1.21 $\int \frac{x}{x^2+4} \, dx + 2 \int \frac{1}{x^2+4} \, dx = \frac{1}{2} \ln(x^2+4) + \tan^{-1}(x/2) + C.$

$$8.1.22 \quad \int \frac{\sin x + 1}{\cos x} dx = \int (\tan x + \sec x) dx = \ln |\sec x| + \ln |\sec x + \tan x| + C.$$

8.1.23 Let $u = 3e^x + 4$ so that $du = 3e^x dx$. Substituting gives

$$\frac{1}{3} \int \csc u du = -\frac{1}{3} \ln |\csc u + \cot u| + C = -\frac{1}{3} \ln |\csc(3e^x + 4) + \cot(3e^x + 4)| + C.$$

$$8.1.24 \quad \int_4^9 x dx - \int_4^9 x^{-1} dx = \frac{x^2}{2} \Big|_4^9 - \ln x \Big|_4^9 = \frac{81}{2} - \frac{16}{2} - (\ln 9 - \ln 4) = \frac{65}{2} - \ln \left(\frac{9}{4}\right).$$

8.1.25 We may rewrite the integrand as $\int_0^{\pi/4} \left(\frac{\sec \theta}{\sec \theta \csc \theta} + \frac{\csc \theta}{\sec \theta \csc \theta} \right) d\theta$. This can then be simplified as

$$\int_0^{\pi/4} (\sin \theta + \cos \theta) d\theta = (-\cos \theta + \sin \theta) \Big|_0^{\pi/4} = \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - (-1 - 0) = 1.$$

$$8.1.26 \quad \int 4e^{-3x} dx + \int e^{-5x} dx = (-4/3)e^{-3x} + (-1/5)e^{-5x} + C.$$

8.1.27

$$\int \frac{2}{\sqrt{1-x^2}} dx - 3 \int \frac{x}{\sqrt{1-x^2}} dx = 2 \sin^{-1} x - 3 \int \frac{x}{\sqrt{1-x^2}} dx.$$

Let $u = 1 - x^2$ so that $du = -2x dx$. Substituting gives

$$2 \sin^{-1} x + \frac{3}{2} \int u^{-1/2} du = 2 \sin^{-1} x + 3\sqrt{u} + C = 2 \sin^{-1} x + 3\sqrt{1-x^2} + C.$$

8.1.28

$$\int \frac{3x}{\sqrt{4-x^2}} dx + \int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{3x}{\sqrt{4-x^2}} dx + \sin^{-1}(x/2).$$

Let $u = 4 - x^2$ so that $du = -2x dx$. Substituting gives

$$-\frac{3}{2} \int u^{-1/2} du + \sin^{-1}(x/2) = -3\sqrt{u} + \sin^{-1}(x/2) + C = -3\sqrt{4-x^2} + \sin^{-1}(x/2) + C.$$

$$8.1.29 \quad \int_{\pi/4}^{\pi/2} \sqrt{1 + \cot^2 x} dx = \int_{\pi/4}^{\pi/2} \csc x dx = -\ln |\csc x + \cot x| \Big|_{\pi/4}^{\pi/2} = -\ln |1+0| + \ln |\sqrt{2}+1| = \ln(\sqrt{2}+1).$$

8.1.30 Let $u = \ln x$. Then $du = \frac{1}{x} dx$. Substituting gives

$$\int_{\pi/4}^{\pi/3} \cot u du = \ln |\sin u| \Big|_{\pi/4}^{\pi/3} = \ln \frac{\sqrt{3}}{2} - \ln \frac{\sqrt{2}}{2} = \frac{\ln 3 - \ln 2}{2} = \frac{1}{2} \ln \frac{3}{2}.$$

8.1.31 Note that by completing the square, we have $x^2 - 2x + 10 = (x^2 - 2x + 1) + 9 = (x-1)^2 + 9$. So

$$\int \frac{dx}{(x-1)^2 + 3^2} = \frac{1}{3} \tan^{-1} \left(\frac{x-1}{3} \right) + C.$$

8.1.32 Note that by completing the square, we have $x^2 + 4x + 8 = (x+2)^2 + 4$. Thus we have

$$\int_0^2 \frac{(x+2)-2}{(x+2)^2 + 4} dx = \int_2^4 \frac{u}{u^2 + 4} du - \int_2^4 \frac{2}{u^2 + 4} du$$

where $u = x + 2$. This is equal to

$$\left(\frac{1}{2} \ln |u^2 + 4| - \tan^{-1}(u/2) \right) \Big|_2^4 = \frac{1}{2} \ln(5/2) + \frac{\pi}{4} - \tan^{-1} 2.$$

8.1.33 We reduce the integrand by long division to $x + 1 + \frac{2x+1}{x^2+x+2}$. Then we have

$$\int \left(x + 1 + \frac{2x+1}{x^2+x+2} \right) dx = \frac{x^2}{2} + x + \int \frac{2x+1}{x^2+x+1} dx.$$

For the remaining integral, we let $u = x^2 + x + 1$ so that $du = (2x + 1) dx$. The last integral is therefore equal to $\ln|u| + C = \ln|x^2 + x + 1| + C$. So our final result is

$$\frac{x^2}{2} + x + \ln|x^2 + x + 1| + C = \frac{x^2}{2} + x + \ln(x^2 + x + 1) + C.$$

8.1.34 Note that long division gives $\frac{x^2+2}{x-1} = x + 1 + \frac{3}{x-1}$. Thus we have

$$\int_2^4 \left(x + 1 + \frac{3}{x-1} \right) dx = \left(x^2/2 + x + 3 \ln|x-1| \right) \Big|_2^4 = 8 + 4 + 3 \ln 3 - (2 + 2 + 0) = 8 + 3 \ln 3.$$

8.1.35 By long division, we can write the integrand as $t^2 + t + \frac{1}{t^2+1}$. Then we have

$$\int_0^1 \left(t^2 + t + \frac{1}{t^2+1} \right) dt = \left(\frac{t^3}{3} + \frac{t^2}{2} + \tan^{-1} t \right) \Big|_0^1 = \left(\frac{1}{3} + \frac{1}{2} + \frac{\pi}{4} \right) - 0 = \frac{3\pi + 10}{12}.$$

8.1.36 Note that long division gives $\frac{t^3-2}{t+1} = t^2 - t + 1 - \frac{3}{t+1}$. Our integral is therefore

$$\int \left(t^2 - t + 1 - \frac{3}{t+1} \right) dt = t^3/3 - t^2/2 + t - 3 \ln|t+1| + C.$$

8.1.37 Note that $27 - 6\theta - \theta^2 = -(\theta^2 + 6\theta + 9 - 36) = -((\theta+3)^2 - 36) = 36 - (\theta+3)^2$. Thus our integral is

$$\int \frac{d\theta}{\sqrt{36 - (\theta+3)^2}} = \sin^{-1}((\theta+3)/6) + C.$$

8.1.38 The integral can be written as $\int \frac{x}{(x^2+1)^2} dx$. Let $u = x^2 + 1$ so that $du = 2x dx$. Substitution gives

$$\frac{1}{2} \int u^{-2} du = \frac{-1}{2u} + C = \frac{-1}{2(x^2+1)} + C.$$

8.1.39

$$\begin{aligned} \int \frac{1}{1+\sin\theta} \cdot \frac{1-\sin\theta}{1-\sin\theta} d\theta &= \int \frac{1-\sin\theta}{1-\sin^2\theta} d\theta = \int \frac{1-\sin\theta}{\cos^2\theta} d\theta \\ &= \int \sec^2\theta d\theta - \int \frac{\sin\theta}{\cos^2\theta} d\theta = \tan\theta - \int \frac{\sin\theta}{\cos^2\theta} d\theta. \end{aligned}$$

Let $u = \cos\theta$ so that $du = -\sin\theta d\theta$. Substituting gives

$$\tan\theta + \int u^{-2} du = \tan\theta - \frac{1}{u} + C = \tan\theta - \sec\theta + C.$$

8.1.40 $\int \frac{1-x}{1-\sqrt{x}} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} dx = \int \frac{(1-x)(1+\sqrt{x})}{1-x} dx = \int (1+\sqrt{x}) dx = x + 2x^{3/2}/3 + C.$

8.1.41 $\int \frac{1}{\sec x - 1} \cdot \frac{\sec x + 1}{\sec x + 1} dx = \int \frac{\sec x + 1}{\sec^2 x - 1} dx = \int \frac{\sec x + 1}{\tan^2 x} dx = \int \frac{\sec x}{\tan^2 x} dx + \int \cot^2 x dx =$
 $\int \cot x \csc x dx + \int \cot^2 x dx = -\csc x + \int (\csc^2 x - 1) dx = -\csc x - \cot x - x + C.$

$$\begin{aligned} 8.1.42 \quad \int \frac{1}{1 - \csc \theta} \cdot \frac{1 + \csc \theta}{1 + \csc \theta} d\theta &= \int \frac{1 + \csc \theta}{1 - \csc^2 \theta} d\theta = \int \frac{1 + \csc \theta}{-\cot^2 \theta} d\theta = \int \frac{-\csc \theta}{\cot^2 \theta} d\theta - \int \frac{1}{\cot^2 \theta} d\theta = \\ &= \int -\tan \theta \sec \theta d\theta - \int \tan^2 \theta d\theta = -\sec \theta - \int (\sec^2 \theta - 1) d\theta = -\sec \theta - \tan \theta + \theta + C. \end{aligned}$$

8.1.43 Let $u = 1 + \sinh 3x$. Then $du = 3 \cosh 3x dx$. Substituting gives

$$\frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |1 + \sinh x| + C.$$

8.1.44 Let $u = x^2 + 1$. Then $du = 2x dx$. Note that $0^2 + 1 = 1$ and $\sqrt{3}^2 + 1 = 4$, and $6x^3 dx = 3x^2 2x dx = 3(u - 1) du$. Substituting gives

$$3 \int_1^4 \frac{u - 1}{\sqrt{u}} du = 3 \int_1^4 (u^{1/2} - u^{-1/2}) du = 3 \left(\frac{2}{3} u^{3/2} - 2u^{1/2} \right) \Big|_1^4 = 3 \left(\frac{16}{3} - 4 - \frac{2}{3} + 2 \right) = 8.$$

8.1.45 We rewrite the integral by multiplying the numerator and denominator of the integrand by e^x . We have

$$\int \frac{e^x}{e^x - 2e^{-x}} dx = \int \frac{e^x}{e^x - 2e^{-x}} \cdot \frac{e^x}{e^x} dx = \int \frac{e^{2x}}{e^{2x} - 2} dx.$$

Now let $u = e^{2x} - 2$ so that $du = 2e^{2x} dx$. Substituting gives

$$\frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |e^{2x} - 2| + C.$$

8.1.46 $\int \frac{e^{2z}}{e^{2z} - 4e^{-z}} \cdot \frac{e^z}{e^z} dz = \int \frac{e^{3z}}{e^{3z} - 4} dz$. Let $u = e^{3z} - 4$ so that $du = 3e^{3z} dz$. Substituting gives

$$\frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |e^{3z} - 4| + C.$$

8.1.47 $\int \frac{dx}{x^{-1} + 1} = \int \frac{1}{x^{-1} + 1} \cdot \frac{x}{x} dx = \int \frac{x}{x + 1} dx$. Using long division, we have $\frac{x}{x + 1} = 1 - \frac{1}{x + 1}$. Then

$$\int \left(1 - \frac{1}{x + 1} \right) dx = x - \ln |x + 1| + C.$$

8.1.48 $\int \frac{dy}{y^{-1} + y^{-3}} = \int \frac{1}{y^{-1} + y^{-3}} \cdot \frac{y^3}{y^3} dy = \int \frac{y^3}{y^2 + 1} dy$. Using long division, we have

$$\frac{y^3}{y^2 + 1} = y - \frac{y}{y^2 + 1}.$$

Thus our integral is equal to

$$\int \left(y - \frac{y}{y^2 + 1} \right) dy = \frac{y^2}{2} - \int \frac{y}{y^2 + 1} dy.$$

To compute this last integral, let $u = y^2 + 1$ so that $du = 2y dy$. We have

$$\frac{y^2}{2} - \frac{1}{2} \int \frac{1}{u} du = \frac{y^2}{2} - \frac{1}{2} \ln |u| + C = \frac{y^2}{2} - \frac{1}{2} \ln |y^2 + 1| + C.$$

8.1.49 Let $u = 9 + \sqrt{t + 1}$. Then $du = \frac{dt}{2\sqrt{t + 1}}$, or $dt = 2(u - 9) du$. Our integral is

$$\begin{aligned} \int \sqrt{9 + \sqrt{t + 1}} dt &= \int \sqrt{u} (2(u - 9)) du = 2 \int (u^{3/2} - 9u^{1/2}) du = 2 \left(\frac{2}{5} u^{5/2} - 6u^{3/2} \right) + C \\ &= \frac{4}{5} u^{3/2} (u - 15) + C = \frac{4}{5} (9 + \sqrt{t + 1})^{3/2} (9 + \sqrt{t + 1} - 15) + C \\ &= \frac{4}{5} (9 + \sqrt{t + 1})^{3/2} (\sqrt{t + 1} - 6) + C. \end{aligned}$$

8.1.50 Let $u = \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}} dx$, or $dx = 2u du$. Substituting gives

$$\int_2^3 \frac{2u}{1-u} du = - \int_2^3 \frac{2u}{u-1} du.$$

By long division, $\frac{2u}{u-1} = 2 + \frac{2}{u-1}$. Thus we have

$$- \int_2^3 \left(2 + \frac{2}{u-1} \right) du = (-2u - 2 \ln |u-1|) \Big|_2^3 = -6 - 2 \ln 2 - (-4 - 0) = -2 - 2 \ln 2.$$

8.1.51 $\int_{-1}^0 \frac{x}{x^2+2x+2} dx = \int_{-1}^0 \frac{x}{(x+1)^2+1} dx$. Let $u = x+1$ so that $du = dx$. Substituting gives

$$\begin{aligned} \int_0^1 \frac{u-1}{u^2+1} du &= \int_0^1 \frac{u}{u^2+1} du - \int_0^1 \frac{1}{u^2+1} du \\ &= \left((1/2) \ln(u^2+1) - \tan^{-1}(u) \right) \Big|_0^1 = (1/2) \ln 2 - \frac{\pi}{4} = \frac{1}{4} (\ln 4 - \pi). \end{aligned}$$

8.1.52 $\int_{\pi/6}^{\pi/2} (\csc y) dy = -\ln |\csc y + \cot y| \Big|_{\pi/6}^{\pi/2} = -\ln |1+0| + \ln |2+\sqrt{3}| = \ln(2+\sqrt{3})$.

8.1.53 Let $u = e^x + 1$ so that $du = e^x dx$. Substituting gives

$$\int \sec u du = \ln |\sec u + \tan u| + C = \ln |\sec(e^x + 1) + \tan(e^x + 1)| + C.$$

8.1.54 Let $u = 1 + \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}} dx$, or $dx = 2\sqrt{x} du = 2(u-1) du$. Substituting gives

$$\begin{aligned} 2 \int_1^2 \sqrt{u}(u-1) du &= 2 \int_1^2 (u^{3/2} - u^{1/2}) du = 2 \left(2u^{5/2}/5 - 2u^{3/2}/3 \right) \Big|_1^2 \\ &= 2(8\sqrt{2}/5 - 4\sqrt{2}/3 - (2/5 - 2/3)) = \frac{8}{15} (1 + \sqrt{2}). \end{aligned}$$

8.1.55 Using the identity $\sin 2x = 2 \sin x \cos x$, we have $2 \int \sin^2 x \cos x dx$. Let $u = \sin x$ so that $du = \cos x dx$. We have $2 \int u^2 du = 2u^3/3 + C = 2(\sin^3 x)/3 + C$.

8.1.56 Using the double angle identity $\cos 2x = 2 \cos^2 x - 1$, we have

$$\int_0^{\pi/2} \sqrt{2 \cos^2 x} dx = \int_0^{\pi/2} \sqrt{2} \cos x dx = \sqrt{2} \sin x \Big|_0^{\pi/2} = \sqrt{2}(1-0) = \sqrt{2}.$$

8.1.57 Rewrite the integral as $\int \frac{1}{\sqrt{x}} \cdot \frac{1}{1+(\sqrt{x})^2} dx$ and let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} dx$, and substituting gives

$$2 \int \frac{1}{1+u^2} du = 2 \tan^{-1} u + C = 2 \tan^{-1} \sqrt{x} + C.$$

8.1.58 Let $u = \sqrt{p}$ so that $u^2 = p$ and $2u du = dp$. Substituting gives

$$\begin{aligned} \int_0^1 \frac{2u}{4-u} du &= 2 \int_0^1 \left(-1 - \frac{4}{u-4} \right) du \\ &= 2(-u - 4 \ln |u-4|) \Big|_0^1 = 2(-1 - 4 \ln 3 - (0 - 4 \ln 4)) = 2(\ln(256/81) - 1). \end{aligned}$$

8.1.59 Note that $x^2 + 6x + 13 = (x^2 + 6x + 9) + 4 = (x + 3)^2 + 4$. Also note that we can write the numerator $x - 2 = x + 3 - 5 = \frac{1}{2}(2x + 6) - 5$. We have

$$\int \frac{\frac{1}{2}(2x + 6)}{x^2 + 6x + 13} dx - \int \frac{5}{(x + 3)^2 + 4} dx.$$

For the first integral, let $u = x^2 + 6x + 13$ so that $du = (2x + 6) dx$. We have (for just the first integral)

$$\frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln(x^2 + 6x + 13) + C.$$

The second integrand has antiderivative equal to $\frac{5}{2} \tan^{-1}((x + 3)/2)$, so the original integral is equal to

$$\frac{1}{2} \ln(x^2 + 6x + 13) - \frac{5}{2} \tan^{-1}((x + 3)/2) + C.$$

8.1.60 $3 \int_0^{\pi/4} \sqrt{1 + \sin 2x} \cdot \frac{\sqrt{1 - \sin 2x}}{\sqrt{1 - \sin 2x}} dx = 3 \int_0^{\pi/4} \frac{\cos 2x}{\sqrt{1 - \sin 2x}} dx$. Let $u = 1 - \sin 2x$ so that $du = -2 \cos 2x dx$. Substituting gives

$$\frac{-3}{2} \int_1^0 u^{-1/2} du = 3 \sqrt{u} \Big|_0^1 = 3.$$

8.1.61 Let $u = e^x$ so that $du = e^x dx$. Substituting gives

$$\int \frac{1}{u^2 + 2u + 1} du = \int (u + 1)^{-2} du = -\frac{1}{u + 1} + C = -\frac{1}{e^x + 1} + C.$$

8.1.62 By long division, the integrand can be written as $-x^3 - x + 1 + \frac{4x + 2}{x^2 + x + 1}$. Then

$$\int \left(-x^3 - x + 1 + \frac{2(2x + 1)}{x^2 + x + 1} \right) = -\frac{x^4}{4} - \frac{x^2}{2} + x + 2 \ln |x^2 + x + 1| + C.$$

8.1.63 The denominator factors as $(x + 1)^2$.

$$\int_1^3 \frac{2}{(x + 1)^2} dx = -2 \left(\frac{1}{x + 1} \right) \Big|_1^3 = -2 \left(\frac{1}{4} - \frac{1}{2} \right) = \frac{1}{2}.$$

8.1.64 The denominator factors as $(s + 1)^3$.

$$\int_0^2 \frac{2}{(s + 1)^3} ds = - (s + 1)^{-2} \Big|_0^2 = - \left(\frac{1}{9} - 1 \right) = \frac{8}{9}.$$

8.1.65

- False. This seem to use the untrue “identity” that $\frac{a}{b + c} = \frac{a}{b} + \frac{a}{c}$.
- False. The degree of the numerator is already less than the degree of the denominator, so long division won’t help.
- False. This is false because $\frac{d}{dx} \ln |\sin x + 1| + C \neq \frac{1}{\sin x + 1}$. The substitution $u = \sin x + 1$ can’t be carried out because $du = \cos x dx$ can’t be accounted for.
- False. In fact, $\int e^{-x} dx = -e^{-x} + C \neq \ln e^x + C$.

8.1.66 $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$. Let $u = \sin x$ so that $du = \cos x \, dx$. We then have

$$\int \frac{1}{u} \, du = \ln |u| + C = \ln |\sin x| + C.$$

8.1.67

$$\int \csc x \, dx = \int (\csc x) \left(\frac{\csc x + \cot x}{\csc x + \cot x} \right) dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx.$$

Let $u = \csc x + \cot x$ so that $du = -\csc^2 x - \csc x \cot x \, dx$. Substituting then yields

$$-\int \frac{1}{u} \, du = -\ln |u| + C = -\ln |\csc x + \cot x| + C.$$

8.1.68

a. If $u = \cot x$, then $du = -\csc^2 x \, dx$. Substituting gives $-\int u \, du = -\frac{u^2}{2} + C = -\frac{\cot^2 x}{2} + C$.

b. If $u = \csc x$, then $du = -\csc x \cot x \, dx$. Substituting gives $-\int u \, du = -\frac{u^2}{2} + C = -\frac{\csc^2 x}{2} + C$.

c. The seemingly different answers are the same, since $-(\cot^2 x)/2$ and $-(\csc^2 x)/2$ differ by a constant.

In fact, $-\frac{\cot^2 x}{2} - \left(-\left(\frac{\csc^2 x}{2} \right) \right) = \frac{1}{2}$.

8.1.69

a. If $u = \tan x$ then $du = \sec^2 x \, dx$. Substituting gives $\int u \, du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C$.

b. If $u = \sec x$, then $du = \sec x \tan x \, dx$. Substituting gives $\int u \, du = \frac{u^2}{2} + C = \frac{\sec^2 x}{2} + C$.

c. The seemingly different answers are the same, since $\frac{\tan^2 x}{2}$ and $\frac{\sec^2 x}{2}$ differ by a constant. In fact,

$$\frac{\tan^2 x}{2} - \frac{\sec^2 x}{2} = -\frac{1}{2}.$$

8.1.70

a. Note that long division gives $\frac{x+2}{x+4} = 1 - \frac{2}{x+4}$. Thus our integral is equal to

$$\int \left(1 - \frac{2}{x+4} \right) dx = x - 2 \ln |x+4| + C.$$

b. Let $u = x+4$. Then $du = dx$ and $x+2 = u-2$. Substituting gives

$$\int \frac{u-2}{u} \, du = \int \left(1 - \frac{2}{u} \right) du = u - 2 \ln |u| + C = x+2 - 2 \ln |x+4| + C = x - 2 \ln |x+4| + 2 + C.$$

c. In part b, $2+C$ is a constant, so we can replace $2+C$ by C to obtain the answer to part a.

8.1.71

a. Let $u = x+1$ so that $du = dx$. Note that $x = u-1$, so that $x^2 = (u-1)^2$. Substituting gives

$$\int \frac{u^2 - 2u + 1}{u} \, du = \int (u - 2 + (1/u)) \, du = u^2/2 - 2u + \ln |u| + C = (x+1)^2/2 - 2(x+1) + \ln |x+1| + C.$$

b. By long division, $\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$. Thus,

$$\int \frac{x^2}{x+1} dx = \int \left(x - 1 + \frac{1}{x+1} \right) dx = x^2/2 - x + \ln|x+1| + C.$$

c. The seemingly different answers are the same, because they differ by a constant. In fact,

$$\frac{(x+1)^2}{2} - 2(x+1) + \ln|x+1| - \left(\frac{x^2}{2} - x + \ln|x+1| \right) = -\frac{3}{2}.$$

8.1.72 The curves intersect when $x^3 = 8x$, so at $x = 0$ and $x = \pm\sqrt{8}$. By symmetry, we have $A = 2 \int_0^{\sqrt{8}} \frac{8x - x^3}{x^2 + 1} dx$. Using long division, we can write $\frac{8x - x^3}{x^2 + 1} = -x + \frac{9x}{x^2 + 1}$. Thus,

$$\begin{aligned} A &= 2 \int_0^{\sqrt{8}} \left(-x + \frac{9x}{x^2 + 1} \right) dx = 2 \int_0^{\sqrt{8}} (-x) dx + 2 \int_0^{\sqrt{8}} \frac{9x}{x^2 + 1} dx \\ &= -2 x^2/2 \Big|_0^{\sqrt{8}} + 2 \int_0^{\sqrt{8}} \frac{9x}{x^2 + 1} dx = -8 + 2 \int_0^{\sqrt{8}} \frac{9x}{x^2 + 1} dx. \end{aligned}$$

To compute this last integral, let $u = x^2 + 1$ so that $du = 2x dx$. Then we have

$$A = -8 + 9 \int_1^9 \frac{1}{u} du = -8 + 9 \ln u \Big|_1^9 = -8 + 9 \ln 9 \approx 11.775.$$

8.1.73

$$A = \int_2^4 \frac{x^2 - 1}{x^3 - 3x} dx.$$

Let $u = x^3 - 3x$ so that $du = 3x^2 - 3 dx$. Substituting gives

$$A = \frac{1}{3} \int_2^{52} \frac{1}{u} du = \frac{1}{3} \ln u \Big|_2^{52} = \frac{1}{3} (\ln 52 - \ln 2) = \frac{\ln 26}{3}.$$

8.1.74

$$\begin{aligned} V &= 2\pi \int_0^3 \frac{x}{x+2} dx = 2\pi \int_0^3 \left(1 - \frac{2}{x+2} \right) dx \\ &= 2\pi (x - 2 \ln(x+2)) \Big|_0^3 = 2\pi(3 - 2 \ln 5 - (0 - 2 \ln 2)) = 2\pi(3 - 2 \ln(2/5)). \end{aligned}$$

8.1.75

a. $V = \pi \int_0^2 (x^2 + 1) dx = \pi (x^3/3 + x) \Big|_0^2 = \pi(8/3 + 2) = \frac{14\pi}{3}.$

b. $V = 2\pi \int_0^2 x \sqrt{x^2 + 1} dx$. Let $u = x^2 + 1$ so that $du = 2x dx$. Substituting gives

$$\pi \int_1^5 u^{1/2} du = \frac{2\pi}{3} (u^{3/2}) \Big|_1^5 = \frac{2\pi}{3} (5\sqrt{5} - 1).$$

8.1.76

- a. Note that $x - x^2 = -(x^2 - x + 1/4 - 1/4) = -((x - 1/2)^2 - 1/4) = 1/4 - (x - 1/2)^2$. We can write our integral as

$$\int \frac{dx}{\sqrt{1/4 - (x - 1/2)^2}} = 2 \int \frac{dx}{\sqrt{1 - (2x - 1)^2}}.$$

Let $u = 2x - 1$. Then $du = 2 dx$. Substituting gives

$$\int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + C = \sin^{-1}(2x - 1) + C.$$

- b. We can write our integral as $\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$. Let $u = \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}} dx$. Substituting gives
- $$2 \int \frac{du}{\sqrt{1 - u^2}} = 2 \sin^{-1} u + C = 2 \sin^{-1} \sqrt{x} + C.$$

- c. By parts a and b, it follows that both $\sin^{-1}(2x - 1)$ and $2 \sin^{-1}(\sqrt{x})$ are antiderivatives of $\frac{1}{\sqrt{x - x^2}}$. Therefore, $2 \sin^{-1} \sqrt{x} - \sin^{-1}(2x - 1) = C$ for some constant C . To determine C , we let $x = 0$, giving $2 \sin^{-1}(0) - \sin^{-1}(-1) = C$. Thus $0 - \left(-\frac{\pi}{2}\right) = C$, so $C = \frac{\pi}{2}$.

8.1.77

$$\begin{aligned} A &= 2\pi \int_0^1 \sqrt{x+1} \sqrt{1 + \frac{1}{4(x+1)}} dx = 2\pi \int_0^1 \sqrt{x+5/4} dx \\ &= 2\pi \left((2/3)(x+5/4)^{3/2} \right) \Big|_0^1 = \frac{4\pi}{3} (27/8 - 5\sqrt{5}/8) = \frac{9\pi}{2} - \frac{5\sqrt{5}\pi}{6}. \end{aligned}$$

8.1.78 $A = 2\pi \int_0^{\ln 2} (e^x + e^{-x}/4) \sqrt{1 + (e^x - e^{-x}/4)^2} dx$. Note that

$$1 + (e^x - e^{-x}/4)^2 = 1 + e^{2x} - 1/2 + e^{-2x}/16 = e^{2x} + 1/2 + e^{-2x}/16 = (e^x + e^{-x}/4)^2.$$

Thus we have

$$\begin{aligned} 2\pi \int_0^{\ln 2} (e^x + e^{-x}/4)^2 dx &= 2\pi \int_0^{\ln 2} (e^{2x} + 1/2 + e^{-2x}/16) dx = 2\pi (e^{2x}/2 + x/2 - e^{-2x}/32) \Big|_0^{\ln 2} \\ &= 2\pi (2 + (\ln 2)/2 - 1/128 - (1/2 - 1/32)) = \pi \left(\frac{195}{64} + \ln 2 \right). \end{aligned}$$

8.1.79 $L = \int_0^1 \sqrt{1 + \frac{25x^{1/2}}{16}} dx$. Let $u^2 = 1 + \frac{25x^{1/2}}{16}$. Then $2u du = \frac{25}{32\sqrt{x}} dx$. Note that $\sqrt{x} = \frac{16}{25}(u^2 - 1)$, and that $dx = \frac{64\sqrt{x}}{25} u du = \frac{1024}{625}(u^3 - u) du$. Substituting gives

$$\begin{aligned} L &= \int_1^{\sqrt{41/16}} \frac{1024}{625} (u^4 - u^2) du = \frac{1024}{625} (u^5/5 - u^3/3) \Big|_1^{\sqrt{41/16}} \\ &= \frac{1024}{625} ((\sqrt{41/16})^5/5 - (\sqrt{41/16})^3/3 - (1/5 - 1/3)) = \frac{1024}{625} \left(\frac{2}{15} + \frac{1763\sqrt{41}}{15360} \right) \\ &= \frac{2048 + 1763\sqrt{41}}{9375} \approx 1.423. \end{aligned}$$

8.1.80

$$d(t) = \int_0^t v(y) dy = \int_0^t v_T \left(\frac{e^{ay} - 1}{e^{ay} + 1} \right) dy = v_T \int_0^t \frac{e^{ay}}{e^{ay} + 1} dy + -v_T \int_0^t \frac{1}{e^{ay} + 1} dy.$$

The first integral can be computed by letting $u = e^{ay} + 1$ so that $du = ae^{ay} dy$. The first integral is then equal to $(v_T/a) \int_2^{e^{at}+1} \frac{1}{u} du = (v_T/a)(\ln(e^{at} + 1) - \ln(2))$. The second integral is $-v_T \int_0^t \frac{1}{e^{ay} + 1} \cdot \frac{e^{ay}}{e^{ay}} dy$. Again, let $u = e^{ay} + 1$ and note that $du = ae^{ay} dy$. Substitution gives

$$\begin{aligned} -(v_T/a) \int_2^{e^{at}+1} \frac{1}{u(u-1)} du &= -(v_T/a) \int_2^{e^{at}+1} \left(\frac{1}{u-1} - \frac{1}{u} \right) dy \\ &= -(v_T/a) (\ln(u-1) - \ln(u)) \Big|_2^{e^{at}+1} = -(v_T/a)(at - \ln(e^{at} + 1) - (0 - \ln 2)). \end{aligned}$$

Adding the results of the two integrations gives

$$d(t) = (v_T/a)(\ln(e^{at} + 1) - \ln(2)) + -(v_T/a)(at - \ln(e^{at} + 1) - (0 - \ln 2)) = (v_T/a)(2\ln(e^{at} + 1) - at - \ln 4)$$

8.2 Integration by Parts

8.2.1 It is based on the product rule. In fact, the rule can be obtained by writing down the product rule, then integrating both sides and rearranging the terms in the result.

8.2.2 $u = x$ so $du = dx$, and $dv = \cos x dx$ so $v = \sin x$. Then the integration by parts formula gives

$$x \sin x - \int \sin x dx = x \sin x + \cos x + C.$$

8.2.3 $u = \ln x$ so $du = \frac{dx}{x}$, and $dv = x dx$ so $v = \frac{x^2}{2}$. Then the integration by parts formula gives

$$\frac{x^2 \ln x}{2} - \int \frac{x^2}{2x} dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2} dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C = \frac{x^2(2 \ln x - 1)}{4} + C.$$

8.2.4 One can use integration by parts for definite integrals via the formula

$$\int_a^b u(x)v'(x) dx = u(x)v(x) \Big|_a^b - \int_a^b v(x)u'(x) dx.$$

8.2.5 Those for which the choice for dv is easily integrated and when the resulting new integral is no more difficult than the original.

8.2.6 It is generally a good idea to let dv be something easy to integrate. In this case, we would let $dv = e^{ax} dx$, leaving $u = x^n$. Note that differentiating x^n results in something simpler (lower degree,) while integrating it make it more complicated (higher degree). However, differentiating or integrating e^{ax} yields essentially the same thing (a constant times the function e^{ax}).

8.2.7 Let $u = \tan x + 2$, so that $du = \sec^2 x dx$. Then

$$\int \sec^2 x \ln(\tan x + 2) dx = \int \ln u du = u \ln u - u + C = (\tan x + 2) \ln(\tan x + 2) - \tan x + C.$$

8.2.8 Let $u = \sin x$ so that $du = \cos x dx$. Then

$$\int \cos x \ln(\sin x) dx = \int \ln u du = u \ln u - u + C = \sin x \ln(\sin x) - \sin x + C.$$

8.2.9 Let $u = x$ and $dv = \cos 5x \, dx$. Then $du = dx$ and $v = \frac{1}{5} \sin 5x$. Then

$$\int x \cos 5x \, dx = \frac{1}{5} x \sin 5x - \frac{1}{5} \int \sin 5x \, dx = \frac{1}{5} x \sin 5x + \frac{1}{25} \cos 5x + C.$$

8.2.10 Let $u = x$ and $dv = \sin 2x \, dx$. Then $du = dx$ and $v = -\frac{1}{2} \cos(2x)$. So

$$\int x \sin 2x \, dx = -\frac{1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x \, dx = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C.$$

8.2.11 Let $u = t$ and $dv = e^{6t} \, dt$. Then $du = dt$ and $v = \frac{1}{6} \cdot e^{6t}$. Then

$$\int t e^{6t} \, dt = \frac{1}{6} t e^{6t} - \frac{1}{6} \int e^{6t} \, dt = \frac{1}{6} t e^{6t} - \frac{1}{36} e^{6t} + C.$$

8.2.12 Let $u = 2x$ and $dv = e^{3x} \, dx$. Then $du = 2 \, dx$ and $v = \frac{e^{3x}}{3}$. Then

$$\int 2x e^{3x} \, dx = \frac{2x e^{3x}}{3} - \frac{2}{3} \int e^{3x} \, dx = \frac{2x e^{3x}}{3} - \frac{2e^{3x}}{9} + C.$$

8.2.13 Let $u = \ln 10x$ and $dv = x \, dx$. Then $du = \frac{1}{x} \, dx$ and $v = \frac{x^2}{2}$. Then

$$\int x \ln 10x \, dx = \frac{x^2}{2} \ln 10x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \ln 10x - \frac{x^2}{4} + C = \frac{x^2}{4} (2 \ln 10x - 1) + C.$$

8.2.14 Let $u = s$ and $dv = e^{-2s} \, ds$. Then $du = ds$ and $v = -\frac{1}{2} e^{-2s}$. Then

$$\int s e^{-2s} \, ds = -\frac{1}{2} s e^{-2s} + \frac{1}{2} \int e^{-2s} \, ds = -\frac{1}{2} s e^{-2s} - \frac{1}{4} e^{-2s} + C.$$

8.2.15 Let $u = 2w + 4$ and $dv = \cos 2w \, dw$. Then $du = 2 \, dw$ and $v = \frac{1}{2} \sin 2w$. The integration by parts formula gives

$$(w + 2) \sin 2w - \int \sin 2w \, dw = (w + 2) \sin 2w + \frac{1}{2} \cos 2w + C.$$

8.2.16 Let $u = \theta$ and $dv = \sec^2 \theta \, d\theta$. Then $du = d\theta$ and $v = \tan \theta$. Then

$$\int \theta \sec^2 \theta \, d\theta = \theta \tan \theta - \int \tan \theta \, d\theta = \theta \tan \theta + \ln |\cos \theta| + C.$$

8.2.17 Let $u = x$ and $dv = 3^x \, dx$. Then $du = dx$ and $v = \frac{3^x}{\ln 3}$. Then we have

$$\frac{x 3^x}{\ln 3} - \frac{1}{\ln 3} \int 3^x \, dx = \frac{x 3^x}{\ln 3} - \frac{3^x}{\ln^2 3} + C = \frac{3^x}{\ln 3} \left(x - \frac{1}{\ln 3} \right) + C.$$

8.2.18 Let $u = \ln x$ and $dv = x^9 \, dx$. Then $du = \frac{1}{x} \, dx$ and $v = \frac{1}{10} x^{10}$. Then we have

$$\frac{1}{10} x^{10} \ln x - \frac{1}{10} \int x^9 \, dx = \frac{1}{10} x^{10} \ln x - \frac{1}{100} x^{10} + C = \frac{1}{100} x^{10} (10 \ln x - 1) + C.$$

8.2.19 Let $u = \ln x$ and $dv = x^{-10} \, dx$. Then $du = \frac{1}{x} \, dx$ and $v = -\frac{1}{9} x^{-9}$. Then

$$\int \frac{\ln x}{x^{10}} \, dx = -\frac{1}{9x^9} \ln x + \frac{1}{9} \int x^{-10} \, dx = -\frac{1}{9x^9} \ln x + -\frac{1}{81x^9} + C.$$

8.2.20 Let $u = \sin^{-1} x$ and $dv = dx$. Then $du = \frac{1}{\sqrt{1-x^2}} dx$ and $v = x$. Then

$$\int \sin^{-1} x \, dx = x \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}} dx = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

The fact that $-\int \frac{x}{\sqrt{1-x^2}} dx = \sqrt{1-x^2} + C$ follows from the ordinary substitution $u = 1 - x^2$.

8.2.21

$$\int x \sin x \cos x \, dx = \frac{1}{2} \int x \cdot (2 \sin x \cos x) \, dx = \frac{1}{2} \int x \sin 2x \, dx.$$

Now using the result of problem 10, we have

$$\int x \sin x \cos x \, dx = \frac{1}{2} \cdot \left(-\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right) + C = -\frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C.$$

8.2.22 Let $u = e^x$ and $dv = e^x \sin e^x \, dx$. Then $du = e^x \, dx$ and $v = -\cos e^x$. Then we have

$$-e^x \cos e^x + \int e^x \cos e^x \, dx = -e^x \cos e^x + \sin e^x + C = \sin e^x - e^x \cos e^x + C.$$

8.2.23 Let $u = x^2$ and $dv = \sin 2x \, dx$. Then $du = 2x \, dx$ and $v = -\frac{1}{2} \cos 2x$. Then we have

$$\int x^2 \sin 2x \, dx = -\frac{1}{2} x^2 \cos 2x + \int x \cos 2x \, dx.$$

Now we consider computing this last term $\int x \cos 2x \, dx$ as a new problem. Let $u = x$ and $dv = \cos 2x \, dx$.

Then $du = dx$ and $v = \frac{1}{2} \sin 2x$. So

$$\int x \cos 2x \, dx = \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x \, dx = \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C.$$

Combining these results we have

$$\int x^2 \sin 2x \, dx = -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C.$$

8.2.24 Let $u = x^2$ and $dv = e^{4x} \, dx$. Then $du = 2x \, dx$ and $v = \frac{e^{4x}}{4}$. Then we have

$$\int x^2 e^{4x} \, dx = \frac{1}{4} x^2 e^{4x} - \frac{1}{2} \int x e^{4x} \, dx.$$

Now we consider computing this last integral $\int x e^{4x} \, dx$ as a new problem. Let $u = x$ and $dv = e^{4x} \, dx$.

Then $du = dx$ and $v = \frac{e^{4x}}{4}$. Then we have

$$\int x e^{4x} \, dx = \frac{1}{4} x e^{4x} - \frac{1}{4} \int e^{4x} \, dx = \frac{1}{4} x e^{4x} - \frac{1}{16} e^{4x} + C.$$

Combining these results gives

$$\int x^2 e^{4x} \, dx = \frac{1}{4} x^2 e^{4x} - \frac{1}{8} x e^{4x} + \frac{1}{32} e^{4x} + C = e^{4x} \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32} \right) + C.$$

8.2.25 Let $u = t^2$ and $dv = e^{-t} dt$. Then $du = 2t dt$ and $v = -e^{-t}$. We have

$$\int t^2 e^{-t} dt = -t^2 e^{-t} + 2 \int t e^{-t} dt.$$

To compute this last integral, we let $u = t$ and $dv = e^{-t} dt$. Then

$$\int t e^{-t} dt = -t e^{-t} + \int e^{-t} dt = -t e^{-t} - e^{-t} + C.$$

Putting these results together, we obtain

$$\int t^2 e^{-t} dt = -t^2 e^{-t} + 2(-t e^{-t} - e^{-t}) + C = -e^{-t}(t^2 + 2t + 2) + C.$$

8.2.26 Let $u = t^3$ and $dv = \sin t dt$. Then $du = 3t^2 dt$ and $v = -\cos t$. Then

$$\int t^3 \sin t dt = -t^3 \cos t + 3 \int t^2 \cos t.$$

We then consider the last integral. We now let $u = t^2$ and $dv = \cos t dt$. Then $du = 2t dt$ and $v = \sin t$. We have

$$3 \int t^2 \cos t dt = 3t^2 \sin t - 6 \int t \sin t dt.$$

For the last integral, we let $u = t$ and $dv = \sin t dt$. Then $du = dt$ and $v = -\cos t$. We have

$$-6 \int t \sin t dt = 6t \cos t - 6 \int \cos t = 6t \cos t - 6 \sin t + C.$$

Putting these results together, we have

$$\int t^3 \sin t dt = -t^3 \cos t + 3t^2 \sin t + 6t \cos t - 6 \sin t + C.$$

8.2.27 Let $u = \cos x$ and $dv = e^x dx$. Then $du = -\sin x dx$ and $v = e^x$. We have

$$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx.$$

Now in order to compute the integral which comprises this last term, we let $u = \sin x$ and $dv = e^x dx$. Then $du = \cos x dx$ and $v = e^x$. Thus,

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx.$$

Putting these results together gives

$$\begin{aligned} \int e^x \cos x dx &= e^x \cos x + e^x \sin x - \int e^x \cos x dx \\ 2 \int e^x \cos x dx &= e^x (\cos x + \sin x) + C \\ \int e^x \cos x dx &= \frac{e^x}{2} (\cos x + \sin x) + C. \end{aligned}$$

8.2.28 Let $u = \cos 2x$ and $dv = e^{3x} dx$. Then $du = -2 \sin 2x dx$ and $v = \frac{1}{3} e^{3x}$. We have

$$\int e^{3x} \cos 2x dx = \frac{1}{3} e^{3x} \cos 2x + \frac{2}{3} \int e^{3x} \sin 2x dx.$$

Now in order to compute the integral which comprises this last term, we let $u = \sin 2x$ and $dv = e^{3x} dx$. Then $du = 2 \cos 2x dx$ and $v = \frac{1}{3}e^{3x}$. Thus,

$$\int e^{3x} \sin 2x dx = \frac{1}{3}e^{3x} \sin 2x - \frac{2}{3} \int e^{3x} \cos 2x dx.$$

Putting these results together gives

$$\begin{aligned} \int e^{3x} \cos 2x dx &= \frac{1}{3}e^{3x} \cos 2x + \frac{2}{9}e^{3x} \sin 2x - \frac{4}{9} \int e^{3x} \cos 2x dx \\ \frac{13}{9} \int e^{3x} \cos 2x dx &= \frac{1}{3}e^{3x} \left(\cos 2x + \frac{2}{3} \sin 2x \right) + C \\ \int e^{3x} \cos 2x dx &= \frac{3}{13}e^{3x} \left(\cos 2x + \frac{2}{3} \sin 2x \right) + C. \end{aligned}$$

8.2.29 Let $u = \sin 4x$ and $dv = e^{-x} dx$. Then $du = 4 \cos 4x dx$ and $v = -e^{-x}$. We have

$$\int e^{-x} \sin 4x dx = -e^{-x} \sin 4x + 4 \int e^{-x} \cos 4x dx.$$

Now in order to compute the integral which comprises this last term, we let $u = \cos 4x$ and $dv = e^{-x} dx$. Then $du = -4 \sin 4x dx$ and $v = -e^{-x}$. Thus,

$$\int e^{-x} \cos 4x dx = -e^{-x} \cos 4x - 4 \int e^{-x} \sin 4x dx.$$

Putting these results together gives

$$\begin{aligned} \int e^{-x} \sin 4x dx &= -e^{-x} \sin 4x - 4e^{-x} \cos 4x - 16 \int e^{-x} \sin 4x dx \\ 17 \int e^{-x} \sin 4x dx &= -e^{-x} \sin 4x - 4e^{-x} \cos 4x + C \\ \int e^{-x} \sin 4x dx &= -\frac{e^{-x}}{17} (\sin 4x + 4 \cos 4x) + C. \end{aligned}$$

8.2.30 Let $u = \sin 6\theta$ and $dv = e^{-2\theta} d\theta$. Then $du = 6 \cos 6\theta d\theta$ and $v = -\frac{1}{2}e^{-2\theta}$. We have

$$\int e^{-2\theta} \sin 6\theta d\theta = -\frac{1}{2}e^{-2\theta} \sin 6\theta + 3 \int e^{-2\theta} \cos 6\theta d\theta.$$

Now in order to compute the integral which comprises this last term, we let $u = \cos 6\theta$ and $dv = e^{-2\theta} d\theta$. Then $du = -6 \sin 6\theta d\theta$ and $v = -\frac{1}{2}e^{-2\theta}$. Thus,

$$\int e^{-2\theta} \cos 6\theta d\theta = -\frac{1}{2}e^{-2\theta} \cos 6\theta - 3 \int e^{-2\theta} \sin 6\theta d\theta.$$

Putting these results together gives

$$\begin{aligned} \int e^{-2\theta} \sin 6\theta d\theta &= -\frac{1}{2}e^{-2\theta} \sin 6\theta - \frac{3}{2}e^{-2\theta} \cos 6\theta - 9 \int e^{-2\theta} \sin 6\theta d\theta \\ 10 \int e^{-2\theta} \sin 6\theta d\theta &= -\frac{1}{2}e^{-2\theta} (\sin 6\theta + 3 \cos 6\theta) + C \\ \int e^{-2\theta} \sin 6\theta d\theta &= -\frac{e^{-2\theta}}{20} (\sin 6\theta + 3 \cos 6\theta) + C. \end{aligned}$$

8.2.31 Let $u = e^{2x}$ and $dv = e^x \sin e^x dx$. Then $du = 2e^{2x} dx$ and $v = -\cos e^x$. Then we have

$$\int e^{3x} \sin e^x dx = -e^{2x} \cos e^x + 2 \int e^{2x} \cos e^x dx.$$

To compute the last integral, we let $u = e^x$ and $dv = e^x \cos e^x dx$. Then $du = e^x dx$ and $v = \sin e^x$. Then the last integral is

$$2 \int e^{2x} \cos e^x dx = 2e^x \sin e^x - 2 \int e^x \sin e^x dx = 2e^x \sin e^x + 2 \cos e^x + C.$$

Combining these results gives a final answer of

$$\int e^{3x} \sin e^x dx = -e^{2x} \cos e^x + 2e^x \sin e^x + 2 \cos e^x + C.$$

8.2.32 Let $u = x^2$ and $dv = 2^x dx$, so that $du = 2x dx$ and $v = \frac{2^x}{\ln 2}$. Then we have

$$\left. \frac{x^2 2^x}{\ln 2} \right|_0^1 - \frac{2}{\ln 2} \int_0^1 x 2^x dx = \frac{2}{\ln 2} - \frac{2}{\ln 2} \int_0^1 x 2^x dx.$$

Now we let $u = x$ and $dv = 2^x dx$, so that $du = dx$ and $v = \frac{2^x}{\ln 2}$. The given integral is then equal to

$$\begin{aligned} \frac{2}{\ln 2} - \frac{2}{\ln 2} \left(\left. \frac{x 2^x}{\ln 2} \right|_0^1 - \frac{1}{\ln 2} \int_0^1 2^x dx \right) &= \frac{2}{\ln 2} - \frac{2}{\ln 2} \left(\frac{2}{\ln 2} - \frac{1}{\ln 2} \left(\left. \frac{2^x}{\ln 2} \right|_0^1 \right) \right) \\ &= \frac{2}{\ln 2} - \frac{2}{\ln 2} \left(\frac{2}{\ln 2} - \left(\frac{2}{\ln^2 2} - \frac{1}{\ln^2 2} \right) \right). \end{aligned}$$

This can be written as

$$\frac{2}{\ln 2} - \frac{2}{\ln 2} \left(\frac{2}{\ln 2} - \frac{1}{\ln^2 2} \right) = \frac{2 \ln^2 2}{\ln^3 2} - \frac{2(2 \ln 2 - 1)}{\ln^3 2} = \frac{2(\ln 2 - 1)^2}{\ln^3 2}.$$

8.2.33 Let $u = x$ and $dv = \sin x dx$. Then $du = dx$ and $v = -\cos x$. Then

$$\int_0^\pi x \sin x dx = -x \cos x \Big|_0^\pi + \int_0^\pi \cos x dx = \pi - 0 + \sin x \Big|_0^\pi = \pi - 0 + 0 - 0 = \pi.$$

8.2.34 First note that $\int_1^e \ln 2x dx = \int_1^e \ln 2 dx + \int_1^e \ln x dx = \ln 2(e-1) + \int_1^e \ln x dx$.

Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$. Then

$$\int_1^e \ln x dx = x \ln x \Big|_1^e - \int_1^e dx = e - (e-1) = 1.$$

Thus $\int_1^e \ln 2x dx = \ln 2(e-1) + 1$.

8.2.35 Let $u = x$ and $dv = \cos 2x dx$. Then $du = dx$ and $v = \frac{1}{2} \sin 2x$. Then

$$\int_0^{\pi/2} x \cos 2x dx = \frac{1}{2} x \sin 2x \Big|_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \sin 2x dx = 0 - \left(\frac{1}{2} \cdot \frac{(-\cos 2x)}{2} \right) \Big|_0^{\pi/2} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

8.2.36 Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$. Then

$$\int_0^{\ln 2} x e^x dx = x e^x \Big|_0^{\ln 2} - \int_0^{\ln 2} e^x dx = 2 \ln 2 - (e^x) \Big|_0^{\ln 2} = 2 \ln 2 - (2 - 1) = 2 \ln 2 - 1.$$

8.2.37 Let $u = \ln x$ and $dv = x^2 dx$. Then $du = \frac{1}{x} dx$ and $v = \frac{x^3}{3}$. Then

$$\int_1^{e^2} x^2 \ln x dx = \frac{1}{3} x^3 \ln x \Big|_1^{e^2} - \frac{1}{3} \int_1^{e^2} x^2 dx = \frac{2}{3} e^6 - \frac{1}{9} x^3 \Big|_1^{e^2} = \frac{2}{3} e^6 - \frac{1}{9} (e^6 - 1) = \frac{5}{9} e^6 + \frac{1}{9}.$$

8.2.38 Let $u = \ln^2 x$ and $dv = x^2 dx$. Then $du = \frac{2 \ln x}{x}$ and $v = \frac{x^3}{3}$. We have

$$\int x^2 \ln^2 x dx = \frac{1}{3} x^3 \ln^2 x - \frac{2}{3} \int x^2 \ln x dx.$$

To compute this last integral, let $u = \ln x$ and $dv = x^2 dx$. Then $du = \frac{1}{x} dx$ and $v = \frac{x^3}{3}$. Then

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C = \frac{x^3}{9} (3 \ln x - 1) + C.$$

$$\text{Thus, } \int x^2 \ln^2 x dx = \frac{x^3}{3} \ln^2 x - \frac{2x^3}{27} (3 \ln x - 1) + C = \frac{x^3}{27} (9 \ln^2 x - 6 \ln x + 2) + C.$$

8.2.39 By problem 20, $\int \sin^{-1} y dy = y \sin^{-1} y + \sqrt{1 - y^2}$. Thus,

$$\int_0^1 \sin^{-1} y dy = \left(y \sin^{-1} y + \sqrt{1 - y^2} \right) \Big|_0^1 = \left(\frac{\pi}{2} + 0 \right) - (0 + 1) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}.$$

8.2.40 Let $u = \sqrt{x}$ and $dv = \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$. Then $du = \frac{dx}{2\sqrt{x}}$ and $v = 2e^{\sqrt{x}}$. Then

$$\int e^{\sqrt{x}} dx = \int \frac{\sqrt{x} e^{\sqrt{x}}}{\sqrt{x}} dx = 2\sqrt{x} e^{\sqrt{x}} - \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2\sqrt{x} e^{\sqrt{x}} - 2e^{\sqrt{x}} + C = 2e^{\sqrt{x}} (\sqrt{x} - 1) + C.$$

8.2.41

a. Let $u = \tan^{-1} x$ and $dv = dx$. Then $du = \frac{dx}{1 + x^2}$ and $v = x$. Then

$$\int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{1 + x^2} dx = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C.$$

b. Let $u = x^2$. Then $du = 2x dx$. Then

$$\int x \tan^{-1} x^2 dx = \int \frac{1}{2} \tan^{-1} u du = \frac{1}{2} u \tan^{-1} u - \frac{1}{4} \ln(1 + u^2) + C = \frac{1}{2} x^2 \tan^{-1} x^2 - \frac{1}{4} \ln(1 + x^4) + C.$$

8.2.42 The volume is given by $V = \int_0^1 2\pi y e^y dy$. Let $u = y$ and $dv = e^y dy$. Then $du = dy$ and $v = e^y$. Then

$$V = 2\pi (y e^y) \Big|_0^1 - 2\pi \int_0^1 e^y dy = 2\pi (e - 0) - 2\pi (e^y) \Big|_0^1 = 2\pi e - 2\pi (e - 1) = 2\pi.$$

8.2.43 Using shells, we have $\frac{V}{2\pi} = \int_0^{\ln 2} x e^{-x} dx$. Let $u = x$ and $dv = e^{-x} dx$, so that $du = dx$ and $v = -e^{-x}$. Then

$$\frac{V}{2\pi} = -x e^{-x} \Big|_0^{\ln 2} + \int_0^{\ln 2} e^{-x} dx = -\frac{1}{2} \ln 2 - e^{-x} \Big|_0^{\ln 2} = -\frac{\ln 2}{2} - \left(\frac{1}{2} - 1\right) = \frac{1}{2}(1 - \ln 2).$$

Thus $V = \pi(1 - \ln 2)$.

8.2.44 Using shells, we have $\frac{V}{2\pi} = \int_0^{\pi} x \sin x dx$. Let $u = x$ and $dv = \sin x dx$, so that $du = dx$ and $v = -\cos x$. Then

$$\frac{V}{2\pi} = -x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x dx = \pi + \sin x \Big|_0^{\pi} = \pi.$$

Thus $V = 2\pi^2$.

8.2.45 We have $V = \int_1^e \pi \ln x dx = \pi(x \ln x - x) \Big|_1^e = \pi(e - e) - \pi(0 - 1) = \pi$.

8.2.46 Using shells, we have $\frac{V}{2\pi} = \int_0^{\ln 2} (\ln 2 - x) e^{-x} dx = \ln 2 \int_0^{\ln 2} e^{-x} dx - \int_0^{\ln 2} x e^{-x} dx$.

In the course of solving problem 43, we deduced that $\int_0^{\ln 2} x e^{-x} dx = \frac{1 - \ln 2}{2}$. Thus,

$$\frac{V}{2\pi} = \ln 2 (-e^{-x}) \Big|_0^{\ln 2} - \frac{1 - \ln 2}{2} = \ln 2 \left(-\frac{1}{2} + 1\right) - \frac{1 - \ln 2}{2} = \ln 2 - \frac{1}{2}.$$

Thus, $V = 2\pi(\ln 2 - \frac{1}{2}) = \pi(\ln 4 - 1)$.

8.2.47 Using disks, we have $\frac{V}{\pi} = \int_1^{e^2} x^2 \ln^2 x dx$. By problem 38, we have $\int x^2 \ln^2 x dx = \frac{1}{3} x^3 \ln^2 x - \frac{2}{9} x^3 \ln x + \frac{2}{27} x^3 + C$. Thus,

$$\frac{V}{\pi} = \left(\frac{1}{3} x^3 \ln^2 x - \frac{2}{9} x^3 \ln x + \frac{2}{27} x^3 \right) \Big|_1^{e^2} = \left(\frac{4}{3} e^6 - \frac{4}{9} e^6 + \frac{2}{27} e^6 \right) - \left(\frac{2}{27} \right) = \frac{26}{27} e^6 - \frac{2}{27}.$$

Thus, $V = \frac{\pi}{27} (26e^6 - 2)$.

8.2.48 Let $u = \sec x$ and $dv = \sec^2 x dx$, so that $du = \sec x \tan x dx$ and $v = \tan x$. Then $\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$.

Thus $2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx$, so

$$\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x dx.$$

8.2.49

a. False. For example, suppose $u = x$ and $dv = x dx$. Then $\int uv' dx = \int x^2 dx = \frac{x^3}{3} + C$, but

$$\int u dx \int v' dx = \left(\int x dx \right)^2 = \left(\frac{x^2}{2} + C \right)^2.$$

b. True. This is one way to write the integration by parts formula.

c. True. This is the integration by parts formula with the roles of u and v reversed.

8.2.50 Let $u = x^n$ and $dv = e^{ax} dx$. Then $du = nx^{n-1} dx$ and $v = \frac{e^{ax}}{a}$. Then

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx.$$

8.2.51 Let $u = x^n$ and $dv = \cos ax dx$. Then $du = nx^{n-1} dx$ and $v = \frac{\sin ax}{a}$. Then

$$\int x^n \cos ax dx = \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} \sin ax dx.$$

8.2.52 Let $u = x^n$ and $dv = \sin ax dx$. Then $du = nx^{n-1} dx$ and $v = -\frac{\cos ax}{a}$. Then

$$\int x^n \sin ax dx = -\frac{x^n \cos ax}{a} + \frac{n}{a} \int x^{n-1} \cos ax dx.$$

8.2.53 Let $u = \ln^n x$ and $dv = dx$. Then $du = \frac{n \ln^{n-1}(x)}{x} dx$ and $v = x$. Then

$$\int \ln^n(x) dx = x \ln^n x - n \int \ln^{n-1}(x) dx.$$

8.2.54

$$\begin{aligned} \int x^2 e^{3x} dx &= \frac{x^2 e^{3x}}{3} - \frac{2}{3} \int x e^{3x} dx \\ &= \frac{x^2 e^{3x}}{3} - \frac{2}{3} \left(\frac{x e^{3x}}{3} - \frac{1}{3} \int e^{3x} dx \right) \\ &= \frac{1}{3} \left(x^2 e^{3x} - \frac{2}{3} x e^{3x} + \frac{2}{9} e^{3x} \right) + C \\ &= \frac{e^{3x}}{3} \left(x^2 - \frac{2}{3} x + \frac{2}{9} \right) + C. \end{aligned}$$

8.2.55

$$\begin{aligned} \int x^2 \cos 5x dx &= \frac{x^2 \sin 5x}{5} - \frac{2}{5} \int x \sin 5x dx \\ &= \frac{x^2 \sin 5x}{5} - \frac{2}{5} \left(-\frac{x \cos 5x}{5} + \frac{1}{5} \int \cos 5x dx \right) \\ &= \frac{1}{5} \left(x^2 \sin 5x + \frac{2}{5} x \cos 5x - \frac{2}{25} \sin 5x \right) + C. \end{aligned}$$

8.2.56

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + 3 \int x^2 \cos x dx \\ &= -x^3 \cos x + 3 \left(x^2 \sin x - 2 \int x \sin x dx \right) \\ &= -x^3 \cos x + 3x^2 \sin x - 6 \left(-x \cos x + \int \cos x dx \right) \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C. \end{aligned}$$

8.2.57

$$\begin{aligned}
 \int_1^e \ln^3 x \, dx &= x \ln^3 x \Big|_1^e - 3 \int_1^e \ln^2 x \, dx \\
 &= e - 3 \left(x \ln^2 x \right) \Big|_1^e + 6 \int_1^e \ln x \, dx \\
 &= e - 3e + 6 \left(x \ln x - x \right) \Big|_1^e \\
 &= e - 3e + 6((e - e) - (0 - 1)) = 6 - 2e.
 \end{aligned}$$

8.2.58

a. Let $u = \ln \cos x$ and $dv = \sin x \, dx$. Then $du = -\tan x \, dx$ and $v = -\cos x$. We have

$$\begin{aligned}
 \int_0^{\pi/3} \sin x \ln(\cos x) \, dx &= -\cos x \ln \cos x \Big|_0^{\pi/3} - \int_0^{\pi/3} \sin x \, dx = -\frac{1}{2} \ln \frac{1}{2} + \left(\cos x \Big|_0^{\pi/3} \right) \\
 &= -\frac{1}{2} \ln \frac{1}{2} + \frac{1}{2} - 1 = \frac{1}{2} \ln 2 - \frac{1}{2} - \frac{\ln 2 - 1}{2}.
 \end{aligned}$$

b. Let $u = \cos x$. Then $du = -\sin x \, dx$. Substituting gives

$$-\int_1^{1/2} \ln u \, du = (u \ln u - u) \Big|_{1/2}^1 = (0 - 1) - \left(\frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \right) = \frac{\ln 2 - 1}{2}.$$

8.2.59

a. Let $u = x$ and $dv = \frac{dx}{\sqrt{x+1}}$. Then $du = dx$ and $v = 2\sqrt{x+1}$. Then

$$\int \frac{x}{\sqrt{x+1}} \, dx = 2x\sqrt{x+1} - \int 2\sqrt{x+1} \, dx = 2x\sqrt{x+1} - \frac{4}{3}(x+1)^{3/2} + C = \frac{2}{3}\sqrt{x+1}(x-2) + C.$$

b. Let $u = x + 1$. Then $du = dx$ and $x = u - 1$. Substituting gives

$$\int \frac{x}{\sqrt{x+1}} \, dx = \int \frac{u-1}{\sqrt{u}} \, du = \int (u^{1/2} - u^{-1/2}) \, du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C.$$

c. The answer to b can be written as the answer to a:

$$\frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C = \frac{2}{3}(x+1)^{1/2}(x+1-3) + C = \frac{2}{3}\sqrt{x+1}(x-2) + C.$$

8.2.60

a. Let $u = x^2$, so that $du = 2x \, dx$. Then

$$\int x \ln x^2 \, dx = \frac{1}{2} \int \ln u \, du = \frac{1}{2} (u \ln u - u) + C = \frac{1}{2} (x^2 \ln(x^2) - x^2) + C.$$

b. Let $u = \ln x$ and $dv = x \, dx$. Then $du = \frac{1}{x} \, dx$ and $v = \frac{x^2}{2}$. Then

$$\int x \ln x^2 \, dx = 2 \int x \ln x \, dx = 2 \left(\frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx \right) = x^2 \ln x - \frac{x^2}{2} + C.$$

c. The answer to the first part is $\frac{1}{2} (x^2 \ln(x^2) - x^2) + C = x^2 \ln(x) - \frac{x^2}{2} + C$, which is the answer to the second part.

8.2.61 Using the change of base formula, we have $\int \log_b x \, dx = \int \frac{\ln x}{\ln b} \, dx = \frac{1}{\ln b} (x \ln x - x) + C$.

8.2.62 By parts: Let $u = \sin x$ and $dv = \cos x \, dx$, so that $du = \cos x \, dx$ and $v = \sin x$. Then

$$\int \sin x \cos x \, dx = \sin^2 x - \int \sin x \cos x \, dx,$$

$$\text{so } \int \sin x \cos x \, dx = \frac{\sin^2 x}{2} + C.$$

By substitution: Let $u = \sin x$, so that $du = \cos x \, dx$. Then we have $\int \sin x \cos x \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$. The two answers are the same.

8.2.63 Let $z = \sqrt{x}$, so that $dz = \frac{1}{2\sqrt{x}} \, dx$. Substituting yields $2 \int \frac{\sqrt{x} \cos \sqrt{x}}{2\sqrt{x}} \, dx = 2 \int z \cos z \, dz$. Now let $u = z$ and $dv = \cos z \, dz$. then $du = dz$ and $v = \sin z$. Then by Integration by Parts, we have

$$2 \int z \cos z \, dz = 2 \left(z \sin z - \int \sin z \, dz \right) = 2z \sin z + 2 \cos z + C.$$

Thus, the original given integral is equal to $2(\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x}) + C$.

8.2.64 Let $z = \sqrt{x}$, so that $dz = \frac{1}{2\sqrt{x}} \, dx$. Substituting yields $\int_0^{\pi^2/4} \sin \sqrt{x} \, dx = 2 \int_0^{\pi/2} z \sin z \, dz$. Now let $u = z$ and $dv = \sin z \, dz$. Then $du = dz$ and $v = -\cos z$. Then by Integration by Parts, we have

$$2 \int_0^{\pi/2} z \sin z \, dz = 2(-z \cos z) \Big|_0^{\pi/2} + 2 \int_0^{\pi/2} \cos z \, dz = 0 + 2 \sin z \Big|_0^{\pi/2} = 2.$$

8.2.65 Let $u = x$ and $dv = f''(x) \, dx$. Then $du = dx$ and $v = f'(x)$. We have

$$\int_a^b x f''(x) \, dx = x f'(x) \Big|_a^b - \int_a^b f'(x) \, dx = (b f'(b) - a f'(a)) - (f(b) - f(a)) = f(a) - f(b).$$

8.2.66 Let $u = f(x)$ and $dv = f'(x) \, dx$. Then $du = f'(x) \, dx$ and $v = f(x)$. We have

$$\int_a^b f(x) f'(x) \, dx = f(x)^2 \Big|_a^b - \int_a^b f(x) f'(x) \, dx.$$

Thus, $2 \int_a^b f(x) f'(x) \, dx = f(x)^2 \Big|_a^b$, so $\int_a^b f(x) f'(x) \, dx = \frac{1}{2} f(x)^2 \Big|_a^b = \frac{1}{2} (f(b)^2 - f(a)^2)$.

8.2.67 By the Fundamental Theorem, $f'(x) = \sqrt{\ln^2 x - 1}$. So the arc length is $\int_e^{e^3} \sqrt{1 + (f'(x))^2} \, dx = \int_e^{e^3} \ln x \, dx = (x \ln x - x) \Big|_e^{e^3} = 3e^3 - e^3 - (e - e) = 2e^3$.

8.2.68 Suppose $m \neq -1$ and let $u = \ln x$ and $dv = x^m \, dx$. Then $du = \frac{1}{x} \, dx$ and $v = \frac{x^{m+1}}{m+1}$. Then $\int x^m \ln x \, dx = \frac{x^{m+1}}{m+1} \ln x - \frac{1}{m+1} \int x^m \, dx = \frac{x^{m+1}}{m+1} \left(\ln x - \frac{1}{m+1} \right) + C$.

For the case $m = -1$ we are computing $\int \frac{1}{x} \ln x \, dx$, so letting $u = \ln x$ so that $du = \frac{1}{x} \, dx$ yields $\int u \, du = \frac{u^2}{2} + C = \frac{\ln^2 x}{2} + C$.

8.2.69 Let V_1 be the volume generated when R is revolved about the x -axis, and V_2 the volume generated when R is revolved about the y -axis.

Using disks, we have

$$\frac{V_1}{\pi} = \int_0^\pi \sin^2 x \, dx = \frac{1}{2} \int_0^\pi (1 - \cos 2x) \, dx = \frac{1}{2} \left(\pi - \frac{1}{2} \int_0^\pi 2 \cos 2x \, dx \right) = \frac{1}{2} \left(\pi - \frac{1}{2} \int_0^{2\pi} \cos u \, du \right) = \frac{\pi}{2},$$

where the ordinary substitution $u = 2x$ was made. So $V_1 = \frac{\pi^2}{2}$.

Using shells to compute V_2 , we have $\frac{V_2}{2\pi} = \int_0^\pi x \sin x \, dx$. Letting $u = x$ and $dv = \sin x \, dx$, we have $du = dx$ and $v = -\cos x$. Thus, $\frac{V_2}{2\pi} = -x \cos x \Big|_0^\pi + \int_0^\pi \cos x \, dx = \pi$. Thus, $V_2 = 2\pi^2$, and $V_2 > V_1$.

8.2.70

a. Let $u = x$ and $dv = f'(x) \, dx$. Then $du = dx$ and $v = f(x)$. So $\int x f'(x) \, dx = x f(x) - \int f(x) \, dx$.

b. Letting $f'(x) = e^{3x}$ we have $\int x e^{3x} \, dx = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C$.

8.2.71 Using shells, we have $\frac{V}{2\pi} = \int_0^{\pi/2} x \cos x \, dx$. Let $u = x$ and $dv = \cos x \, dx$, so that $du = dx$ and $v = \sin x$. We have $\frac{V}{2\pi} = x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} - 1$.
Thus, $V = \pi(\pi - 2)$.

8.2.72 We are looking to compute $\int_0^{1/2} (\sin^{-1} x - \sin x) \, dx = \int_0^{1/2} \sin^{-1} x \, dx + \cos 1 - \cos(1/2)$.

Consider $\int \sin^{-1} x$. Let $u = \sin^{-1} x$ and $dv = dx$. Then $du = \frac{1}{\sqrt{1-x^2}} \, dx$ and $v = x$. Then

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx.$$

Now we perform the ordinary substitution $u = 1 - x^2$, so that $du = -2x \, dx$. We have

$$x \sin^{-1} x + \frac{1}{2} \int u^{-1/2} \, du = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

Thus $\int_0^{1/2} \sin^{-1} x \, dx = \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} \right) - 1$, and

$$\int_{1/2}^1 (\sin^{-1} x - \sin x) \, dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 - (\cos(1/2) - 1) = \frac{\pi}{12} + \frac{\sqrt{3}}{2} + \cos(1/2) - 2.$$

8.2.73 Let $u = \sin bx$ and $dv = e^{ax} \, dx$ so that $du = b \cos bx \, dx$ and $v = \frac{e^{ax}}{a}$. Then $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a} \sin bx - \frac{b}{a} \int e^{ax} \cos bx \, dx$. Now let $u = \cos bx$ and $dv = e^{ax}$, so that $du = -b \sin bx$ and $v = \frac{e^{ax}}{a}$. Then $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx$. Putting these together yields

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a} \sin bx - \frac{b}{a} \left(\frac{e^{ax}}{a} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx \right).$$

Multiplying through by a^2 and combining like terms yields

$$(a^2 + b^2) \int e^{ax} \sin bx \, dx = e^{ax} (a \sin bx - b \cos bx) + C,$$

so $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$ as desired.

Now, returning to the integral mentioned above, and incorporating this last result, we have

$$\begin{aligned} \int e^{ax} \cos bx \, dx &= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \cdot e^{ax} \cdot \frac{a \sin bx - b \cos bx}{a^2 + b^2} + C = \frac{e^{ax}}{a} \cdot \frac{(a^2 + b^2) \cos bx + b(a \sin bx - b \cos bx)}{a^2 + b^2} + C \\ &= e^{ax} \cdot \frac{a \cos bx + b \sin bx}{a^2 + b^2} + C. \end{aligned}$$

8.2.74

a. Let $y = f^{-1}(x)$, so that $dy = \frac{1}{f'(y)} dx$, $f'(y) \neq 0$. Then $\int f^{-1}(x) \, dx = \int f'(y) f^{-1}(x) \frac{1}{f'(y)} \, dx = \int y f'(y) \, dy$.

b. Using the result of exercise 73, $\int f^{-1}(x) \, dx = \int y f'(y) \, dy = y f(y) - \int f(y) \, dy$.

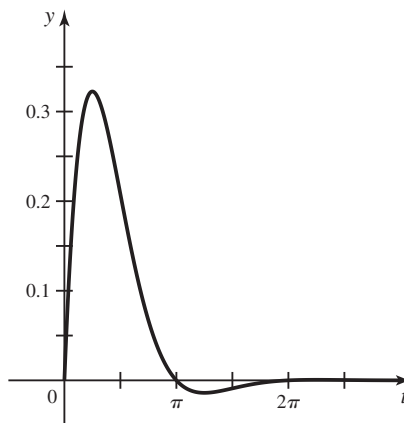
c. $\int \ln x \, dx = x \ln x - \int e^y \, dy = x \ln x - e^y + C = x \ln x - x + C$.

d. $\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \sin y \, dy = x \sin^{-1} x + \cos y + C = x \sin^{-1} x + \sqrt{1 - x^2} + C$.

e. $\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \tan y \, dy = x \tan^{-1} x + \ln |\cos y| + C = x \tan^{-1} x + \ln \left(\frac{1}{\sqrt{1 + x^2}} \right) + C = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C$.

8.2.75

We have $s(t) = 0$ when $\sin t = 0$, which
a. occurs for $t = k\pi$, where k is an integer.



b. This is given by $\frac{1}{\pi} \int_0^\pi e^{-t} \sin t \, dt$. Using the previous problem, this is equal to

$$\frac{1}{\pi} e^{-t} \cdot \frac{-\sin t - \cos t}{2} \Big|_0^\pi = \frac{1}{2\pi} (e^{-\pi} + 1).$$

c. This is given by $\frac{1}{\pi} \int_{n\pi}^{(n+1)\pi} e^{-t} \sin t \, dt$. Using the previous problem, this is equal to

$$\begin{aligned} \frac{1}{\pi} e^{-t} \cdot \frac{-\sin t - \cos t}{2} \Big|_{n\pi}^{(n+1)\pi} &= \frac{1}{2\pi} \left(e^{-(n+1)\pi} (-\sin(n+1)\pi - \cos(n+1)\pi) - e^{-n\pi} (-\sin n\pi - \cos n\pi) \right) \\ &= \frac{1}{2\pi} \left(-e^{-(n+1)\pi} \cos(n+1)\pi + e^{-n\pi} \cos n\pi \right) = \frac{e^{-n\pi}}{2\pi} (\cos n\pi - e^{-\pi} \cos(n+1)\pi) \\ &= \frac{e^{-n\pi}}{2\pi} ((-1)^n - e^{-\pi}(-1)^{n+1}) = (-1)^n \frac{e^{-n\pi}}{2\pi} (1 + e^{-\pi}). \end{aligned}$$

d. Each a_n is $e^{-\pi}$ times a_{n-1} .

8.2.76 The fallacy comes from thinking that $\int \frac{dx}{x}$ represents a unique quantity, as opposed to a family of functions which differ by a constant. When we subtract this quantity from both sides, we should write the arbitrary constant, giving $0 + C_1 = 1 + C_2$, which is a valid statement.

8.2.77

a. If $u = f(x)$ then $du = f'(x) \, dx$, and if $dv = g(x) \, dx$, then $v = G_1(x)$. The integration by parts formula gives

$$\int f(x)g(x) \, dx = f(x)G_1(x) - \int f'(x)G_1(x) \, dx.$$

b. Letting $u = f'(x)$ yields $du = f''(x) \, dx$ and letting $dv = G_1(x) \, dx$ yields $v = G_2(x)$. Then continuing from part (a) we have

$$\begin{aligned} \int f(x)g(x) \, dx &= f(x)G_1(x) - \left(f'(x)G_2(x) - \int f''(x)G_2(x) \, dx \right) \\ &= f(x)G_1(x) - f'(x)G_2(x) + \int f''(x)G_2(x) \, dx. \end{aligned}$$

Note that we end up with three terms which consist of products as indicated by the arrows, with the last product inside an integral sign. The signs alternate as indicated on the chart.

c. For the trailing integral from part (b), we let $u = f''(x)$ so that $du = f'''(x) \, dx$, and let $dv = G_2(x) \, dx$ so that $v = G_3(x)$. Then we have

$$\begin{aligned} \int f(x)g(x) \, dx &= f(x)G_1(x) - f'(x)G_2(x) + \int f''(x)G_2(x) \, dx \\ &= f(x)G_1(x) - f'(x)G_2(x) + f''(x)G_3(x) - \int f'''(x)G_3(x) \, dx. \end{aligned}$$

<i>f</i> and its derivatives		<i>g</i> and its integrals
$f(x)$	$\searrow +$	$g(x)$
$f'(x)$	$\searrow -$	$G_1(x)$
$f''(x)$	$\searrow +$	$G_2(x)$
$f'''(x)$	$\searrow -$	$G_3(x)$

d. The table is as follows:

<i>f</i> and its derivatives		<i>g</i> and its integrals
x^2	$\searrow +$	$e^{0.5x}$
$2x$	$\searrow -$	$2e^{0.5x}$
2	$\searrow +$	$4e^{0.5x}$
0	$\searrow -$	$8e^{0.5x}$

So

$$\int x^2 e^{0.5x} dx = 2x^2 e^{0.5x} - 8x e^{0.5x} + 16e^{0.5x} + C.$$

Note that the last integral is $\int 0 \cdot 8e^{0.5x} dx = \int 0 dx = C$.

e. We have:

f and its derivatives	g and its integrals
x^3 +	$\cos x$
$3x^2$ -	$\sin x$
$6x$ +	$-\cos x$
6 -	$-\sin x$
0 +	$\cos x$

$$\text{So } \int x^3 \cos x dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C.$$

f. $\frac{d^k}{dx^k}(p_n(x)) = 0$ for $k \geq n + 1$, so the process will end.

8.2.78

a. The chart looks like this:

f and its derivatives	g and its integrals
x^4 +	e^x
$4x^3$ -	e^x
$12x^2$ +	e^x
$24x$ -	e^x
24 +	e^x

$$\text{Thus } \int x^4 e^x dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + C.$$

b. The chart looks like this:

f and its derivatives	g and its integrals
$7x$ +	e^{3x}
7 -	$\frac{1}{3}e^{3x}$
0 +	$\frac{1}{9}e^{3x}$

$$\text{Thus } \int 7x e^{3x} dx = \frac{7}{3}x e^{3x} - \frac{7}{9}e^{3x} + C.$$

c. The chart looks like this:

f and its derivatives	g and its integrals
$2x^2$ +	$\sqrt{x+1}$
$4x$ -	$\frac{2}{3}(x+1)^{3/2}$
4 +	$\frac{4}{15}(x+1)^{5/2}$
0 -	$\frac{8}{105}(x+1)^{7/2}$

Thus $\int 2x^2\sqrt{x+1} dx = \frac{4}{3}x^2(x+1)^{3/2} - \frac{16}{15}x(x+1)^{5/2} + \frac{32}{105}(x+1)^{7/2} + C$. Evaluating at -1 and 0 gives

$$\int_{-1}^0 2x^2\sqrt{x+1} = \frac{32}{105}.$$

d. The chart looks like this:

f and its derivatives	g and its integrals
$x^3 - 2x$ +	$\sin 2x$
$3x^2 - 2$ -	$-\frac{1}{2}\cos 2x$
$6x$ +	$-\frac{1}{4}\sin 2x$
6 -	$\frac{1}{8}\cos 2x$
0 +	$\frac{1}{16}\sin 2x$

$$\text{Thus } \int (x^3 - 2x) \sin 2x dx = -\frac{1}{2}(x^3 - 2x) \cos 2x + \frac{1}{4}(3x^2 - 2) \sin 2x + \frac{3}{4}x \sin 2x - \frac{3}{8} \sin 2x + C.$$

e. The chart looks like this:

f and its derivatives	g and its integrals
$2x^2 - 3x$ +	$(x-1)^{-3}$
$4x - 3$ -	$-\frac{1}{2}(x-1)^{-2}$
4 +	$\frac{1}{2}(x-1)^{-1}$
0 -	$\frac{1}{2}\ln x-1 $

$$\text{Thus } \int \frac{2x^2 - 3x}{(x-1)^3} dx = -\frac{1}{2}(2x^2 - 3x)(x-1)^{-2} - \frac{4x-3}{2}(x-1)^{-1} + 2\ln|x-1| + C.$$

f. The chart looks like this:

f and its derivatives	g and its integrals
$x^2 + 3x + 4$ +	$(2x+1)^{-1/3}$
$2x + 3$ -	$\frac{3}{4}(2x+1)^{2/3}$
2 +	$\frac{9}{40}(2x+1)^{5/3}$
0 -	$\frac{27}{640}(2x+1)^{8/3}$

$$\text{Thus } \int \frac{x^2 + 3x + 4}{\sqrt[3]{2x+1}} dx = \frac{3}{4}(x^2 + 3x + 4)(2x+1)^{2/3} - \frac{9}{40}(2x+3)(2x+1)^{5/3} + \frac{27}{320}(2x+1)^{8/3} + C.$$

g. The chart looks like this:

f and its derivatives	g and its integrals
x +	$\frac{1}{\sqrt{1-x^2}}$
1 -	$\sin^{-1} x$
0 +	$x \sin^{-1} x + \sqrt{1-x^2}$

So $\int \frac{x}{\sqrt{1-x^2}} dx = x \sin^{-1} x - (x \sin^{-1} x + \sqrt{1-x^2}) + C = -\sqrt{1-x^2} + C$. However, this is a little ridiculous because finding the integrals in the 2nd column of the table is more difficult than just integrating the given function using the substitution $u = 1 - x^2$. This gives $du = -2x dx$, so we have $-\frac{1}{2} \int u^{-1/2} du = -\sqrt{u} + C = -\sqrt{1-x^2} + C$.

8.2.79

a. $\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$

b. Adding $\int e^x \cos x \, dx$ to both sides of the previous equation gives $2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$,
so

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

c. The chart looks like this:

f and its derivatives	g and its integrals
e^{-2x} +	$\sin 3x$
$-2e^{-2x}$ -	$-\frac{1}{3} \cos 3x$
$4e^{-2x}$ +	$-\frac{1}{9} \sin 3x$

So we have

$$\int e^{-2x} \sin 3x \, dx = -\frac{1}{3} e^{-2x} \cos 3x - \frac{2}{9} e^{-2x} \sin 3x - \frac{4}{9} e^{-2x} \sin 3x.$$

Adding $\frac{4}{9} \int e^{-2x} \sin 3x \, dx$ to both sides gives

$$\frac{13}{9} \int e^{-2x} \sin 3x \, dx = -\frac{1}{3} e^{-2x} \cos 3x - \frac{2}{9} e^{-2x} \sin 3x + D,$$

where D is an arbitrary constant so

$$\int e^{-2x} \sin 3x \, dx = -\frac{3}{13} e^{-2x} \cos 3x - \frac{2}{13} e^{-2x} \sin 3x + C.$$

8.2.80 Let $u = g(x)$ and $dv = f''(x) \, dx$. Then $du = g'(x) \, dx$ and $v = f'(x)$. Then

$$\int_0^1 f''(x)g(x) \, dx = f'(x)g(x) \Big|_0^1 - \int_0^1 f'(x)g'(x) \, dx = - \int_0^1 f'(x)g'(x) \, dx.$$

Now let $u = g'(x)$ and $dv = f'(x) \, dx$. Then $du = g''(x) \, dx$ and $v = f(x)$. Then

$$\int_0^1 f'(x)g'(x) \, dx = f(x)g'(x) \Big|_0^1 - \int_0^1 f(x)g''(x) \, dx = - \int_0^1 f(x)g''(x) \, dx.$$

Thus, $\int_0^1 f''(x)g(x) \, dx = \int_0^1 f(x)g''(x) \, dx.$

8.2.81

a. To compute I_1 , we let $u = -x^2$, so that $du = -2x \, dx$. Then an ordinary substitution yields

$$-\frac{1}{2} \int e^u \, du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C.$$

b. To compute I_3 , we let $u = x^2$ and $dv = xe^{-x^2} \, dx$, so that $du = 2x \, dx$ and $v = -\frac{1}{2} e^{-x^2}$ (by part (a)).

Then $I_3 = -\frac{1}{2} x^2 e^{-x^2} + \int x e^{-x^2} \, dx = -\frac{1}{2} x^2 e^{-x^2} - \frac{1}{2} e^{-x^2} + C = -\frac{1}{2} e^{-x^2} (x^2 + 1) + C.$

- c. To compute I_5 , we let $u = x^4$ and $dv = xe^{-x^2} dx$. Then $du = 4x^3 dx$ and $v = -\frac{1}{2}e^{-x^2}$. Then
$$I_5 = -\frac{1}{2}x^4e^{-x^2} + 2 \int x^3e^{-x^2} dx = -\frac{1}{2}x^4e^{-x^2} + 2I_3 = -\frac{1}{2}e^{-x^2}(x^4 + 2x^2 + 2).$$
- d. Suppose that n is odd and that it is true that $I_n = -\frac{1}{2}e^{-x^2}p_{n-1}(x)$ where $p_{n-1}(x)$ is an even polynomial of degree $n-1$. We will show that I_{n+2} also has this property, so the result will follow by induction. Let $u = x^{n+1}$ and let $dv = xe^{-x^2} dx$, so that $du = (n+1)x^n dx$ and $v = -\frac{1}{2}e^{-x^2}$. Then
$$I_{n+2} = -\frac{1}{2}x^{n+1}e^{-x^2} + \frac{n+1}{2}I_n = -\frac{1}{2}e^{-x^2}\left(x^{n+1} + \frac{n+1}{2}p_{n-1}(x)\right) + C = -\frac{1}{2}e^{-x^2}p_{n+1}(x) + C.$$
 Note that if $p_{n-1}(x)$ is an even polynomial of degree $n-1$ then $\frac{n+1}{2}p_{n-1}(x) + x^{n+1}$ is an even polynomial of degree $n+1$.
- e. Note that $I_2 = -\frac{1}{2}xe^{-x^2} + \frac{1}{2}I_0$. (Using a similar technique to that above). Now if I_2 were expressible in terms of elementary functions, then I_0 would be as well, but we are given that it isn't. Similarly, we can express I_{2k} in terms of I_{2k-2} using Integration by Parts, and if any of these were expressible in terms of elementary functions, then the even numbered one below it would be. So the inability to express I_0 that way implies the inability to express I_2 that way, which implies the inability to express I_4 that way, and so on.

8.2.82

- a. This is given by $\int_0^4 xe^{-x} dx$. Let $u = x$ and $dv = e^{-x} dx$. Then $du = dx$ and $v = -e^{-x}$. Then
$$\int_0^4 xe^{-x} dx = -xe^{-x} \Big|_0^4 + \int_0^4 e^{-x} dx = -\frac{4}{e^4} + (-e^{-x}) \Big|_0^4 = -\frac{4}{e^4} - \frac{1}{e^4} + 1 = 1 - \frac{5}{e^4}.$$
- b. This is given by $\int_0^4 xe^{-ax} dx$. Let $u = x$ and $dv = e^{-ax} dx$. Then $du = dx$ and $v = -\frac{1}{a}e^{-ax}$. Then
$$\int_0^4 xe^{-ax} dx = -\frac{1}{a}xe^{-ax} \Big|_0^4 + \frac{1}{a} \int_0^4 e^{-ax} dx = -\frac{4}{ae^{4a}} + \left(-\frac{1}{a^2}e^{-ax}\right) \Big|_0^4 = -\frac{4}{ae^{4a}} - \frac{1}{a^2} \cdot \left(\frac{1}{e^{4a}} - 1\right) = \frac{1}{a^2} \left(1 - \frac{4a+1}{e^{4a}}\right).$$
- c. This is given by $\int_0^b xe^{-ax} dx$. Let $u = x$ and $dv = e^{-ax} dx$. Then $du = dx$ and $v = -\frac{1}{a}e^{-ax}$. Then
$$\int_0^b xe^{-ax} dx = -\frac{1}{a}xe^{-ax} \Big|_0^b + \frac{1}{a} \int_0^b e^{-ax} dx = -\frac{b}{ae^{ba}} + \left(-\frac{1}{a^2}e^{-ax}\right) \Big|_0^b = -\frac{b}{ae^{ba}} - \frac{1}{a^2} \cdot \left(\frac{1}{e^{ba}} - 1\right) = \frac{1}{a^2} \left(1 - \frac{ba+1}{e^{ba}}\right).$$
- d. $A(1, \ln b) = 1 - \frac{\ln b + 1}{e^{\ln b}} = 1 - \frac{\ln b + 1}{b}.$
 $A(2, (\ln b)/2) = \frac{1}{4} \left(1 - \frac{\ln b + 1}{e^{\ln b}}\right) = \frac{1}{4} \left(1 - \frac{\ln b + 1}{b}\right) = \frac{1}{4}A(1, \ln b).$ So $A(1, \ln b) = 4A(2, (\ln b)/2).$
- e. Yes. $A(a, (\ln b)/a) = \frac{1}{a^2} \left(1 - \frac{\ln b + 1}{e^{\ln b}}\right) = \frac{1}{a^2} \left(1 - \frac{\ln b + 1}{b}\right) = \frac{1}{a^2}A(1, \ln b).$
 So $A(1, \ln b) = a^2A(a, (\ln b)/a).$

8.3 Trigonometric Integrals

8.3.1 The half-angle identities for sine and cosine:

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2},$$

which are conjugates.

8.3.2 The Pythagorean Identities:

$$\cos^2 x + \sin^2 x = 1 \quad \text{and} \quad 1 + \tan^2 x = \sec^2 x \quad \text{and} \quad \cot^2 x + 1 = \csc^2 x,$$

where the second follows from the first by dividing through by $\cos^2 x$, and the third follows from the first by dividing through by $\sin^2 x$.

8.3.3 To integrate $\sin^3 x$, write $\sin^3 x = \sin x \sin^2 x = \sin x(1 - \cos^2 x)$, and let $u = \cos x$ so that $du = -\sin x dx$.

8.3.4 To integrate $\sin^m x \cos^n x$ where m is even and n is odd, write $\sin^m x \cos^n x = \sin^{m-1} x \cos^{n-1} x \cos x = \sin^{m-1} x (1 - \sin^2 x)^{(n-1)/2} \cos x$ and let $u = \sin x$ so that $du = \cos x dx$.

8.3.5 A reduction formula is a recursive formula involving integrals. Using it, one can rewrite an integral of a certain type in a simpler form – which can then perhaps be evaluated or further reduced.

8.3.6 One would compute this integral by writing $\cos^2 x \sin^3 x$ as $\cos^2 x (\sin^2 x) \sin x = \cos^2 x (1 - \cos^2 x) \sin x$ and then performing the ordinary substitution $u = \cos x$.

8.3.7 One would compute this integral by letting $u = \tan x$, so that $du = \sec^2 x dx$. This substitution leads to the integral $\int u^{10} du$, which can easily be evaluated.

8.3.8 One would compute this integral by letting $u = \sec x$, so that $du = \sec x \tan x dx$. This substitution leads to the integral $\int u^{11} du$, which can easily be evaluated.

8.3.9 $\int \cos^3 x dx = \int \cos x (1 - \sin^2 x) dx$. Let $u = \sin x$. Then $du = \cos x dx$. Substituting gives

$$\int (1 - u^2) du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C.$$

8.3.10 $\int \sin^3 x dx = \int \sin x (1 - \cos^2 x) dx$. Let $u = \cos x$. Then $du = -\sin x dx$. Substituting gives

$$\int (u^2 - 1) du = \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C.$$

8.3.11 $\int \sin^2 3x dx = \frac{1}{2} \int (1 - \cos 6x) dx = \frac{1}{2} \left(x - \frac{1}{6} \sin 6x \right) + C = \frac{x}{2} - \frac{1}{12} \sin 6x + C.$

8.3.12

$$\begin{aligned} \int \cos^4 2\theta d\theta &= \int \left(\frac{1 + \cos 4\theta}{2} \right)^2 d\theta = \int \frac{1}{4} (1 + 2\cos 4\theta + \cos^2 4\theta) d\theta \\ &= \frac{1}{4} \int \left(1 + 2\cos 4\theta + \frac{1 + \cos 8\theta}{2} \right) d\theta = \frac{1}{4} \int \left(\frac{3}{2} + 2\cos 4\theta + \frac{1}{2} \cos 8\theta \right) d\theta \\ &= \frac{1}{4} \left(\frac{3\theta}{2} + \frac{\sin 4\theta}{2} + \frac{\sin 8\theta}{16} \right) + C = \frac{1}{8} \left(3\theta + \sin 4\theta + \frac{1}{8} \sin 8\theta \right) + C. \end{aligned}$$

8.3.13 $\int \sin^5 x \, dx = \int (\sin^2 x)^2 \sin x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx$. Let $u = \cos x$ so that $du = -\sin x \, dx$. Substituting yields

$$-\int (1 - u^2)^2 \, du = \int (-u^4 + 2u^2 - 1) \, du = \frac{-u^5}{5} + \frac{2u^3}{3} - u + C = \frac{-\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C.$$

8.3.14 $\int \cos^3 20x \, dx = \int \cos^2 20x \cos 20x \, dx = \int (1 - \sin^2 20x) \cos 20x \, dx$. Let $u = \sin 20x$, so that $du = 20 \cos 20x \, dx$. Substituting yields

$$\frac{1}{20} \int (1 - u^2) \, du = \frac{1}{20} \left(u - \frac{u^3}{3} \right) + C = \frac{1}{20} \left(\sin 20x - \frac{\sin^3 20x}{3} \right) + C.$$

8.3.15 $\int \sin^3 x \cos^2 x \, dx = \int \sin x (1 - \cos^2 x) (\cos^2 x) \, dx$. Let $u = \cos x$ so that $du = -\sin x \, dx$. Substituting gives

$$\int (u^4 - u^2) \, du = u^5/5 - u^3/3 + C = (\cos^5 x)/5 - (\cos^3 x)/3 + C.$$

8.3.16 $\int \sin^2 \theta \cos^5 \theta \, d\theta = \int (\sin^2 \theta) (1 - \sin^2 \theta)^2 \cos \theta \, d\theta$. Let $u = \sin \theta$ so that $du = \cos \theta \, d\theta$. Substituting gives

$$\int u^2 (1 - u^2)^2 \, du = \int (u^2 - 2u^4 + u^6) \, du = u^3/3 - 2u^5/5 + u^7/7 + C = (\sin^3 \theta)/3 - 2(\sin^5 \theta)/5 + (\sin^7 \theta)/7 + C.$$

8.3.17 $\int \cos^3 x \sqrt{\sin x} \, dx = \int \cos x (1 - \sin^2 x) \sqrt{\sin x} \, dx$. Let $u = \sin x$ so that $du = \cos x \, dx$. Substituting gives

$$\int (1 - u^2) u^{1/2} \, du = \int (u^{1/2} - u^{5/2}) \, du = 2u^{3/2}/3 - 2u^{7/2}/7 + C = 2(\sin^{3/2} x)/3 - 2(\sin^{7/2} x)/7 + C.$$

8.3.18 $\int \frac{\sin^3 \theta}{\cos^2 \theta} \, d\theta = \int (\sin \theta) \left(\frac{1 - \cos^2 \theta}{\cos^2 \theta} \right) \, d\theta$. Let $u = \cos \theta$ so that $du = -\sin \theta \, d\theta$. Substituting gives

$$\int \frac{u^2 - 1}{u^2} \, du = \int (1 - u^{-2}) \, du = u + \frac{1}{u} + C = \cos \theta + \sec \theta + C.$$

8.3.19 $\int_0^{\pi/3} \sin^5 x \cos^{-2} x \, dx = \int_0^{\pi/3} (\sin x) \left(\frac{(1 - \cos^2 x)^2}{\cos^2 x} \right) \, dx$. Let $u = \cos x$ so that $du = -\sin x \, dx$. Substituting yields

$$\begin{aligned} -\int_1^{1/2} \frac{(1 - u^2)^2}{u^2} \, du &= \int_1^{1/2} (-u^{-2} + 2 - u^2) \, du = \left(\frac{1}{u} + 2u - \frac{u^3}{3} \right) \Big|_1^{1/2} \\ &= \left(2 + 1 - \frac{1}{24} - \left(1 + 2 - \frac{1}{3} \right) \right) = \frac{7}{24}. \end{aligned}$$

8.3.20 $\int \sin^{-3/2} x \cos^3 x \, dx = \int \sin^{-3/2} x \cos^2 x \cos x \, dx = \int (\sin^{-3/2} x) (1 - \sin^2 x) \cos x \, dx$. Let $u = \sin x$ so that $du = \cos x \, dx$. Then a substitution yields

$$\int u^{-3/2} (1 - u^2) \, du = \int u^{-3/2} - u^{1/2} \, du = \frac{-2}{u^{1/2}} - \frac{2u^{3/2}}{3} + C = \frac{-2}{\sqrt{\sin x}} - \frac{2\sin^{3/2} x}{3} + C.$$

8.3.21 $\int_0^{\pi/2} \cos^3 x \sqrt{\sin^3 x} dx = \int_0^{\pi/2} (1 - \sin^2 x) \sin^{3/2} x \cos x dx$. Let $u = \sin x$ so that $du = \cos x dx$. Substituting gives

$$\int_0^1 (1 - u^2) u^{3/2} du = \int_0^1 (u^{3/2} - u^{7/2}) du = \left(\frac{2}{5} u^{5/2} - \frac{2}{9} u^{9/2} \right) \Big|_0^1 = \frac{2}{5} - \frac{2}{9} = \frac{8}{45}.$$

8.3.22 $\int_{\pi/4}^{\pi/2} \sin^2 2x \cos^3 2x dx = \int_{\pi/4}^{\pi/2} \sin^2 2x \cos^2 2x \cos 2x dx = \int_{\pi/4}^{\pi/2} \sin^2 2x (1 - \sin^2 2x) \cos 2x dx$. Let $u = \sin 2x$ so that $du = 2 \cos 2x dx$. Substituting gives

$$\frac{1}{2} \int_1^0 (u^2 - u^4) du = \frac{1}{2} \left(\frac{u^3}{3} - \frac{u^5}{5} \right) \Big|_1^0 = \frac{1}{2} \left(\frac{1}{5} - \frac{1}{3} \right) = -\frac{1}{15}.$$

8.3.23

$$\begin{aligned} \int \sin^2 x \cos^2 x dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{1}{4} \int 1 - \cos^2 2x dx \\ &= \frac{1}{4} \int \left(1 - \frac{1 + \cos 4x}{2} \right) dx = \frac{1}{4} \int \left(\frac{1}{2} - \frac{1}{2} \cos 4x \right) dx = \frac{1}{4} \left(\frac{x}{2} - \frac{\sin 4x}{8} \right) + C. \end{aligned}$$

8.3.24 $\int \sin^3 x \cos^5 x dx = \int \sin^2 x \cos^5 x \sin x dx = \int (1 - \cos^2 x) \cos^5 x \sin x dx$. Let $u = \cos x$ so that $du = -\sin x dx$. Substituting gives

$$-\int (u^5 - u^7) du = -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C.$$

8.3.25 $\int \sin^2 x \cos^4 x dx = \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx = \frac{1}{8} \int (1 - \cos^2 2x)(1 + \cos 2x) dx = \frac{1}{8} \int 1 + \cos 2x - \cos^2 2x - \cos^3 2x dx = \frac{1}{8} \int 1 + \cos 2x - \frac{1 + \cos 4x}{2} - \cos^3 2x dx = \frac{1}{8} \int \frac{1}{2} + \cos 2x - \frac{1}{2} \cos 4x dx - \frac{1}{8} \int (1 - \sin^2 2x) \cos 2x dx = \frac{x}{16} + \frac{\sin 2x}{16} - \frac{\sin 4x}{64} - \frac{1}{8} \int (1 - \sin^2 2x) \cos 2x dx$. To compute this last integral, we let $u = \sin 2x$, so that $du = 2 \cos 2x dx$. Then

$$\int (1 - \sin^2 2x) \cos 2x dx = \frac{1}{2} \int (1 - u^2) du = \frac{1}{2} \left(u - \frac{u^3}{3} \right) + C = \frac{1}{2} \left(\sin 2x - \frac{\sin^3 2x}{3} \right) + C.$$

Thus, our original given integral is equal to

$$\frac{x}{16} + \frac{\sin 2x}{16} - \frac{\sin 4x}{64} - \frac{1}{8} \left(\frac{1}{2} \left(\sin 2x - \frac{\sin^3 2x}{3} \right) \right) + C = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$$

8.3.26 $\int \sin^3 x \cos^{3/2} x dx = \int (\sin x)(1 - \cos^2 x) \cos^{3/2} x dx$. Let $u = \cos x$ so that $du = -\sin x dx$. Then substituting yields

$$-\int (1 - u^2) u^{3/2} du = \int u^{7/2} - u^{3/2} du = \frac{2u^{9/2}}{9} - \frac{2u^{5/2}}{5} + C = \frac{2 \cos^{9/2} x}{9} - \frac{2 \cos^{5/2} x}{5} + C.$$

8.3.27 $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$.

8.3.28 $\int 6 \sec^4 x dx = \int 6 \sec^2 x (\tan^2 x + 1) dx$. Let $u = \tan x$, so that $du = \sec^2 x dx$. Then we have

$$6 \int (u^2 + 1) du = 2u^3 + 6u + C = 2 \tan^3 x + 6 \tan x + C.$$

8.3.29 $\int \cot^4 x \, dx = \int \cot^2 x (\csc^2 x - 1) \, dx = \int (\cot^2 x \csc^2 x - (\csc^2 x - 1)) \, dx = \int \cot^2 x \csc^2 x \, dx + \cot x + x$. Let $u = \cot x$ so that $du = -\csc^2 x \, dx$. Substituting gives

$$-\int u^2 \, du + \cot x + x = -u^3/3 + \cot x + x + C = -(\cot^3 x)/3 + \cot x + x + C.$$

8.3.30 $\int \tan^3 \theta \, d\theta = \int \tan \theta (\sec^2 \theta - 1) \, d\theta = \int (\tan \theta \sec^2 \theta - \tan \theta) \, d\theta = \int \tan \theta \sec^2 \theta \, d\theta + \ln |\cos \theta|$. Let $u = \tan \theta$ so that $du = \sec^2 \theta \, d\theta$. Substituting gives

$$\int u \, du + \ln |\cos \theta| = u^2/2 + \ln |\cos \theta| + C = (\tan^2 \theta)/2 + \ln |\cos \theta| + C.$$

Note that this can also be written as $\sec^2 \theta/2 + \ln |\cos \theta| + C$, because $\sec^2 \theta$ and $\tan^2 \theta$ differ by a constant.

8.3.31

$$\begin{aligned} \int 20 \tan^6 x \, dx &= 20 \int (\tan^4 x)(\sec^2 x - 1) \, dx = 20 \int ((\tan^4 x) \sec^2 x - (\tan^2 x)(\sec^2 x - 1)) \, dx \\ &= 20 \int (\tan^4 x \sec^2 x - \tan^2 x \sec^2 x + \sec^2 x - 1) \, dx. \end{aligned}$$

Let $u = \tan x$ so that $du = \sec^2 x \, dx$. We have

$$\begin{aligned} 20 \left(\int (u^4 - u^2) \, du + \tan x - x \right) + C &= 4u^5 - \frac{20u^3}{3} + 20 \tan x - 20x + C \\ &= 4 \tan^5 x - \frac{20 \tan^3 x}{3} + 20 \tan x - 20x + C. \end{aligned}$$

8.3.32

$$\begin{aligned} \int \cot^5 3x \, dx &= \int (\cot^3 3x)(\csc^2 3x - 1) \, dx = \int (\cot^3 3x \csc^2 3x - (\cot 3x)(\csc^2 3x - 1)) \, dx \\ &= \cot^3 3x \csc^2 3x - \cot 3x \csc^2 3x \, dx + \frac{\ln |\sin 3x|}{3} + C. \end{aligned}$$

Now let $u = \cot 3x$, so that $du = -3 \csc^2 3x \, dx$. Substituting gives

$$\begin{aligned} -\frac{1}{3} \int (u^3 - u) \, du + \frac{\ln |\sin 3x|}{3} + C &= -\frac{u^4}{12} + \frac{u^2}{6} + \frac{\ln |\sin 3x|}{3} + C \\ &= -\frac{\cot^4 3x}{12} + \frac{\cot^2 3x}{6} + \frac{\ln |\sin 3x|}{3} + C. \end{aligned}$$

8.3.33 Let $u = \tan x$ so that $du = \sec^2 x \, dx$. Substituting gives $\int 10u^9 \, du = u^{10} + C = \tan^{10} x + C$.

8.3.34 $\int \tan^9 x (\tan^2 x + 1) \sec^2 x \, dx$. Let $u = \tan x$ so that $du = \sec^2 x \, dx$. Substituting gives

$$\int u^9 (u^2 + 1) \, du = \int (u^{11} + u^9) \, du = u^{12}/12 + u^{10}/10 + C = (\tan^{12} x)/12 + (\tan^{10} x)/10 + C.$$

8.3.35 Let $u = \sec x$ so that $du = \sec x \tan x \, dx$. Substituting gives

$$\int u^2 \, du = u^3/3 + C = (\sec^3 x)/3 + C.$$

8.3.36 $\int \tan 4x \sec^{3/2} 4x \, dx = \int \sec^{1/2} 4x \sec 4x \tan 4x \, dx$. Let $u = \sec 4x$. Then $du = 4 \sec 4x \tan 4x$. Substituting gives

$$\frac{1}{4} \int u^{1/2} \, du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{6} \sec^{3/2} 4x + C.$$

8.3.37 Let $u = \ln \theta$ so that $du = \frac{1}{\theta} d\theta$. Substituting yields $\int \sec^4 u \, du = \int (\sec^2 u)(1 + \tan^2 u) \, du$. Let $w = \tan u$ so that $dw = \sec^2 u \, du$. Substituting again gives

$$\int (1 + w^2) \, dw = w + \frac{w^3}{3} + C = \tan(\ln(\theta)) + \frac{\tan^3(\ln(\theta))}{3} + C.$$

8.3.38 $\int \tan^5 \theta \sec^4 \theta \, d\theta = \int (\tan^5 \theta)(\tan^2 \theta + 1)(\sec^2 \theta) \, d\theta$. Let $u = \tan \theta$ so that $du = \sec^2 \theta \, d\theta$. Substituting yields $\int u^7 + u^5 \, du = \frac{u^8}{8} + \frac{u^6}{6} + C = \frac{\tan^8 \theta}{8} + \frac{\tan^6 \theta}{6} + C$.

8.3.39

$$\begin{aligned} \int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 \theta - 1} \, d\theta &= 2 \int_0^{\pi/3} \sqrt{\sec^2 \theta - 1} \, d\theta = 2 \int_0^{\pi/3} \tan \theta \, d\theta \\ &= -2 \ln |\cos \theta| \Big|_0^{\pi/3} = -2 \ln(1/2) + 2 \ln(1) = 2 \ln 2. \end{aligned}$$

8.3.40 $\int_0^{\pi/6} \tan^5 2x \sec 2x \, dx = \int_0^{\pi/6} (\tan^2 2x)^2 \tan 2x \sec 2x \, dx = \int_0^{\pi/6} (\sec^2 2x - 1)^2 \tan 2x \sec 2x \, dx$. Let $u = \sec 2x$. Then $du = 2 \sec 2x \tan 2x \, dx$. Substituting gives

$$\begin{aligned} \frac{1}{2} \int_1^2 (u^2 - 1)^2 \, du &= \frac{1}{2} \int_1^2 (u^4 - 2u^2 + 1) \, du = \frac{1}{2} \left(\frac{u^5}{5} - \frac{2u^3}{3} + u \right) \Big|_1^2 \\ &= \frac{1}{2} \left(\frac{32}{5} - \frac{16}{3} + 2 - \left(\frac{1}{5} - \frac{2}{3} + 1 \right) \right) = \frac{19}{15}. \end{aligned}$$

8.3.41 $\int_0^{\pi/4} \sec^7 x \sin x \, dx = \int_0^{\pi/4} \sec^6 x \tan x \, dx = \int_0^{\pi/4} \sec^5 x \sec x \tan x \, dx$. Let $u = \sec x$ so that $du = \sec x \tan x \, dx$. Substituting gives

$$\int_1^{\sqrt{2}} u^5 \, du = \frac{u^6}{6} \Big|_1^{\sqrt{2}} = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.$$

8.3.42 $\int \sqrt{\tan x} \sec^4 x \, dx = \int \sqrt{\tan x} (\tan^2 x + 1)(\sec^2 x) \, dx$. Let $u = \tan x$ so that $du = \sec^2 x \, dx$. Substituting gives

$$\int \sqrt{u}(u^2 + 1) \, du = \int (u^{5/2} + u^{1/2}) \, du = 2u^{7/2}/7 + 2u^{3/2}/3 + C = 2(\tan^{7/2} x)/7 + 2(\tan^{3/2} x)/3 + C.$$

8.3.43

$$\begin{aligned} \int \tan^3 4x \, dx &= \int (\tan 4x)(\sec^2 4x - 1) \, dx = \int (\tan 4x) \sec^2 4x \, dx - \int \tan 4x \, dx \\ &= \int (\tan 4x) \sec^2 4x \, dx + \frac{\ln |\cos 4x|}{4} + C. \end{aligned}$$

Let $u = \tan 4x$ so that $du = 4 \sec^2 4x \, dx$. Substituting gives

$$\frac{1}{4} \int u \, du + \frac{\ln |\cos 4x|}{4} + C = \frac{u^2}{8} + \frac{\ln |\cos 4x|}{4} + C = \frac{\tan^2 4x}{8} + \frac{\ln |\cos 4x|}{4} + C.$$

8.3.44 Let $u = \tan x$ so that $du = \sec^2 x \, dx$. Substituting gives

$$\int u^{-5} \, du = -\frac{u^{-4}}{4} + C = -\frac{1}{4 \tan^4 x} + C.$$

8.3.45 Let $u = \tan x$ so that $du = \sec^2 x \, dx$. Then

$$\int \sec^2 x \tan^{1/2} x \, dx = \int u^{1/2} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} \tan^{3/2} x + C.$$

8.3.46

$$\begin{aligned} \int \sec^{-2} x \tan^3 x \, dx &= \int \sec^{-2} (\sec^2 x - 1) \tan x \, dx = \int (\tan x - \sec^{-2} x \tan x) \, dx \\ &= \int \left(\frac{\sin x}{\cos x} - \cos x \sin x \right) \, dx. \end{aligned}$$

Let $u = \cos x$ so that $du = -\sin x \, dx$. Then we have

$$\int \left(-\frac{1}{u} + u \right) \, du = -\ln |u| + \frac{u^2}{2} + C = -\ln |\cos x| + \frac{1}{2} \cos^2 x + C.$$

8.3.47 $\int \frac{\csc^4 x}{\cot^2 x} \, dx = \int (\csc^2 x) \left(\frac{\cot^2 x + 1}{\cot^2 x} \right) \, dx$. Let $u = \cot x$ so that $du = -\csc^2 x \, dx$. Substituting gives

$$-\int \frac{u^2 + 1}{u^2} \, du = \int -1 - u^{-2} \, du = -u + \frac{1}{u} + C = -\cot x + \tan x + C.$$

8.3.48 Let $u = \csc x$ so that $du = -\csc x \cot x \, dx$. Substituting gives

$$-\int u^9 \, du = -u^{10}/10 + C = -(\csc^{10} x)/10 + C.$$

8.3.49 Let $u = \cot 5w$. Then $du = -5 \csc^2 5w$. Substituting gives

$$-\frac{1}{5} \int_1^0 u^4 \, du = -\frac{1}{25} u^5 \Big|_1^0 = -\frac{1}{25} (0 - 1) = \frac{1}{25}.$$

8.3.50 $\int \csc^{10} x \cot^3 x \, dx = \int \csc^9 x \cot^2 x \csc x \cot x \, dx = \int \csc^9 x (\csc^2 x - 1) \csc x \cot x \, dx$. Let $u = \csc x$ so that $du = -\csc x \cot x \, dx$. Substituting gives

$$-\int u^9 (u^2 - 1) \, du = -\int (u^{11} - u^9) \, du = -(u^{12}/12 - u^{10}/10) + C = \frac{-\csc^{12} x}{12} + \frac{\csc^{10} x}{10} + C.$$

8.3.51 $\int (\csc^2 x + \csc^4 x) \, dx = \int (1 + \csc^2 x) \csc^2 x \, dx$. Using the identity $\csc^2 x = 1 + \cot^2 x$, we can write this integral as $\int (2 + \cot^2 x) \csc^2 x \, dx$. Substituting $u = \cot x$ so that $du = -\csc^2 x \, dx$ gives

$$-\int (2 + u^2) \, du = -2u - \frac{u^3}{3} + C = -2 \cot x - \frac{\cot^3 x}{3} + C.$$

8.3.52

$$\begin{aligned} \int_0^{\pi/8} (\tan 2x + \tan^3 2x) \, dx &= \int_0^{\pi/8} (1 + \tan^2 2x) \tan 2x \, dx = \int_0^{\pi/8} \sec^2 2x \tan 2x \, dx \\ &= \int_0^{\pi/8} \sec 2x \sec 2x \tan 2x \, dx. \end{aligned}$$

Let $u = \sec 2x$. Then $du = 2 \sec 2x \tan 2x \, dx$. Substituting gives

$$\frac{1}{2} \int_1^{\sqrt{2}} u \, du = \frac{1}{4} u^2 \Big|_1^{\sqrt{2}} = \frac{1}{4} (2 - 1) = \frac{1}{4}.$$

8.3.53 $\int_0^{\pi/4} \sec^4 \theta d\theta = \int_0^{\pi/4} (\sec^2 \theta)(1 + \tan^2 \theta) d\theta$. Let $u = \tan \theta$ so that $du = \sec^2 \theta d\theta$. Note that when $\theta = 0$ we have $u = 0$ and when $\theta = \frac{\pi}{4}$ we have $u = 1$. So the original integral is equal to

$$\int_0^1 (1 + u^2) du = \left(u + \frac{u^3}{3} \right) \Big|_0^1 = 1 + \frac{1}{3} = \frac{4}{3}.$$

8.3.54 $\int_0^{\sqrt{\pi/2}} x \sin^3(x^2) dx = \frac{1}{2} \int_0^{\pi/2} \sin^3 u du$, where $u = x^2$ and $du = 2x dx$. Now $\frac{1}{2} \int_0^{\pi/2} \sin^3 u du = \frac{1}{2} \int_0^{\pi/2} (\sin u)(1 - \cos^2 u) du$. Now let $w = \cos u$ so that $dw = -\sin u du$. We then have

$$-\frac{1}{2} \int_1^0 (1 - w^2) dw = \frac{1}{2} \left(w - \frac{w^3}{3} \right) \Big|_0^1 = \frac{1}{3}.$$

8.3.55 $\int_{\pi/6}^{\pi/3} \cot^3 \theta d\theta = \int_{\pi/6}^{\pi/3} (\cot \theta)(\csc^2 \theta - 1) d\theta = \int_{\pi/6}^{\pi/3} \cot \theta \csc^2 \theta d\theta - \int_{\pi/6}^{\pi/3} \frac{\cos \theta}{\sin \theta} d\theta$. For the first integral, let $u = \cot \theta$ so that $du = -\csc^2 \theta d\theta$. For the second integral, let $w = \sin \theta$ so that $dw = \cos \theta d\theta$. Substituting gives

$$\begin{aligned} -\int_{\sqrt{3}}^{1/\sqrt{3}} u du - \int_{1/2}^{\sqrt{3}/2} \frac{1}{w} dw &= -\frac{u^2}{2} \Big|_{\sqrt{3}}^{1/\sqrt{3}} - \ln w \Big|_{1/2}^{\sqrt{3}/2} \\ &= -\frac{1}{2} \left(\frac{1}{3} - 3 \right) - (\ln \sqrt{3} - \ln 2 - \ln 1 + \ln 2) = \frac{4}{3} - \frac{\ln 3}{2}. \end{aligned}$$

8.3.56 $\int_0^{\pi/4} \tan^3 \theta \sec^2 \theta d\theta$. Let $u = \tan \theta$ so that $du = \sec^2 \theta d\theta$. Substituting yields $\int_0^1 u^3 du = \frac{u^4}{4} \Big|_0^1 = \frac{1}{4}$.

8.3.57 $\int_0^{\pi} (1 - \cos 2x)^{3/2} dx = \int_0^{\pi} (2 \sin^2 x)^{3/2} dx = 2\sqrt{2} \int_0^{\pi} \sin^3 x dx = 2\sqrt{2} \int_0^{\pi} (\sin x)(1 - \cos^2 x) dx$. Let $u = \cos x$ so that $du = -\sin x dx$. Substituting yields

$$-2\sqrt{2} \int_1^{-1} (1 - u^2) du = 2\sqrt{2} \int_{-1}^1 (1 - u^2) du = 4\sqrt{2} \int_0^1 (1 - u^2) du = 4\sqrt{2} \left(u - \frac{u^3}{3} \right) \Big|_0^1 = \frac{8\sqrt{2}}{3}.$$

8.3.58 $\int_{-\pi/4}^{\pi/4} \sqrt{1 + \cos 4x} dx = 2 \int_0^{\pi/4} \sqrt{1 + \cos 4x} dx = 2\sqrt{2} \int_0^{\pi/4} \cos 2x dx$. Let $u = 2x$, so that $du = 2 dx$. Substituting yields

$$\sqrt{2} \int_0^{\pi/2} \cos u du = \sqrt{2} \sin u \Big|_0^{\pi/2} = \sqrt{2}.$$

8.3.59 $\int_0^{\pi/2} \sqrt{1 - \cos 2x} dx = \sqrt{2} \int_0^{\pi/2} \sin x dx = -\sqrt{2} \cos x \Big|_0^{\pi/2} = \sqrt{2}$.

8.3.60 $\int_0^{\pi/8} \sqrt{1 - \cos 8x} dx = \sqrt{2} \int_0^{\pi/8} \sin 4x dx = -\sqrt{2} \cdot \frac{\cos 4x}{4} \Big|_0^{\pi/8} = \frac{\sqrt{2}}{4}$.

8.3.61 $\int_0^{\pi/4} (1 + \cos 4x)^{3/2} dx = \int_0^{\pi/4} (2 \cos^2 2x)^{3/2} dx = 2\sqrt{2} \int_0^{\pi/4} \cos^3 2x dx = 2\sqrt{2} \int_0^{\pi/4} (\cos 2x)(1 - \sin^2 2x) dx$. Let $u = \sin 2x$ so that $du = 2 \cos 2x dx$. Substituting gives

$$\sqrt{2} \int_0^1 (1 - u^2) du = \sqrt{2} \left(u - \frac{u^3}{3} \right) \Big|_0^1 = \frac{2\sqrt{2}}{3}.$$

8.3.62 If $y = \ln(\sec x)$ then $\frac{dy}{dx} = \tan x$. Thus,

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 x} \, dx = \int_0^{\pi/4} \sec x \, dx = \ln |\sec x + \tan x| \Big|_0^{\pi/4} = \ln(\sqrt{2} + 1).$$

8.3.63

a. True. We have $\int_0^\pi \cos^{2m+1} x \, dx = \int_0^\pi (\cos^2 x)^m \cos x \, dx = \int_0^\pi (1 - \sin^2 x)^m \cos x \, dx$. Let $u = \sin x$ so that $du = \cos x \, dx$. Substituting yields $\int_0^0 (1 - u^2)^m \, du = 0$.

b. False. For example, suppose $m = 1$. Then $\int_0^\pi \sin x \, dx = -\cos x \Big|_0^\pi = -(-1 - 1) = 2 \neq 0$.

8.3.64 Using the disk method, we have

$$\frac{V}{\pi} = \int_0^\pi \sin^2 x \, dx = \frac{1}{2} \int_0^\pi (1 - \cos 2x) \, dx = \frac{\pi}{2} - \frac{1}{2} \int_0^\pi \cos 2x \, dx = \frac{\pi}{2} - \left(\frac{\sin 2x}{4} \right) \Big|_0^\pi = \frac{\pi}{2}.$$

Thus, $V = \frac{\pi^2}{2}$.

8.3.65 $V = \int_0^{\pi/2} \pi \sin^4 x \cos^3 x \, dx = \pi \int_0^{\pi/2} \sin^4 x (1 - \sin^2 x) \cos x \, dx = \int_0^{\pi/2} (\sin^4 x - \sin^6 x) \cos x \, dx$. Let $u = \sin x$. Then $du = \cos x \, dx$ and substitution gives

$$\pi \int_0^1 (u^4 - u^6) \, du = \pi \left(\frac{u^5}{5} - \frac{u^7}{7} \right) \Big|_0^1 = \pi \left(\frac{1}{5} - \frac{1}{7} \right) = \frac{2\pi}{35}.$$

8.3.66 $s(t) = s(0) + \int_0^t \sec^4 \left(\frac{\pi x}{12} \right) \, dx = 0 + \int_0^t \left(\tan^2 \left(\frac{\pi x}{12} \right) + 1 \right) \sec^2 \left(\frac{\pi x}{12} \right) \, dx$. Let $u = \tan \left(\frac{\pi x}{12} \right)$. Then $du = \frac{\pi}{12} \sec^2 \left(\frac{\pi x}{12} \right) \, dx$. Substituting gives

$$\frac{12}{\pi} \int_0^{\tan(\pi t/12)} (u^2 + 1) \, du = \frac{12}{\pi} \left(\frac{\tan^3 \left(\frac{\pi t}{12} \right)}{3} + \tan \left(\frac{\pi t}{12} \right) \right) = \frac{4}{\pi} \left(\tan^3 \left(\frac{\pi t}{12} \right) + 3 \tan \left(\frac{\pi t}{12} \right) \right).$$

8.3.67 $\int \sin 3x \cos 7x \, dx = \frac{1}{2} \left(\int \sin(-4x) \, dx + \int \sin 10x \, dx \right) = \frac{1}{2} \left(\frac{\cos(-4x)}{4} - \frac{\cos 10x}{10} \right) + C = \frac{\cos 4x}{8} - \frac{\cos 10x}{20} + C$.

8.3.68 $\int \sin 5x \sin 7x \, dx = \frac{1}{2} \left(\int \cos(-2x) \, dx - \int \cos 12x \, dx \right) = \frac{1}{2} \left(\int \cos 2x \, dx - \int \cos 12x \, dx \right) = \frac{1}{2} \left(\frac{\sin 2x}{2} - \frac{\sin 12x}{12} \right) + C = \frac{\sin 2x}{4} - \frac{\sin 12x}{24} + C$.

8.3.69 $\int \sin 3x \sin 2x \, dx = \frac{1}{2} \left(\int \cos x \, dx - \int \cos 5x \, dx \right) = \frac{\sin x}{2} - \frac{\sin 5x}{10} + C$.

8.3.70 $\int \cos x \cos 2x \, dx = \frac{1}{2} \left(\int \cos(-x) \, dx + \int \cos 3x \, dx \right) = \frac{\sin x}{2} + \frac{\sin 3x}{6} + C$.

8.3.71

a. $\int_0^\pi \sin mx \sin nx \, dx = \frac{1}{2} \left(\int_0^\pi \cos(m-n)x \, dx - \int_0^\pi \cos(m+n)x \, dx \right) =$
 $\frac{1}{2} \left(\frac{1}{m-n} \int_0^{(m-n)\pi} \cos u \, du - \frac{1}{m+n} \int_0^{(m+n)\pi} \cos v \, dv \right)$ where $u = (m-n)x$ and $v = (m+n)x$. But
 this yields $\frac{1}{2} \left(\frac{1}{m-n} \sin u \Big|_0^{(m-n)\pi} - \frac{1}{m+n} \sin v \Big|_0^{(m+n)\pi} \right) = \frac{1}{2} (0 - 0) = 0$.

b. $\int_0^\pi \cos mx \cos nx \, dx = \frac{1}{2} \left(\int_0^\pi \cos(m-n)x \, dx + \int_0^\pi \cos(m+n)x \, dx \right) = 0$ by the previous part of this problem,

c. $\int_0^\pi \sin mx \cos nx \, dx = \frac{1}{2} \left(\int_0^\pi \sin(m-n)x \, dx + \int_0^\pi \sin(m+n)x \, dx \right) =$
 $\frac{1}{2} \left(\frac{1}{m-n} \int_0^{(m-n)\pi} \sin u \, du + \frac{1}{m+n} \int_0^{(m+n)\pi} \sin v \, dv \right)$ where $u = (m-n)x$ and $v = (m+n)x$. This
 quantity is equal to $\frac{-1}{2} \left(\frac{1}{m-n} \cos u \Big|_0^{(m-n)\pi} + \frac{1}{m+n} \cos v \Big|_0^{(m+n)\pi} \right) =$
 $\frac{-1}{2} \left(\frac{1}{m-n} (\cos(m-n)\pi - 1) + \frac{1}{m+n} (\cos(m+n)\pi - 1) \right) =$
 $\begin{cases} 0 & \text{if } m \text{ and } n \text{ are both even or both odd;} \\ \frac{1}{m-n} + \frac{1}{m+n} = \frac{2m}{m^2 - n^2} & \text{otherwise.} \end{cases}$

8.3.72 $\int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$. Let $u = \sin^{n-1} x$ and $dv = \sin x \, dx$. Then we have $du = (n-1) \sin^{n-2} x \cos x \, dx$ and $v = -\cos x$. We have $\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int (\sin^{n-2} x)(1 - \sin^2 x) \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$. Adding the appropriate quantity to both sides of this last equation gives

$$n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx,$$

so $\int \sin^n x \, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$.

Thus,

$$\int \sin^6 x \, dx = \frac{-\sin^5 x \cos x}{6} + \frac{5}{6} \left(\frac{-\sin^3 x \cos x}{4} + \frac{3}{4} \left(\frac{x}{2} - \frac{\sin 2x}{4} \right) \right) + C.$$

8.3.73 For $n \neq 1$, $\int \tan^n x \, dx = \int (\tan^{n-2} x)(\sec^2 x - 1) \, dx = \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$. Let $u = \tan x$ so that $du = \sec^2 x \, dx$. Then substituting in the first of these last two integrals yields

$$\int u^{n-2} \, du - \int \tan^{n-2} x \, dx = \frac{u^{n-1}}{n-1} - \int \tan^{n-2} x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx.$$

Thus $\int_0^{\pi/4} \tan^3 x \, dx = \frac{\tan^2 x}{2} \Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx = \frac{1}{2} + \ln |\cos x| \Big|_0^{\pi/4} = \frac{1}{2} - \frac{\ln 2}{2}$.

8.3.74 $\int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$. Let $u = \sec^{n-2} x$ and $dv = \sec^2 x \, dx$. Then $du = (n-2) \sec^{n-2} x \tan x \, dx$ and $v = \tan x$. Integration by Parts gives us $\sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx =$

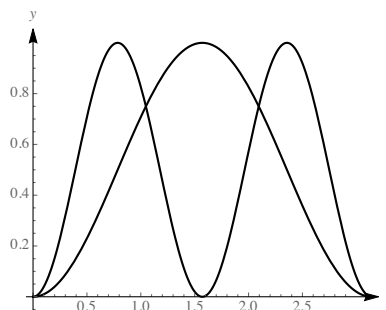
$$\sec^{n-2} x \tan x - (n-2) \int (\sec^{n-2} x)(\sec^2 x - 1) dx = \sec^{n-2} x \tan x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx.$$

Combining like terms then gives $(n-1) \int \sec^n x dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x dx$, so as long as $n \neq 1$ we have

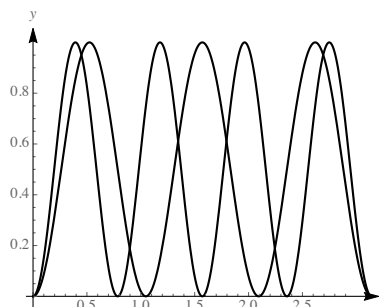
$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

8.3.75

$$\begin{aligned} \int_0^\pi \sin^2 x dx &= \frac{1}{2} \int_0^\pi (1 - \cos 2x) dx = \\ \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) \Big|_0^\pi &= \frac{\pi}{2}. \\ \text{a. } \int_0^\pi \sin^2 2x dx &= \frac{1}{2} \int_0^\pi (1 - \cos 4x) dx = \\ \frac{1}{2} \left(x - \frac{\sin 4x}{4} \right) \Big|_0^\pi &= \frac{\pi}{2}. \end{aligned}$$



$$\begin{aligned} \int_0^\pi \sin^2 3x dx &= \frac{1}{2} \int_0^\pi (1 - \cos 6x) dx = \\ \frac{1}{2} \left(x - \frac{\sin 6x}{6} \right) \Big|_0^\pi &= \frac{\pi}{2}. \\ \text{b. } \int_0^\pi \sin^2 4x dx &= \frac{1}{2} \int_0^\pi (1 - \cos 8x) dx = \\ \frac{1}{2} \left(x - \frac{\sin 8x}{8} \right) \Big|_0^\pi &= \frac{\pi}{2}. \end{aligned}$$



$$\text{c. } \int_0^\pi \sin^2 nx dx = \frac{1}{2} \int_0^\pi (1 - \cos 2nx) dx = \frac{1}{2} \left(x - \frac{\sin 2nx}{2n} \right) \Big|_0^\pi = \frac{\pi}{2}.$$

$$\text{d. Yes. } \int_0^\pi \cos^2 nx dx = \frac{1}{2} \int_0^\pi (1 + \cos 2nx) dx = \frac{1}{2} \left(x + \frac{\sin 2nx}{2n} \right) \Big|_0^\pi = \frac{\pi}{2}.$$

e. Claim: The corresponding integrals are all equal to $\frac{3\pi}{8}$. Proof:

$$\begin{aligned} \int_0^\pi \sin^4 nx dx &= \int_0^\pi \left(\frac{1 - \cos 2nx}{2} \right)^2 dx \\ &= \int_0^\pi \frac{1 - 2\cos 2nx + \cos^2 2nx}{4} dx = \int_0^\pi \frac{1}{4} dx - \frac{1}{2} \int_0^\pi \cos 2nx dx + \frac{1}{4} \int_0^\pi \cos^2 2nx dx \\ &= \frac{\pi}{4} - \frac{1}{2} \left(\frac{\sin 2nx}{2n} \right) \Big|_0^\pi + \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{4} - 0 + \frac{\pi}{8} = \frac{3\pi}{8}. \end{aligned}$$

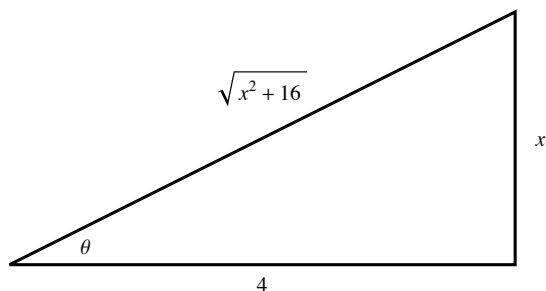
8.4 Trigonometric Substitutions

8.4.1 This would suggest $x = 3 \sec \theta$, because then $\sqrt{x^2 - 9} = 3\sqrt{\sec^2 \theta - 1} = 3\sqrt{\tan^2 \theta} = 3 \tan \theta$, for $\theta \in [0, \pi/2)$.

8.4.2 This would suggest $x = 6 \tan \theta$, because then $\sqrt{x^2 + 36} = 6\sqrt{\tan^2 \theta + 1} = 6\sqrt{\sec^2 \theta} = 6 \sec \theta$, for $|\theta| < \frac{\pi}{2}$.

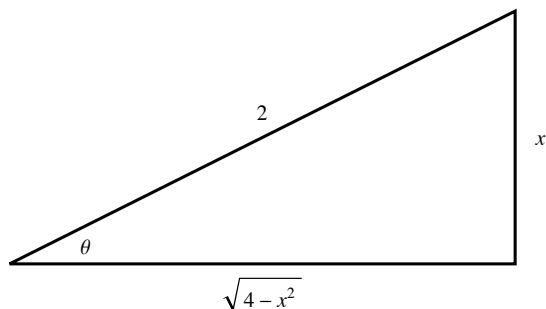
8.4.3 This would suggest $x = 10 \sin \theta$, because then $\sqrt{100 - x^2} = 10\sqrt{1 - \sin^2 \theta} = 10\sqrt{\cos^2 \theta} = 10 \cos \theta$, for $|\theta| \leq \frac{\pi}{2}$.

8.4.4



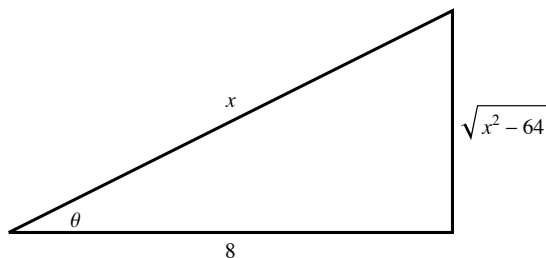
If $\tan \theta = \frac{x}{4}$, then $16 \tan^2 \theta = x^2$, so $16(\sec^2 \theta - 1) = x^2$. Thus $\sec^2 \theta = \frac{x^2 + 16}{16}$ and $\cos^2 \theta = 1 - \sin^2 \theta = \frac{16}{x^2 + 16}$. Thus $\sin^2 \theta = \frac{x^2}{16 + x^2}$ and we have $\sin \theta = \frac{x}{\sqrt{16 + x^2}}$, for $|\theta| < \frac{\pi}{2}$.

8.4.5



If $x = 2 \sin \theta$ then $\frac{x^2}{4} = \sin^2 \theta = \frac{1}{\csc^2 \theta}$. Then $\cot^2 \theta = \csc^2 \theta - 1 = \frac{4}{x^2} - 1 = \frac{4 - x^2}{x^2}$. So $\cot \theta = \frac{\sqrt{4 - x^2}}{x}$ for $0 < |\theta| \leq \frac{\pi}{2}$.

8.4.6



If $x = 8 \sec \theta$, the $\tan^2 \theta = \sec^2 \theta - 1 = \frac{x^2}{64} - 1 = \frac{x^2 - 64}{64}$. Thus $\tan \theta = \frac{\sqrt{x^2 - 64}}{8}$.

8.4.7 Let $x = 5 \sin \theta$, so that $dx = 5 \cos \theta d\theta$. Note that $\sqrt{25 - x^2} = 5 \cos \theta$. Then $\int_0^{5/2} \frac{1}{\sqrt{25 - x^2}} dx = \int_0^{\pi/6} \frac{5 \cos \theta}{5 \cos \theta} d\theta = \frac{\pi}{6}$.

Checking without using a trigonometric substitution:

$$\int_0^{5/2} \frac{dx}{\sqrt{25-x^2}} = \arcsin\left(\frac{x}{5}\right) \Big|_0^{5/2} = \left(\frac{\pi}{6} - 0\right) = \frac{\pi}{6}.$$

8.4.8 Let $x = 3 \sin \theta$ so that $dx = 3 \cos \theta d\theta$. Note that $\sqrt{9-x^2} = 3 \cos \theta$. Then

$$\int_0^{3/2} \frac{1}{(9-x^2)^{3/2}} dx = \int_0^{\pi/6} \frac{3 \cos \theta}{27 \cos^3 \theta} d\theta = \frac{1}{9} \int_0^{\pi/6} \sec^2 \theta d\theta = \frac{1}{9} \tan \theta \Big|_0^{\pi/6} = \frac{1}{9} \left(\frac{1}{\sqrt{3}} - 0 \right) = \frac{\sqrt{3}}{27}.$$

8.4.9 Let $x = 10 \sin \theta$ so that $dx = 10 \cos \theta d\theta$. Note that $\sqrt{100-x^2} = 10 \cos \theta$. Then $\int_5^{5\sqrt{3}} \sqrt{100-x^2} dx = 100 \int_{\pi/6}^{\pi/3} \cos^2 \theta d\theta = 50 \int_{\pi/6}^{\pi/3} (1 + \cos 2\theta) d\theta = 50 \left(\theta + \frac{\sin 2\theta}{2} \right) \Big|_{\pi/6}^{\pi/3} = 50 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right) \right) = \frac{25\pi}{3}.$

8.4.10 Let $x = 2 \sin \theta$, so that $dx = 2 \cos \theta d\theta$. Note that $\sqrt{4-x^2} = 2 \cos \theta$. Thus, $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}} dx = \int_0^{\pi/4} \frac{4 \sin^2 \theta \cdot 2 \cos \theta}{2 \cos \theta} d\theta = 4 \int_0^{\pi/4} \sin^2 \theta d\theta = 2 \left(\frac{\pi}{4} - \int_0^{\pi/4} \cos 2\theta d\theta \right) = \frac{\pi}{2} - 2 \left(\frac{\sin 2\theta}{2} \right) \Big|_0^{\pi/4} = \frac{\pi}{2} - 1.$

8.4.11 Let $x = \sin \theta$ so that $dx = \cos \theta d\theta$. Note that $\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \cos \theta$. Substituting gives

$$\int_{\pi/6}^{\pi/3} \sin^2 \theta d\theta = \int_{\pi/6}^{\pi/3} \frac{1 - \cos 2\theta}{2} d\theta = \frac{\theta}{2} \Big|_{\pi/6}^{\pi/3} - \frac{\sin 2\theta}{4} \Big|_{\pi/6}^{\pi/3} = \frac{\pi}{12}.$$

8.4.12 Let $x = \sin \theta$ so that $dx = \cos \theta d\theta$. Note that $\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \cos \theta$. Substituting gives

$$\int_{\pi/6}^{\pi/2} \cot^2 \theta d\theta = \int_{\pi/6}^{\pi/2} (\csc^2 \theta - 1) d\theta = (-\cot \theta - \theta) \Big|_{\pi/6}^{\pi/2} = 0 - \pi/2 - (-\sqrt{3} - \pi/6) = \sqrt{3} - \frac{\pi}{3}.$$

8.4.13 Let $x = 4 \sin \theta$ so that $dx = 4 \cos \theta d\theta$. Note that $\sqrt{16-x^2} = 4 \cos \theta$. Thus,

$$\int \frac{1}{\sqrt{16-x^2}} dx = \int \frac{4 \cos \theta}{4 \cos \theta} d\theta = \theta + C = \sin^{-1} \left(\frac{x}{4} \right) + C.$$

8.4.14 Let $t = 6 \sin \theta$ so that $dt = 6 \cos \theta d\theta$ and $\sqrt{36-t^2} = 6 \cos \theta$. Then

$$\begin{aligned} \int \sqrt{36-t^2} dt &= \int 36 \cos^2 \theta d\theta = 18 \int (1 + \cos 2\theta) d\theta = 18 \left(\theta + \frac{\sin 2\theta}{2} \right) + C = 18 (\theta + \sin \theta \cos \theta) \\ &= 18 \left(\sin^{-1} \left(\frac{t}{6} \right) + \frac{t}{6} \cdot \frac{\sqrt{36-t^2}}{6} \right) + C = 18 \sin^{-1} \left(\frac{t}{6} \right) + \frac{t\sqrt{36-t^2}}{2} + C. \end{aligned}$$

8.4.15 Let $x = 3 \tan \theta$ so that $dx = 3 \sec^2 \theta d\theta$. Note that $\sqrt{x^2+9} = \sqrt{9(\tan^2 \theta + 1)} = 3 \sec \theta$. Substituting gives

$$\int \frac{1}{9} \cot \theta \csc \theta d\theta = -\frac{1}{9} \csc \theta + C = -\frac{1}{9} \csc(\tan^{-1}(x/3)) + C = -\frac{\sqrt{x^2+9}}{9x} + C.$$

8.4.16 Let $x = 5 \tan \theta$ so that $dx = 5 \sec^2 \theta d\theta$. Note that $25+x^2 = 25 \sec^2 \theta$. Thus, $\int \frac{x^2}{(25+x^2)^2} dx = \int \frac{25 \tan^2 \theta \cdot 5 \sec^2 \theta}{25^2 \sec^4 \theta} d\theta = \frac{1}{5} \int \frac{\tan^2 \theta}{\sec^2 \theta} d\theta = \frac{1}{5} \int \sin^2 \theta d\theta = \frac{1}{10} \int (1 - \cos 2\theta) d\theta = \frac{1}{10} \left(\theta - \frac{1}{2} \sin 2\theta \right) + C = \frac{1}{10} (\tan \theta - \sin \theta \cos \theta) + C = \frac{1}{10} \left(\tan^{-1} \left(\frac{x}{5} \right) - \frac{5x}{25+x^2} \right) + C.$

8.4.17 Let $x = 2 \tan \theta$ so that $dx = 2 \sec^2 \theta d\theta$. Note that $x^2 + 4 = 4 \tan^2 \theta + 4 = 4(\tan^2 \theta + 1) = 4 \sec^2 \theta$. Then

$$\begin{aligned} \int_0^1 \frac{x^2}{x^2 + 4} dx &= \int_0^{\pi/4} \frac{4 \tan^2 \theta}{4 \sec^2 \theta} 2 \sec^2 \theta d\theta = 2 \int_0^{\pi/4} \tan^2 \theta d\theta \\ &= 2 \int_0^{\pi/4} (\sec^2 \theta - 1) d\theta = 2(\tan \theta - \theta) \Big|_0^{\pi/4} = 2 \left(1 - \frac{\pi}{4}\right) = 2 - \frac{\pi}{2}. \end{aligned}$$

8.4.18 Let $x = \tan \theta$ so that $dx = \sec^2 \theta d\theta$. Note that $\sqrt{1 + x^2} = \sqrt{1 + \tan^2 \theta} = \sec \theta$. Substituting gives

$$\int \frac{1}{\sec \theta} d\theta = \int \cos \theta d\theta = \sin \theta + C = \frac{x}{\sqrt{x^2 + 1}} + C.$$

8.4.19 Let $x = 9 \sec \theta$ with $\theta \in (0, \pi/2)$. Then $dx = 9 \sec \theta \tan \theta d\theta$ and $\sqrt{x^2 - 81} = 9 \tan \theta$. Then

$$\int \frac{1}{\sqrt{x^2 - 81}} dx = \int \frac{9 \sec \theta \tan \theta}{9 \tan \theta} d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{x}{9} + \frac{\sqrt{x^2 - 81}}{9} \right| + C.$$

Note that because $x > 9$, the absolute value signs are unnecessary, and the final result can be written as $\ln(\sqrt{x^2 - 81} + x) + C$.

8.4.20 Let $x = 7 \sec \theta$ where $\theta \in (0, \pi/2)$. Then $dx = 7 \sec \theta \tan \theta d\theta$ and $\sqrt{x^2 - 49} = 7 \tan \theta$. Then

$$\int \frac{1}{\sqrt{x^2 - 49}} dx = \int \frac{7 \sec \theta \tan \theta}{7 \tan \theta} d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C = \ln \left| \frac{x}{7} + \frac{\sqrt{x^2 - 49}}{7} \right| + C.$$

(Note also that the absolute value signs can be omitted because $x > 7$, and if we replace $-\ln(7) + C$ by a different arbitrary constant D , we can write the result as $\ln(x + \sqrt{x^2 - 49}) + D$.)

8.4.21 Let $x = 8 \sin \theta$ so that $dx = 8 \cos \theta d\theta$ and $\sqrt{64 - x^2} = 8 \cos \theta$. Then,

$$\begin{aligned} \int \sqrt{64 - x^2} dx &= \int 64 \cos^2 \theta d\theta = 32 \int (1 + \cos 2\theta) d\theta = 32\theta + 16 \sin 2\theta + C \\ &= 32\theta + 32 \sin \theta \cos \theta + C = 32 \sin^{-1} \left(\frac{x}{8} \right) + \frac{x\sqrt{64 - x^2}}{2} + C. \end{aligned}$$

8.4.22 Let $t = 3 \sin \theta$, so that $dt = 3 \cos \theta d\theta$. Note that $\sqrt{9 - t^2} = \sqrt{9(\cos^2 \theta)} = 3 \cos \theta$. Substituting gives

$$\int \frac{1}{9} \csc^2 \theta d\theta = -\frac{1}{9} \cot \theta + C = -\frac{\sqrt{9 - t^2}}{9t} + C.$$

8.4.23 Let $x = 5 \sin \theta$ so that $dx = 5 \cos \theta d\theta$. Note that $\sqrt{25 - x^2} = \sqrt{25 - 25 \sin^2 \theta} = 5 \cos \theta$. Substituting gives

$$\int \frac{5 \cos \theta}{125 \cos^3 \theta} d\theta = \frac{1}{25} \int \sec^2 \theta d\theta = \frac{1}{25} \tan \theta + C = \frac{x}{25\sqrt{25 - x^2}} + C.$$

8.4.24 Let $x = 3 \sin \theta$ so that $dx = 3 \cos \theta d\theta$. Note that $\sqrt{9 - x^2} = 3 \cos \theta$. Thus, $\int \frac{\sqrt{9 - x^2}}{x^2} dx =$

$$\int \frac{3 \cos \theta \cdot 3 \cos \theta}{9 \sin^2 \theta} d\theta = \int \cot^2 \theta d\theta = \int \csc^2 \theta - 1 d\theta = -\cot \theta - \theta + C = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}(x/3) + C.$$

8.4.25 Let $x = 3 \sin \theta$ so that $dx = 3 \cos \theta d\theta$ and $\sqrt{9 - x^2} = 3 \cos \theta$. Then

$$\begin{aligned} \int \frac{\sqrt{9 - x^2}}{x} dx &= \int \frac{3 \cos \theta \cdot 3 \cos \theta}{3 \sin \theta} d\theta = 3 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta \\ &= 3 \left(\int \csc \theta d\theta - \int \sin \theta d\theta \right) d\theta = 3(-\ln |\csc \theta + \cot \theta| + \cos \theta) \\ &= -3 \ln \left| \frac{3}{x} + \frac{\sqrt{9 - x^2}}{x} \right| + \sqrt{9 - x^2} + C. \end{aligned}$$

8.4.26 Let $x = \sec \theta$ so that $dx = \sec \theta \tan \theta d\theta$. Note that $\sqrt{x^2 - 1} = \sqrt{\tan^2 \theta} = \tan \theta$. Substituting gives

$$\int_{\pi/4}^{\pi/3} \tan^2 \theta d\theta = \int_{\pi/4}^{\pi/3} (\sec^2 \theta - 1) d\theta = (\tan \theta - \theta) \Big|_{\pi/4}^{\pi/3} = \sqrt{3} - \pi/3 - (1 - \pi/4) = \sqrt{3} - 1 - \frac{\pi}{12}.$$

8.4.27 Let $x = \frac{1}{3} \tan \theta$ so that $dx = \frac{1}{3} \sec^2 \theta d\theta$. Note that $\sqrt{9x^2 + 1} = \sec \theta$. Thus $\int_0^{1/3} \frac{1}{(9x^2 + 1)^{3/2}} dx =$

$$\int_0^{\pi/4} \frac{\frac{1}{3} \sec^2 \theta}{\sec^3 \theta} d\theta = \frac{1}{3} \int_0^{\pi/4} \cos \theta d\theta = \frac{1}{3} \sin \theta \Big|_0^{\pi/4} = \frac{\sqrt{2}}{6}.$$

8.4.28 Let $z = 6 \tan \theta$ so that $dz = 6 \sec^2 \theta d\theta$. Note that $z^2 + 36 = 36 \sec^2 \theta$. Thus

$$\begin{aligned} \int_0^6 \frac{z^2}{(z^2 + 36)^2} dz &= \int_0^{\pi/4} \frac{36 \tan^2 \theta \cdot 6 \sec^2 \theta}{36^2 \sec^4 \theta} d\theta = \frac{1}{6} \int_0^{\pi/4} \frac{\tan^2 \theta}{\sec^2 \theta} d\theta \\ &= \frac{1}{6} \int_0^{\pi/4} \sin^2 \theta d\theta = \frac{1}{12} \int_0^{\pi/4} (1 - \cos 2\theta) d\theta \\ &= \frac{1}{12} \left(\theta - \frac{\sin 2\theta}{2} \right) \Big|_0^{\pi/4} = \frac{1}{12} \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi - 2}{48} \end{aligned}$$

8.4.29 Let $x = 2 \tan \theta$. Then $dx = 2 \sec^2 \theta d\theta$ and $4 + x^2 = 4(\sec^2 \theta)$. Then

$$\begin{aligned} \int \frac{dx}{(4 + x^2)^2} &= \int \frac{2 \sec^2 \theta}{2^4 \sec^4 \theta} d\theta = \frac{1}{8} \int \cos^2 \theta d\theta = \frac{1}{16} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{16} \left(\theta + \frac{\sin 2\theta}{2} \right) + C = \frac{1}{16} (\theta + \sin \theta \cos \theta) + C = \frac{1}{16} \left(\tan^{-1} \frac{x}{2} + \frac{2x}{x^2 + 4} \right) + C. \end{aligned}$$

8.4.30 Let $x = \sin \theta$. Then $dx = \cos \theta d\theta$ and $\sqrt{1 - x^2} = \sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta$. Substituting gives

$$\begin{aligned} \int x^3 \sqrt{1 - x^2} dx &= \int \sin^3 \theta \sqrt{1 - \sin^2 \theta} \cos \theta d\theta = \int \sin^3 \theta \cos^2 \theta d\theta \\ &= \int (1 - \cos^2 \theta) \cos^2 \theta \sin \theta d\theta = - \int (u^2 - u^4) du \\ &= - \left(\frac{1}{3} u^3 - \frac{1}{5} u^5 \right) + C = - \left(\frac{1}{3} \cos^3 \theta - \frac{1}{5} \cos^5 \theta \right) + C \\ &= -\frac{1}{3} (1 - x^2)^{3/2} + \frac{1}{5} (1 - x^2)^{5/2} + C = -\frac{1}{15} (1 - x^2)^{3/2} (3x^2 + 2) + C. \end{aligned}$$

Along the way we made the substitution $u = \cos \theta$.

8.4.31 Let $x = 4 \sin \theta$ so that $dx = 4 \cos \theta d\theta$. Note that $\sqrt{16 - x^2} = 4 \cos \theta$. Then $\int \frac{x^2}{\sqrt{16 - x^2}} dx =$

$$\int \frac{16 \sin^2 \theta \cdot 4 \cos \theta}{4 \cos \theta} d\theta = 16 \int \sin^2 \theta d\theta = 8 \int (1 - \cos 2\theta) d\theta = 8 \left(\theta - \frac{\sin 2\theta}{2} \right) + C = 8\theta - 8 \sin \theta \cos \theta + C =$$

$$8 \sin^{-1} \left(\frac{x}{4} \right) - \frac{x \sqrt{16 - x^2}}{2} + C.$$

8.4.32 Let $x = 6 \sec \theta$ with $\theta \in (0, \pi/2)$. Then $dx = 6 \sec \theta \tan \theta d\theta$, and $\sqrt{x^2 - 36} = 6 \tan \theta$. Then

$$\int \frac{1}{(x^2 - 36)^{3/2}} dx = \int \frac{6 \sec \theta \tan \theta}{6^3 \tan^3 \theta} d\theta = \frac{1}{36} \int \frac{\sec \theta}{\tan^2 \theta} d\theta = \frac{1}{36} \int \frac{\cos \theta}{\sin^2 \theta} d\theta.$$

Let $u = \sin \theta$ so that $du = \cos \theta d\theta$. Then we have

$$\frac{1}{36} \int u^{-2} du = -\frac{1}{36u} + C = -\frac{1}{36 \sin \theta} + C = -\frac{x}{36 \sqrt{x^2 - 36}} + C.$$

8.4.33 Let $x = 3 \sec \theta$ where $\theta \in (0, \pi/2)$. Then $dx = 3 \sec \theta \tan \theta d\theta$ and $\sqrt{x^2 - 9} = 3 \tan \theta$. Thus we have $\int \frac{\sqrt{x^2 - 9}}{x} dx = \int \frac{3 \sec \theta \tan \theta \cdot 3 \tan \theta}{3 \sec \theta} d\theta = 3 \int \tan^2 \theta d\theta = 3 \int \sec^2 \theta - 1 d\theta = 3(\tan \theta - \theta) + C = \sqrt{x^2 - 9} - 3 \tan^{-1} \left(\frac{\sqrt{x^2 - 9}}{3} \right) + C = \sqrt{x^2 - 9} - 3 \sec^{-1}(x/3) + C$.

8.4.34 Let $x = \sec \theta$ where $\theta \in (0, \pi/2)$. Then $dx = \sec \theta \tan \theta d\theta$ and $\sqrt{x^2 - 1} = \tan \theta$. Thus,

$$\begin{aligned} \int \frac{1}{x^3 \sqrt{x^2 - 1}} dx &= \int \frac{\sec \theta \tan \theta}{\sec^3 \theta \tan \theta} d\theta = \int \cos^2 \theta d\theta = \frac{1}{2} \int 1 + \cos 2\theta d\theta = \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) + C \\ &= \frac{1}{2} (\theta + \sin \theta \cos \theta) + C = \frac{1}{2} \left(\tan^{-1} \sqrt{x^2 - 1} + \frac{\sqrt{x^2 - 1}}{x^2} \right) + C. \end{aligned}$$

8.4.35 Let $x = \sec \theta$ where $\theta \in (0, \pi/2)$. Then $dx = \sec \theta \tan \theta d\theta$ and $\sqrt{x^2 - 1} = \tan \theta$. Then

$$\begin{aligned} \int \frac{1}{x(x^2 - 1)^{3/2}} dx &= \int \frac{\sec \theta \tan \theta}{\sec \theta \tan^3 \theta} d\theta = \int \cot^2 \theta d\theta \\ &= \int \csc^2 \theta - 1 d\theta = -\cot \theta - \theta + C = -\frac{1}{\sqrt{x^2 - 1}} - \sec^{-1} x + C. \end{aligned}$$

8.4.36 Let $x = 8 \sec \theta$ so that $dx = 8 \sec \theta \tan \theta d\theta$ and $\sqrt{x^2 - 64} = 8 \tan \theta$. Then $\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2 - 64}} = \int_{\pi/4}^{\pi/3} \frac{8 \sec \theta \tan \theta}{8 \tan \theta} d\theta = \int_{\pi/4}^{\pi/3} \sec \theta d\theta = \ln |\sec \theta + \tan \theta| \Big|_{\pi/4}^{\pi/3} = \ln(2 + \sqrt{3}) - \ln(\sqrt{2} + 1) = \ln \left(\frac{2 + \sqrt{3}}{1 + \sqrt{2}} \right)$.

8.4.37 Let $x = \tan \theta$ so that $dx = \sec^2 \theta d\theta$ and $\sqrt{1 + x^2} = \sec \theta$. Substituting gives

$$\int_{\pi/6}^{\pi/4} \cot \theta \csc \theta d\theta = (-\csc \theta) \Big|_{\pi/6}^{\pi/4} = -(\sqrt{2} - 2) = 2 - \sqrt{2}.$$

8.4.38 Let $x = 2 \sin \theta$, so that $dx = 2 \cos \theta d\theta$. Note that when $x = 1$ we have $\theta = \frac{\pi}{6}$ and when $x = \sqrt{2}$ we have $\theta = \frac{\pi}{4}$. Also $\sqrt{4 - x^2} = 2\sqrt{\cos^2 \theta} = 2 \cos \theta$. Substituting gives

$$\frac{1}{4} \int_{\pi/6}^{\pi/4} \csc^2 \theta d\theta = -\frac{1}{4} (\cot \theta) \Big|_{\pi/6}^{\pi/4} = -\frac{1}{4} (1 - \sqrt{3}) = \frac{\sqrt{3} - 1}{4}.$$

8.4.39 Let $x = 10 \sin \theta$ so that $dx = 10 \cos \theta d\theta$. Note that $\sqrt{100 - x^2} = 10 \cos \theta$. Thus,

$$\begin{aligned} \int \frac{x^2}{(100 - x^2)^{3/2}} dx &= \int \frac{100 \sin^2 \theta \cdot 10 \cos \theta}{1000 \cos^3 \theta} d\theta = \int \tan^2 \theta d\theta \\ &= \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + C = \frac{x}{\sqrt{100 - x^2}} - \sin^{-1}(x/10) + C. \end{aligned}$$

8.4.40 Let $y = 5 \sec \theta$ so that $dy = 5 \sec \theta \tan \theta d\theta$. Then $\sqrt{y^2 - 25} = 5 \tan \theta$. Then, $\int_{10/\sqrt{3}}^{10} \frac{1}{\sqrt{y^2 - 25}} dy = \int_{\pi/6}^{\pi/3} \frac{5 \sec \theta \tan \theta}{5 \tan \theta} d\theta = \int_{\pi/6}^{\pi/3} \sec \theta d\theta = \ln |\sec \theta + \tan \theta| \Big|_{\pi/6}^{\pi/3} = \ln(2 + \sqrt{3}) - \ln(\sqrt{3}) = \ln \left(\frac{2 + \sqrt{3}}{\sqrt{3}} \right)$.

8.4.41 Let $x = \frac{\tan \theta}{2}$ so that $dx = \frac{\sec^2 \theta}{2} d\theta$ and $\sqrt{1 + 4x^2} = \sec \theta$. Then

$$\int \frac{1}{(1 + 4x^2)^{3/2}} dx = \frac{1}{2} \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta = \frac{1}{2} \int \cos \theta d\theta = \frac{\sin \theta}{2} + C = \frac{x}{\sqrt{1 + 4x^2}} + C.$$

8.4.42 Let $x = \frac{1}{3} \sec \theta$, where $\theta \in (0, \pi/2)$. Then $dx = \frac{1}{3} \sec \theta \tan \theta d\theta$. Note that $\sqrt{9x^2 - 1} = \tan \theta$. Then

$$\int \frac{1}{x^2 \sqrt{9x^2 - 1}} dx = \int \frac{\frac{1}{3} \sec \theta \tan \theta}{\frac{1}{9} \sec^2 \theta \tan \theta} d\theta = 3 \int \cos \theta d\theta = 3 \sin \theta + C = \frac{\sqrt{9x^2 - 1}}{x} + C.$$

8.4.43 Let $x = 4 \tan \theta$ so that $dx = 4 \sec^2 \theta d\theta$. Note that $\sqrt{x^2 + 16} = 4 \sec \theta$. Thus, $\int_0^{4/\sqrt{3}} \frac{1}{\sqrt{x^2 + 16}} dx = \int_0^{\pi/6} \frac{4 \sec^2 \theta}{4 \sec \theta} d\theta = \int_0^{\pi/6} \sec \theta d\theta = \ln |\sec \theta + \tan \theta| \Big|_0^{\pi/6} = \ln \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) - \ln 1 = \ln \frac{3}{\sqrt{3}} = \ln 3 - \frac{1}{2} \ln 3 = \frac{1}{2} \ln 3$.

8.4.44 Let $x = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$ and $\sqrt{16 + 4x^2} = 4 \sec \theta$. Then

$$\int \frac{1}{\sqrt{16 + 4x^2}} dx = \int \frac{2 \sec^2 \theta}{4 \sec \theta} d\theta = \frac{1}{2} \int \sec \theta d\theta = \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = \frac{1}{2} \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C.$$

8.4.45 Let $x = 9 \sin \theta$ so that $dx = 9 \cos \theta d\theta$. Note that $81 - x^2 = 81 \cos^2 \theta$. Thus,

$$\begin{aligned} \int \frac{x^3}{(81 - x^2)^2} dx &= \int \frac{9^3 \sin^3 \theta \cdot 9 \cos \theta}{9^4 \cos^4 \theta} d\theta = \int \tan^3 \theta d\theta \\ &= \int (\tan \theta)(\sec^2 \theta - 1) d\theta = \int \sec^2 \theta \tan \theta d\theta - \int \tan \theta d\theta \\ &= \frac{\sec^2 \theta}{2} + \ln |\cos \theta| + C = \frac{81}{2(81 - x^2)} + \ln \left| \frac{\sqrt{81 - x^2}}{9} \right| + C. \end{aligned}$$

This can be written as $\frac{81}{2(81 - x^2)} + \ln \sqrt{81 - x^2} + C$.

8.4.46 Let $x = (1/\sqrt{2}) \sin \theta$ so that $dx = (1/\sqrt{2}) \cos \theta d\theta$ and $\sqrt{1 - 2x^2} = \cos \theta$. Then

$$\int \frac{1}{\sqrt{1 - 2x^2}} dx = \frac{1}{\sqrt{2}} \int \frac{\cos \theta}{\cos \theta} d\theta = \frac{1}{\sqrt{2}} \cdot \theta + C = \frac{1}{\sqrt{2}} \sin^{-1}(\sqrt{2}x) + C.$$

8.4.47 Let $x = 2 \sec \theta$ so that $dx = 2 \sec \theta \tan \theta d\theta$ and $x^2 - 4 = 4 \tan^2 \theta$. Thus,

$$\begin{aligned} \int_{4/\sqrt{3}}^4 \frac{1}{x^2(x^2 - 4)} dx &= \int_{\pi/6}^{\pi/3} \frac{2 \sec \theta \tan \theta}{4 \sec^2 \theta \cdot 4 \tan^2 \theta} d\theta = \frac{1}{8} \int_{\pi/6}^{\pi/3} \frac{\cos^2 \theta}{\sin \theta} d\theta = \frac{1}{8} \int_{\pi/6}^{\pi/3} \frac{1 - \sin^2 \theta}{\sin \theta} d\theta = \\ &= \frac{1}{8} \int_{\pi/6}^{\pi/3} \csc \theta - \sin \theta d\theta = \frac{1}{8} (-\ln |\csc \theta + \cot \theta| + \cos \theta) \Big|_{\pi/6}^{\pi/3} = \frac{1}{8} \left(-\ln(\sqrt{3}(2 - \sqrt{3})) + \frac{1 - \sqrt{3}}{2} \right) = \\ &= \frac{1}{16} (1 - \sqrt{3} - \ln(21 - 12\sqrt{3})). \end{aligned}$$

8.4.48 Let $x = \frac{3}{2} \sin \theta$, so that $dx = \frac{3}{2} \cos \theta d\theta$. Note that $\sqrt{9 - 4x^2} = 3 \cos \theta$. Thus $\int \sqrt{9 - 4x^2} dx = \frac{3}{2} \int \cos \theta \cdot 3 \cos \theta d\theta = \frac{9}{2} \int \cos^2 \theta d\theta = \frac{9}{4} \int 1 + \cos 2\theta d\theta = \frac{9}{4} \left(\theta + \frac{\sin 2\theta}{2} \right) + C = \frac{9\theta}{4} + \frac{9 \sin \theta \cos \theta}{4} + C = \frac{9 \sin^{-1}(2x/3)}{4} + \frac{x\sqrt{9 - 4x^2}}{2} + C$.

8.4.49 Let $x = \tan \theta$ so that $dx = \sec^2 \theta d\theta$. Note that $\sqrt{x^2 + 1} = \sqrt{\tan^2 \theta + 1} = \sqrt{\sec^2 \theta} = \sec \theta$. Substituting gives $\int_0^{\pi/6} \sec^3 \theta d\theta$. Recall from section 8.2 number 48 that $\int \sec^3 \theta d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta|$. Thus the original integral is equal to $\left(\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) \Big|_0^{\pi/6} = \frac{1 \cdot 2 \cdot 1}{2 \cdot \sqrt{3} \cdot \sqrt{3}} + \frac{1}{2} \ln \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) = \frac{1}{3} + \frac{\ln 3}{4}$.

8.4.50 Let $x = 2 \sin \theta$ so that $dx = 2 \cos \theta d\theta$ and $\sqrt{4 - x^2} = 2 \cos \theta$. Then $\int (36 - 9x^2)^{-3/2} dx = \frac{1}{27} \int \frac{1}{(4 - x^2)^{3/2}} dx = \frac{1}{27} \int \frac{2 \cos \theta}{8 \cos^3 \theta} d\theta = \frac{1}{108} \int \sec^2 \theta d\theta = \frac{1}{108} \tan \theta + C = \frac{x}{108\sqrt{4 - x^2}} + C$.

8.4.51 Let $x = 2 \tan \theta$ so that $dx = 2 \sec^2 \theta d\theta$. Note that $\sqrt{4 + x^2} = 2 \sec \theta$. Then $\int \frac{x^2}{\sqrt{4 + x^2}} dx = \int \frac{4 \tan^2 \theta \cdot 2 \sec^2 \theta}{2 \sec \theta} d\theta = 4 \int \tan^2 \theta \sec \theta d\theta = 4 \int (\sec^2 \theta - 1) \sec \theta d\theta = 4 \left(\int \sec^3 \theta d\theta - \int \sec \theta d\theta \right) = 4 \left(\frac{1}{2} \left(\sec \theta \tan \theta + \int \sec \theta d\theta \right) - \int \sec \theta d\theta \right) = 2 \sec \theta \tan \theta - 2 \int \sec \theta d\theta = 2 \sec \theta \tan \theta - 2 \ln |\sec \theta + \tan \theta| + C = \frac{x\sqrt{4 + x^2}}{2} - 2 \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C$. This can be written as $\frac{x\sqrt{4 + x^2}}{2} - 2 \ln(x + \sqrt{4 + x^2}) + C$.

8.4.52 Let $x = \frac{1}{2} \sec \theta$ where $\theta \in [0, \pi/2)$. Then $dx = \frac{1}{2} \sec \theta \tan \theta d\theta$ and $\sqrt{4x^2 - 1} = \tan \theta$. Thus, $\int \frac{\sqrt{4x^2 - 1}}{x^2} dx = \int \frac{\tan \theta \cdot \frac{1}{2} \sec \theta \tan \theta}{\frac{1}{4} \sec^2 \theta} d\theta = 2 \int \frac{\tan^2 \theta}{\sec \theta} d\theta = 2 \int \frac{\sec^2 \theta - 1}{\sec \theta} d\theta = 2 \int \sec \theta - \cos \theta d\theta = 2(\ln |\sec \theta + \tan \theta| - \sin \theta) + C = 2 \ln(2x + \sqrt{4x^2 - 1}) - \frac{\sqrt{4x^2 - 1}}{x} + C$.

8.4.53 Let $x = \frac{5}{3} \sec \theta$ where $\theta \in [0, \pi/2)$. Then $dx = \frac{5}{3} \sec \theta \tan \theta d\theta$ and $\sqrt{9x^2 - 25} = 5 \tan \theta$. Thus, $\int \frac{\sqrt{9x^2 - 25}}{x^3} dx = \int \frac{5 \tan \theta \cdot \frac{5}{3} \sec \theta \tan \theta}{\frac{125}{27} \sec^3 \theta} d\theta = \frac{9}{5} \int \frac{\tan^2 \theta}{\sec^2 \theta} d\theta = \frac{9}{5} \int \frac{\sec^2 \theta - 1}{\sec^2 \theta} d\theta = \frac{9}{5} \int 1 - \cos^2 \theta d\theta = \frac{9}{5} \int \sin^2 \theta d\theta = \frac{9}{10} \int (1 - \cos 2\theta) d\theta = \frac{9\theta}{10} - \frac{9 \sin 2\theta}{20} + C = \frac{9\theta}{10} - \frac{9 \sin \theta \cos \theta}{10} = \frac{9 \cos^{-1}(5/3x)}{10} - \frac{\sqrt{9x^2 - 25}}{2x^2} + C$.

8.4.54 Let $y = \tan \theta$ so that $dy = \sec^2 \theta d\theta$. Note that $1 + y^2 = \sec^2 \theta$. Thus,

$$\begin{aligned} \int \frac{y^4}{1 + y^2} dy &= \int \frac{\tan^4 \theta \sec^2 \theta}{\sec^2 \theta} d\theta = \int \tan^4 \theta d\theta \\ &= \int (\tan^2 \theta)(\sec^2 \theta - 1) d\theta = \int \tan^2 \theta \sec^2 \theta - \tan^2 \theta d\theta \\ &= \int \tan^2 \theta \sec^2 \theta + 1 - \sec^2 \theta d\theta = \int \tan^2 \theta \sec^2 \theta d\theta + \theta - \tan \theta. \end{aligned}$$

Now recall that $y = \tan \theta$, so we have $\int y^2 dy + \theta - \tan \theta = \frac{y^3}{3} + \theta - \tan \theta + C = \frac{y^3}{3} + \tan^{-1}(y) - y + C$.

8.4.55 Let $x = 10 \sec \theta$ where $\theta \in (0, \pi/2)$. Then $dx = 10 \sec \theta \tan \theta d\theta$ and $\sqrt{x^2 - 100} = 10 \tan \theta$. Thus,

$$\begin{aligned} \int \frac{1}{x^3 \sqrt{x^2 - 100}} dx &= \int \frac{10 \sec \theta \tan \theta}{10^3 \sec^3 \theta \cdot 10 \tan \theta} d\theta = \frac{1}{1000} \int \cos^2 \theta d\theta \\ &= \frac{1}{2000} \int (1 + \cos 2\theta) d\theta = \frac{\theta}{2000} + \frac{\sin 2\theta}{4000} + C = \frac{\theta}{2000} + \frac{\sin \theta \cos \theta}{2000} + C \\ &= \frac{\sec^{-1}(x/10)}{2000} + \frac{\sqrt{x^2 - 100}}{200x^2} + C. \end{aligned}$$

8.4.56 Let $x = 4 \sec \theta$ where $\theta \in (0, \pi/2)$. Then $dx = 4 \sec \theta \tan \theta d\theta$ and $\sqrt{x^2 - 16} = 4 \tan \theta$. Then

$$\begin{aligned} \int \frac{x^3}{(x^2 - 16)^{3/2}} dx &= \int \frac{4^3 \sec^3 \theta \cdot 4 \sec \theta \tan \theta}{64 \tan^3 \theta} d\theta = 4 \int \frac{\sec^4 \theta}{\tan^2 \theta} d\theta \\ &= 4 \int \frac{(\sec^2 \theta)(1 + \tan^2 \theta)}{\tan^2 \theta} d\theta. \end{aligned}$$

Let $u = \tan \theta$ so that $du = \sec^2 \theta d\theta$. Then we have

$$\begin{aligned} 4 \int \frac{1+u^2}{u^2} du &= 4 \int u^{-2} + 1 du = -\frac{4}{u} + 4u + C \\ &= -4 \cot \theta + 4 \tan \theta + C = -\frac{16}{\sqrt{x^2-16}} + \sqrt{x^2-16} + C. \end{aligned}$$

8.4.57

- False. In fact, we would have $\csc \theta = \frac{\sqrt{x^2+16}}{x}$.
- True. Almost every number in the interval $[1, 2]$ is not in the domain of $\sqrt{1-x^2}$, so this integral isn't defined.
- False. It does represent a finite real number, because $\sqrt{x^2-1}$ is continuous on the interval $[1, 2]$.
- False. It can be so evaluated. The integral is equivalent to $\int \frac{1}{(x+2)^2+5} dx$, and this can be evaluated by the substitution $x+2 = \sqrt{5} \tan \theta$.

8.4.58 Let A be the area of the ellipse. Using symmetry, we have $\frac{A}{4} = \int_0^b \frac{b}{a} \sqrt{a^2-x^2} dx$. Let $x = a \sin \theta$, so that $dx = a \cos \theta d\theta$. Substituting yields

$$\frac{b}{a} \int_0^{\pi/2} a \cos \theta \cdot a \cos \theta d\theta = ab \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{ab}{2} \int_0^{\pi/2} (1 + \cos 2\theta) d\theta = \frac{ab}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) \Big|_0^{\pi/2} = \frac{\pi ab}{4}.$$

So the total area of the ellipse is $A = \pi ab$.

8.4.59

- Recall that the area of a circular sector subtended by an angle θ is given by $\frac{\theta r^2}{2}$. So the area of the cap is this area minus the area of the isosceles triangle with two sides of length r and angle between them θ . So

$$A_{\text{cap}} = A_{\text{sector}} - A_{\text{triangle}} = \frac{\theta r^2}{2} - \frac{r^2 \sin \theta}{2} = \frac{r^2}{2} (\theta - \sin \theta).$$

- For a cap we have $0 \leq \theta \leq \pi$ so $0 \leq \theta/2 \leq \pi/2$. By symmetry, $\frac{A_{\text{cap}}}{2} = \int_{r \cos \theta/2}^r \sqrt{r^2-x^2} dx$. Let $x = r \cos \alpha/2$ so that $dx = -\frac{r}{2} \sin \alpha/2 d\alpha$. Then we have

$$\begin{aligned} \frac{A_{\text{cap}}}{2} &= \int_{\theta}^0 r \sin(\alpha/2) \cdot -\frac{r}{2} \sin(\alpha/2) d\alpha \\ &= \frac{r^2}{2} \int_0^{\theta} \sin^2(\alpha/2) d\alpha = \frac{r^2}{4} \int_0^{\theta} (1 - \cos \alpha) d\alpha = \frac{r^2}{4} (\alpha - \sin \alpha) \Big|_0^{\theta} = \frac{r^2}{4} (\theta - \sin \theta). \end{aligned}$$

$$\text{Thus } A_{\text{cap}} = \frac{r^2}{2} (\theta - \sin \theta).$$

8.4.60 Note that the given integral can be written $\int \frac{1}{(x-3)^2+25} dx = \int \frac{1}{u^2+25} du$ with $u = x-3$. Now let $u = 5 \tan \theta$ so that $du = 5 \sec^2 \theta d\theta$ and $u^2+25 = 25 \sec^2 \theta$. Thus we have

$$\int \frac{5 \sec^2 \theta}{25 \sec^2 \theta} d\theta = \frac{\theta}{5} + C = \frac{\tan^{-1}((x-3)/5)}{5} + C.$$

8.4.61 $\int \frac{1}{\sqrt{3-2x-x^2}} dx = \int \frac{1}{\sqrt{4-(x+1)^2}} dx = \int \frac{1}{\sqrt{4-u^2}} du$ where $u = x+1$. Then let $u = 2 \sin \theta$ so that $du = 2 \cos \theta d\theta$. We have $\int \frac{2 \cos \theta}{2 \cos \theta} d\theta = \theta + C = \sin^{-1} \left(\frac{x+1}{2} \right) + C$.

8.4.62 Note that the given integral can be written $\frac{1}{2} \int \frac{1}{(u-3)^2+9} du = \frac{1}{2} \int \frac{1}{w^2+9} dw$ where $w = u-3$. Now let $w = 3 \tan \theta$ so that $dw = 3 \sec^2 \theta d\theta$ and $w^2+9 = 9 \sec^2 \theta$. Thus we have

$$\frac{1}{2} \int \frac{3 \sec^2 \theta}{9 \sec^2 \theta} d\theta = \frac{\theta}{6} + C = \frac{\tan^{-1}((u-3)/3)}{6} + C.$$

8.4.63 Note that the given integral can be written $\int \frac{1}{(x+3)^2+9} dx = \int \frac{1}{u^2+9} du$ where $u = x+3$. Now let $u = 3 \tan \theta$ so that $du = 3 \sec^2 \theta d\theta$ and $u^2+9 = 9 \sec^2 \theta$. Thus we have

$$\int \frac{3 \sec^2 \theta}{9 \sec^2 \theta} d\theta = \frac{\theta}{3} + C = \frac{\tan^{-1}((x+3)/3)}{3} + C.$$

8.4.64 Note that the given integral can be written as $\int \frac{(x-1)^2}{\sqrt{(x-1)^2+9}} dx = \int \frac{u^2}{\sqrt{u^2+9}} du$ where $u = x-1$. Now let $u = 3 \tan \theta$ so that $du = 3 \sec^2 \theta d\theta$ and $u^2+9 = 9 \sec^2 \theta$. Thus we have

$$\begin{aligned} \int \frac{9 \tan^2 \theta \cdot 3 \sec^2 \theta}{3 \sec \theta} d\theta &= 9 \int \tan^2 \theta \sec \theta d\theta = 9 \int (\sec^2 \theta - 1)(\sec \theta) d\theta \\ &= 9 \int (\sec^3 \theta - \sec \theta) d\theta = 9 \left(\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta - \int \sec \theta d\theta \right) \\ &= \frac{9}{2} \left(\sec \theta \tan \theta - \int \sec \theta d\theta \right) \\ &= \frac{9}{2} (\sec \theta \tan \theta - \ln |\sec \theta + \tan \theta|) + C \\ &= \frac{9}{2} \left(\frac{(x-1)\sqrt{(x-1)^2+9}}{9} - \ln \left| \frac{\sqrt{(x-1)^2+9}}{3} + \frac{x-1}{3} \right| \right) + C. \end{aligned}$$

This can be written as $\frac{x-1}{2} \sqrt{x^2-2x+10} - \frac{9}{2} \ln(x-1+\sqrt{x^2-2x+10}) + C$. Note that in the middle of this derivation we used the reduction formula for $\int \sec^3 \theta d\theta$ given in the previous section.

8.4.65 $\int_{1/2}^{(\sqrt{2}+3)/2\sqrt{2}} \frac{1}{8x^2-8x+11} dx = \int_{1/2}^{(\sqrt{2}+3)/2\sqrt{2}} \frac{1}{8(x-1/2)^2+9} dx$. Let $u = x-1/2$, so that our integral becomes $\int_0^{3/2\sqrt{2}} \frac{1}{8u^2+9} du$. Now let $u = \frac{3}{\sqrt{8}} \tan \theta$ so that $du = \frac{3}{\sqrt{8}} \sec^2 \theta d\theta$. Substituting gives

$$\int_0^{\pi/4} \frac{\frac{3}{\sqrt{8}} \sec^2 \theta}{9 \sec^2 \theta} d\theta = \frac{1}{6\sqrt{2}} \int d\theta = \frac{1}{6\sqrt{2}} \theta \Big|_0^{\pi/4} = \frac{\pi\sqrt{2}}{48}.$$

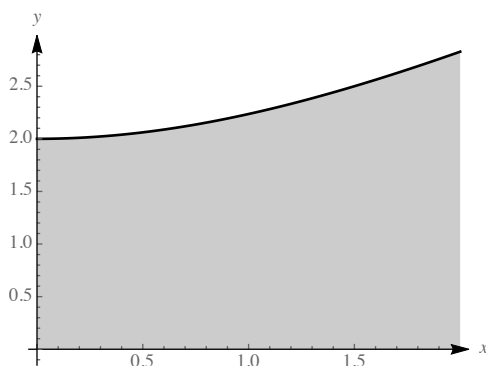
8.4.66 $\int_1^4 \frac{1}{t^2-2t+10} dt = \int_1^4 \frac{1}{(t-1)^2+9} dt$. Let $3 \tan \theta = t-1$, so that $dt = 3 \sec^2 \theta d\theta$. Note that $(t-1)^2+9 = 9 \sec^2 \theta$. Then we have $\int_0^{\pi/4} \frac{3 \sec^2 \theta}{9 \sec^2 \theta} d\theta = \frac{1}{3} \theta \Big|_0^{\pi/4} = \frac{\pi}{12}$.

8.4.67 Note that the given integral can be written as $\int \frac{(x-4)^2}{(25-(x-4)^2)^{3/2}} dx$. Let $u = x-4$, and note that we have $\int \frac{u^2}{(25-u^2)^{3/2}} du$. Now let $u = 5 \sin \theta$ so that $du = 5 \cos \theta d\theta$, and note that $\sqrt{25-u^2} = 5 \cos \theta$. Thus we have

$$\int \frac{25 \sin^2 \theta \cdot 5 \cos \theta}{5^3 \cos^3 \theta} d\theta = \int \tan^2 \theta d\theta = \int \sec^2 \theta - 1 d\theta = \tan \theta - \theta + C = \frac{x-4}{\sqrt{25-(x-4)^2}} - \sin^{-1} \left(\frac{x-4}{5} \right) + C.$$

$$\mathbf{8.4.68} \quad \int \frac{1}{\sqrt{(x-1)(3-x)}} dx = \int \frac{1}{\sqrt{1-(x-2)^2}} dx = \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + C = \sin^{-1}(x-2) + C.$$

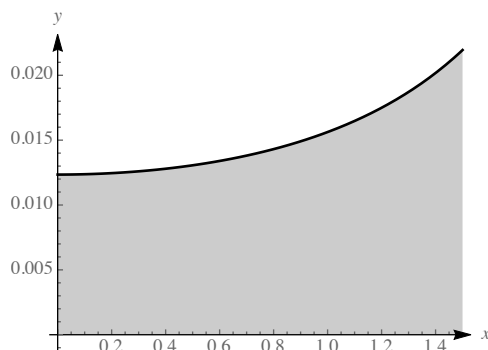
$$\mathbf{8.4.69} \quad \int_{2+\sqrt{2}}^4 \frac{1}{\sqrt{(x-1)(x-3)}} dx = \int_{2+\sqrt{2}}^4 \frac{1}{\sqrt{(x-2)^2-1}} dx = \int_{\sqrt{2}}^2 \frac{1}{\sqrt{u^2-1}} du, \text{ where } u = x-2. \text{ Now let } u = \sec \theta, \text{ so that } du = \sec \theta \tan \theta d\theta. \text{ Then } \int_{\pi/4}^{\pi/3} \frac{\sec \theta \tan \theta}{\tan \theta} d\theta = \int_{\pi/4}^{\pi/3} \sec \theta d\theta = \ln(\sec \theta + \tan \theta) \Big|_{\pi/4}^{\pi/3} = \ln(2 + \sqrt{3}) - \ln(\sqrt{2} + 1) = \ln \left(\frac{2 + \sqrt{3}}{\sqrt{2} + 1} \right) = \ln((2 + \sqrt{3})(\sqrt{2} - 1)).$$

8.4.70

The area is given by $\int_0^2 \sqrt{4+x^2} dx$. Let $x = 2 \tan \theta$ so that $dx = 2 \sec^2 \theta d\theta$. Then we have

$$\int_0^{\pi/4} 2 \sec^2 \theta \cdot 2 \sec \theta d\theta = 4 \int_0^{\pi/4} \sec^3 \theta d\theta = 2 (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \Big|_0^{\pi/4} = 2 (\sqrt{2} + \ln(\sqrt{2} + 1)).$$

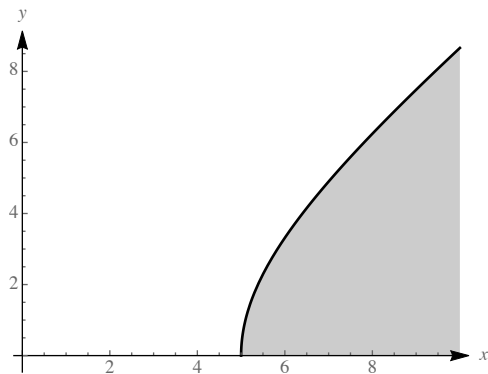
8.4.71



The area is given by $\int_0^{3/2} \frac{1}{(9-x^2)^2} dx$. Let $x = 3 \sin \theta$ so that $dx = 3 \cos \theta d\theta$. Then we have

$$\begin{aligned} \int_0^{\pi/6} \frac{3 \cos \theta}{9^2 \cos^4 \theta} d\theta &= \frac{1}{27} \int_0^{\pi/6} \sec^3 \theta d\theta = \frac{1}{54} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \Big|_0^{\pi/6} \\ &= \frac{1}{54} \left(\frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \ln \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \right) = \frac{1}{81} + \frac{\ln(\sqrt{3})}{54}. \end{aligned}$$

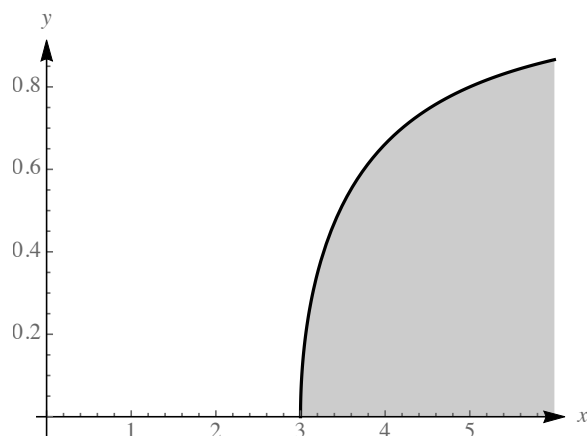
8.4.72



The area is given by $\int_5^{10} \sqrt{x^2 - 25} dx$. Let $x = 5 \sec \theta$ so that $dx = 5 \sec \theta \tan \theta d\theta$. Then we have

$$\begin{aligned} \int_0^{\pi/3} 5 \sec \theta \tan \theta \cdot 5 \tan \theta d\theta &= 25 \int_0^{\pi/3} \sec \theta \tan^2 \theta d\theta = 25 \int_0^{\pi/3} \sec \theta (\sec^2 \theta - 1) d\theta \\ &= 25 \int_0^{\pi/3} (\sec^3 \theta - \sec \theta) d\theta = \frac{25}{2} (\sec \theta \tan \theta - \ln |\sec \theta + \tan \theta|) \Big|_0^{\pi/3} \\ &= \frac{25}{2} (2\sqrt{3} - \ln(2 + \sqrt{3})) = 25\sqrt{3} - \frac{25}{2} \cdot \ln(2 + \sqrt{3}). \end{aligned}$$

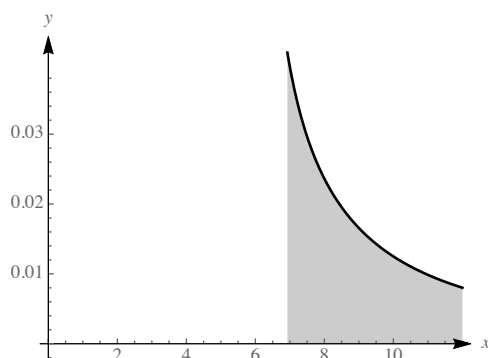
8.4.73



Let $x = 3 \sec \theta$ for $x \in (0, \pi/2)$. Then $dx = 3 \sec \theta \tan \theta d\theta$, and note that $|\tan \theta| = \tan \theta$. We have

$$\int_3^6 \frac{\sqrt{x^2 - 9}}{x} dx = \int_0^{\pi/3} \frac{3 \sec \theta \tan \theta (3 \tan \theta)}{3 \sec \theta} d\theta =$$

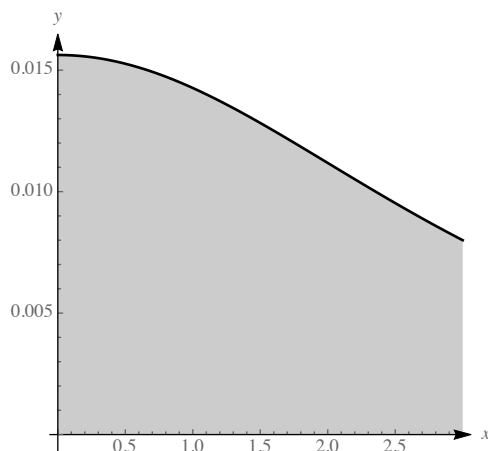
8.4.74



Let $x = 6 \sec \theta$ so that $dx = 6 \sec \theta \tan \theta d\theta$. Then

$$\int_{12/\sqrt{3}}^{12} \frac{1}{x\sqrt{x^2 - 36}} dx = \int_{\pi/6}^{\pi/3} \frac{6 \sec \theta \tan \theta}{6 \sec \theta \cdot 6 \tan \theta} d\theta = \frac{1}{6} \int_{\pi/6}^{\pi/3} d\theta = \frac{\pi}{36}.$$

8.4.75



The area bounded by the curve and the axis on $[0, 3]$ is given by $\int_0^3 \frac{1}{(16+x^2)^{3/2}} dx$.

Let $x = 4 \tan \theta$ so that $dx = 4 \sec^2 \theta d\theta$. Substituting yields

$$\int_0^{\tan^{-1}(3/4)} \frac{4 \sec^2 \theta}{4^3 \sec^3 \theta} d\theta = \frac{1}{16} \int_0^{\tan^{-1}(3/4)} \cos \theta d\theta = \frac{\sin \theta}{16} \Big|_0^{\tan^{-1}(3/4)} = \frac{3}{80}.$$

8.4.76

a. The area is given by $\int_0^4 \frac{1}{\sqrt{9+x^2}} dx$. Let $x = 3 \tan \theta$, so that $dx = 3 \sec^2 \theta d\theta$. Substituting yields

$$\int_0^{\tan^{-1}(4/3)} \frac{3 \sec^2 \theta}{3 \sec \theta} d\theta = \int_0^{\tan^{-1}(4/3)} \sec \theta d\theta = \ln |\sec \theta + \tan \theta| \Big|_0^{\tan^{-1}(4/3)} = \ln \left(\frac{5}{3} + \frac{4}{3} \right) = \ln 3.$$

b. Using disks, we have $\frac{V}{\pi} = \int_0^4 \frac{1}{9+x^2} dx$. Let $x = 3 \tan \theta$ so that $dx = 3 \sec^2 \theta d\theta$. Substituting yields

$$\int_0^{\tan^{-1}(4/3)} \frac{3 \sec^2 \theta}{9 \sec^2 \theta} d\theta = \frac{1}{3} \int_0^{\tan^{-1}(4/3)} 1 d\theta = \frac{1}{3} \theta \Big|_0^{\tan^{-1}(4/3)} = \frac{1}{3} \tan^{-1}(4/3). \text{ Thus } V = \frac{\pi}{3} \tan^{-1}(4/3).$$

c. Using shells, we have $\frac{V}{2\pi} = \int_0^4 \frac{x}{(9+x^2)^{1/2}} dx$. Let $u = 9+x^2$, so that $du = 2x dx$. Then we have

$$\frac{1}{2} \int_9^{25} u^{-1/2} du = \sqrt{u} \Big|_9^{25} = 5 - 3 = 2. \text{ So } V = 4\pi.$$

8.4.77 Because $y = ax^2$, we have $1 + \left(\frac{dy}{dx}\right)^2 = 1 + 4a^2x^2$, so the arc length is given by $\int_0^{10} \sqrt{1+4a^2x^2} dx$.

Let $x = \frac{1}{2a} \tan \theta$ so that $dx = \frac{1}{2a} \sec^2 \theta d\theta$. Then we have

$$\begin{aligned} \frac{1}{2a} \int_0^{\tan^{-1}(20a)} \sec^2 \theta \sec \theta d\theta &= \frac{1}{2a} \int_0^{\tan^{-1}(20a)} \sec^3 \theta d\theta = \frac{1}{4a} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \Big|_0^{\tan^{-1}(20a)} \\ &= \frac{1}{4a} \left(\sqrt{1+400a^2}(20a) + \ln(\sqrt{1+400a^2} + 20a) \right). \end{aligned}$$

8.4.78

a. $\int_0^2 \frac{x^2}{3} dx = \frac{x^3}{9} \Big|_0^2 = \frac{8}{9}.$

b. $\int_0^2 \frac{x^2}{\sqrt{9-x^2}} dx$. Let $x = 3 \sin \theta$ so that $dx = 3 \cos \theta d\theta$. Then

$$\begin{aligned} \int_0^{\sin^{-1}(2/3)} \frac{9 \sin^2 \theta \cdot 3 \cos \theta}{3 \cos \theta} d\theta &= 9 \int_0^{\sin^{-1}(2/3)} \sin^2 \theta d\theta = \frac{9}{2} \int_0^{\sin^{-1}(2/3)} 1 - \cos 2\theta d\theta \\ &= \frac{9}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) \Big|_0^{\sin^{-1}(2/3)} \\ &= \frac{9 \sin^{-1}(2/3)}{2} - \frac{9}{2} \left(\frac{2\sqrt{5}}{3 \cdot 3} \right) = \frac{9}{2} \sin^{-1}(2/3) - \sqrt{5} \approx 1.05. \end{aligned}$$

c. The area under g is larger than the area under f .

8.4.79 If $1 < x = \sec \theta$ then we need $\theta \in (0, \pi/2)$. Alternatively, if $-1 > x = \sec \theta$, then we need $\theta \in (\pi/2, \pi)$. So, in the former, $\int \frac{1}{x\sqrt{x^2-1}} dx = \int \frac{\sec \theta \tan \theta}{\sec \theta \tan \theta} d\theta = \theta + C = \sec^{-1} x + C$. In the latter case, $\int \frac{1}{x\sqrt{x^2-1}} dx = \int \frac{\sec \theta}{\sec \theta (-\tan \theta)} d\theta = -\theta + C = -\sec^{-1} x + C$.

8.4.80 For $1 \leq x = \sec \theta$ (where $\theta \in [0, \pi/2)$) we have $dx = \sec \theta \tan \theta d\theta$. Note that $|\tan \theta| = \tan \theta$ in this case. We have $\int \frac{\sqrt{x^2-1}}{x^3} dx = \int \frac{\sec \theta \tan \theta |\tan \theta|}{\sec^3 \theta} d\theta = \int \frac{\tan^2 \theta}{\sec^2 \theta} d\theta = \int \sin^2 \theta d\theta = \frac{1}{2} \int 1 - \cos 2\theta d\theta = \frac{1}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) + C = \frac{1}{2} \left(\sec^{-1} x - \frac{\sqrt{x^2-1}}{x^2} \right) + C$.

For the case $-1 \geq x = \sec \theta$, we have that $|\tan \theta| = -\tan \theta$. So in this case, $\int \frac{\sqrt{x^2-1}}{x^3} dx = \int \frac{\sec \theta \tan \theta |\tan \theta|}{\sec^3 \theta} d\theta = -\int \frac{\tan^2 \theta}{\sec^2 \theta} d\theta = -\int \sin^2 \theta d\theta = -\frac{1}{2} \int 1 - \cos 2\theta d\theta = -\frac{1}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) + C = -\frac{1}{2} (\theta - \sin \theta \cos \theta) + C$. Now note that because $\theta \in (\pi/2, \pi]$, we must have that $\cos \theta < 0$. Thus we have $-\frac{1}{2} \left(\sec^{-1} x + \frac{\sqrt{x^2-1}}{x^2} \right) + C$.

8.4.81

a. $E_x(a) = \frac{kQa}{2L} \int_{-L}^L \frac{dy}{(a^2 + y^2)^{3/2}}$. Let $y = a \tan \theta$ so that $dy = a \sec^2 \theta d\theta$. Then note that

$$\begin{aligned} \int_{-L}^L \frac{dy}{(a^2 + y^2)^{3/2}} &= 2 \int_0^L \frac{dy}{(a^2 + y^2)^{3/2}} = 2 \int_0^{\tan^{-1}(L/a)} \frac{a \sec^2 \theta}{a^3 \sec^3 \theta} d\theta = \frac{2}{a^2} \int_0^{\tan^{-1}(L/a)} \cos \theta d\theta \\ &= \frac{2}{a^2} \sin \theta \Big|_0^{\tan^{-1}(L/a)} = \frac{2}{a^2} \cdot \frac{L}{\sqrt{a^2 + L^2}} = \frac{2L}{a^2 \sqrt{a^2 + L^2}}. \end{aligned}$$

$$\text{Thus, } E_x(a) = \frac{kQ}{a\sqrt{a^2 + L^2}}.$$

b. Set $\rho = Q/(2L)$. Then because $\lim_{L \rightarrow \infty} \frac{2L}{a^2 \sqrt{a^2 + L^2}} = \frac{2}{a^2}$, we have

$$E_x(a, 0) \approx \frac{kQa}{2L} \lim_{L \rightarrow \infty} \int_{-L}^L \frac{dy}{(a^2 + y^2)^{3/2}} = \frac{kQa}{2L} \lim_{L \rightarrow \infty} \frac{2L}{a^2 \sqrt{a^2 + L^2}} = \frac{kQa}{2L} \left(\frac{2}{a^2} \right) = \frac{2kQ}{2aL} = \frac{2k\rho}{a}.$$

8.4.82

a. Let β be the angle which forms a linear pair with θ , and let α be the angle in the pictured triangle with vertex at $(a, 0)$. Note that $\theta + \beta = \pi$ and $\alpha + \beta = \frac{\pi}{2}$, so $\theta = \pi - \beta = \pi/2 + \alpha$. Thus, $\sin(\theta) = \sin(\pi/2 + \alpha) = \cos \alpha$. Now, because $r^2 = a^2 + y^2$ and $\cos \alpha = \frac{a}{r}$, we have that

$$\frac{\sin \theta}{r^2} = \frac{a}{r^3} = \frac{a}{(a^2 + y^2)^{3/2}}.$$

Thus

$$\int_{-L}^L \frac{\sin \theta}{r^2} dy = \int_{-L}^L \frac{a}{(a^2 + y^2)^{3/2}} dy = 2 \int_0^L \frac{a}{(a^2 + y^2)^{3/2}} dy.$$

Let $y = a \tan u$ so that $dy = a \sec^2 u du$. Substituting, we have

$$2 \int_0^{\tan^{-1}(L/a)} \frac{a^2 \sec^2 u}{a^3 \sec^3 u} du = \frac{2}{a} \int_0^{\tan^{-1}(L/a)} \cos u du = \frac{2}{a} \sin u \Big|_0^{\pi/2 - \theta_0} = \frac{2}{a} \sin(\pi/2 - \theta_0) = \frac{2}{a} \cos \theta_0$$

where $\tan \theta_0 = \frac{a}{L}$. Thus,

$$B(a) = \frac{\mu_0 I}{4\pi} \cdot \frac{2}{a} \cos \theta_0 = \frac{\mu_0 I}{2\pi a} \cos \theta_0.$$

b. From part (a), $\cos \theta_0 = \frac{L}{\sqrt{a^2 + L^2}}$, and thus $\lim_{L \rightarrow \infty} \cos \theta_0 = \lim_{L \rightarrow \infty} \frac{L}{\sqrt{a^2 + L^2}} = \lim_{L \rightarrow \infty} \frac{1}{\sqrt{(a^2/L^2) + 1}} = 1$.

Therefore, $\lim_{L \rightarrow \infty} B(a) = \frac{\mu_0 I}{2\pi a}$.

8.4.83

a. The area of sector OAB is given by the formula $\frac{\theta}{2}a^2$ where $\sin \theta = x/a$. The area of triangle OBC is $\frac{x}{2}\sqrt{a^2 - x^2}$. Thus,

$$F(x) = \frac{a^2 \sin^{-1}(x/a)}{2} + \frac{x\sqrt{a^2 - x^2}}{2}.$$

b. By the first fundamental theorem of calculus, $F(x)$ is an antiderivative of $\sqrt{a^2 - x^2}$, because $F(x) = \int_0^x \sqrt{a^2 - t^2} dt$. Thus, any other antiderivative differs from this by a constant, so

$$\int \sqrt{a^2 - x^2} dx = \frac{a^2 \sin^{-1}(x/a)}{2} + \frac{x\sqrt{a^2 - x^2}}{2} + C.$$

8.4.84

a. $0 = -\frac{1}{2}ka^2 + y_{\max}$ when $a = \sqrt{\frac{2y_{\max}}{k}}$.

b. $y'(x) = -kx$, so $1 + (y'(x))^2 = 1 + k^2x^2$, so the arc length is given by

$$L = \int_{-a}^a \sqrt{1 + (y'(x))^2} dx = \int_{-a}^a \sqrt{1 + k^2x^2} dx = 2 \int_0^a \sqrt{1 + k^2x^2} dx$$

because the function is an even function.

c. Let $x = \frac{1}{k} \tan \theta$ so that $dx = \frac{1}{k} \sec^2 \theta d\theta$. Then the arc length integral is equal to

$$\begin{aligned} 2 \int_0^{\tan^{-1}(ak)} \frac{1}{k} \sec^2 \theta \cdot \sec \theta d\theta &= \frac{1}{k} \left(\sec \theta \tan \theta \Big|_0^{\tan^{-1}(ak)} + \ln |\sec \theta + \tan \theta| \Big|_0^{\tan^{-1}(ak)} \right) \\ &= \frac{1}{k} \left(ak\sqrt{1 + a^2k^2} + \ln(\sqrt{1 + a^2k^2} + ak) \right). \end{aligned}$$

Noting that $\tan(\theta) = ak$, $a = (V^2/g) \sin \theta \cos \theta$, and $k = \frac{g}{V^2 \cos^2 \theta}$, we can write L as a function of θ as $L(\theta) = (V^2/g) \sin \theta + (V^2/g) \cos^2 \theta (\ln(\sqrt{1 + \tan^2 \theta} + \tan \theta))$, and using an identity for the inverse hyperbolic sine function, this can be written as $L(\theta) = \frac{V^2}{g} (\sin \theta + \cos^2 \theta \sinh^{-1}(\tan \theta))$.

d. Using the expression for $L(\theta)$ above, we have

$L'(\theta) = (V^2/g) (\cos \theta - 2 \cos \theta \sin \theta (\sinh^{-1}(\tan \theta)) - \cos^2 \theta (1 + \tan^2 \theta)^{-1/2} \sec^2 \theta) = \frac{2V^2 \cos \theta}{g} (1 - \sin \theta \sinh^{-1}(\tan \theta))$. This expression is 0 when $\sin \theta (\sinh^{-1} \tan \theta) = 1$, which is equivalent to $\sin \theta (\ln(\tan \theta + \sec \theta)) = 1$. Either the first or second derivative tests can be used to see that this critical number yields the maximum.

e. Some experimenting reveals that the maximum occurs for $\theta \approx 56.5$ degrees.

8.4.85

a. Because $t \in [0, \pi]$ so that $\sin t \geq 0$, we have

$$\int_a^b \sqrt{\frac{1 - \cos t}{g(\cos a - \cos t)}} dt = \int_a^b \sqrt{\frac{(1 - \cos t)(1 + \cos t)}{g(1 + \cos t)(\cos a - \cos t)}} dt = \int_a^b \sin t \sqrt{\frac{1}{g(1 + \cos t)(\cos a - \cos t)}} dt.$$

Let $u = \cos t$ so that $du = -\sin t dt$. Then the given integral is equal to

$$-\frac{1}{\sqrt{g}} \int_{\cos a}^{\cos b} \sqrt{\frac{1}{(1+u)(\cos a - u)}} du.$$

Now we complete the square:

$$\begin{aligned} (1+u)(\cos a - u) &= \cos a + (\cos a - 1)u - u^2 \\ &= -\left(u^2 - (\cos a - 1)u + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\frac{\cos a - 1}{2}\right)^2\right) + \cos a \\ &= \cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(u - \frac{\cos a - 1}{2}\right)^2 = \left(\frac{\cos a + 1}{2}\right)^2 - \left(u - \frac{\cos a - 1}{2}\right)^2. \end{aligned}$$

Thus, setting $v = u - \frac{\cos a - 1}{2}$ we have that the original integral is equal to

$$-\frac{1}{\sqrt{g}} \int_{(\cos a + 1)/2}^{\cos b - \frac{\cos a - 1}{2}} \frac{1}{\sqrt{k^2 - v^2}} dv \text{ where } k = \frac{(\cos a + 1)}{2}.$$

Now, $\int \frac{1}{\sqrt{k^2 - v^2}} dv = \int \frac{k \cos \theta}{k \cos \theta} d\theta = \theta + C = \sin^{-1}(v/k) + C$ where $v = k \sin \theta$.

Therefore, the original integral is equal to

$$\begin{aligned} -\frac{1}{\sqrt{g}} \sin^{-1} \left(\frac{2v}{\cos a + 1} \right) \Big|_{(\cos a + 1)/2}^{\cos b - (\cos a - 1)/2} &= \frac{1}{\sqrt{g}} \left(\sin^{-1} \left(\frac{\cos a + 1}{\cos a + 1} \right) - \sin^{-1} \left(\frac{2 \cos b - \cos a + 1}{\cos a + 1} \right) \right) \\ &= \frac{1}{\sqrt{g}} \left(\frac{\pi}{2} - \sin^{-1} \left(\frac{2 \cos b - \cos a + 1}{\cos a + 1} \right) \right). \end{aligned}$$

b. Letting $b = \pi$, we have that the integral is equal to

$$\begin{aligned} &\frac{1}{\sqrt{g}} \left(\frac{\pi}{2} - \sin^{-1} \left(\frac{-2 - \cos a + 1}{\cos a + 1} \right) \right) \\ &= \frac{1}{\sqrt{g}} \left(\frac{\pi}{2} - \sin^{-1}(-1) \right) = \frac{1}{\sqrt{g}} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = \frac{\pi}{\sqrt{g}}. \end{aligned}$$

8.4.86 Let $x = 2 \tan^{-1} u$ so that $u = \tan(x/2)$ and $\sec^2(x/2) = 1 + \tan^2(x/2) = 1 + u^2$. Also, $\cos^2(x/2) = 1/(1 + u^2)$. By the double angle identity,

$$\cos x = \cos^2(x/2) - \sin^2(x/2) = \cos^2(x/2) - u^2 \cos^2(x/2) = (1 - u^2) \cos^2(x/2) = \frac{1 - u^2}{1 + u^2}.$$

Also, $\sin x = 2 \sin(x/2) \cos(x/2) = 2 \tan(x/2) \cos^2(x/2) = \frac{2u}{1 + u^2}$. Now

$$\begin{aligned} \int \frac{1}{1 + \sin x + \cos x} dx &= 2 \int \frac{1}{1 + u^2} \cdot \frac{1}{1 + \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} du = 2 \int \frac{1}{1 + u^2 + 2u + 1 - u^2} du = 2 \int \frac{1}{2 + 2u} du = \\ &= \int \frac{1}{1 + u} du = \ln |1 + u| + C = \ln |1 + \tan(x/2)| + C. \end{aligned}$$

8.5 Partial Fractions

8.5.1 Proper rational functions can be integrated using partial fraction decomposition.

8.5.2 Your answers may vary.

- a. $x - 1$.
- b. $(x - 1)^3$.
- c. $x^2 + x + 1$.
- d. $(x^2 + x + 1)^2$

8.5.3

- a. $\frac{A}{x - 3}$.
- b. $\frac{A_1}{x - 4}, \frac{A_2}{(x - 4)^2}, \frac{A_3}{(x - 4)^3}$.
- c. $\frac{Ax + B}{x^2 + 2x + 6}$.

8.5.4 The first step is to divide the numerator by the denominator via long division in order to write the quotient as the sum of a polynomial and a proper rational function. Thus we would write

$$\frac{x^2 + 2x - 3}{x + 1} = x + 1 - \frac{4}{x + 1}.$$

$$\mathbf{8.5.5} \quad \frac{4x}{(x - 4)(x - 5)} = \frac{A}{x - 4} + \frac{B}{x - 5}.$$

$$\mathbf{8.5.6} \quad \frac{4x + 1}{(2x - 1)(2x + 1)} = \frac{A}{2x - 1} + \frac{B}{2x + 1}.$$

$$\mathbf{8.5.7} \quad \frac{x + 3}{(x - 5)^2} = \frac{A}{x - 5} + \frac{B}{(x - 5)^2}.$$

$$\mathbf{8.5.8} \quad \frac{2}{x(x - 1)^2} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}.$$

$$\mathbf{8.5.9} \quad \frac{4}{x(x + 1)(x - 1)(x + 2)(x - 2)} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x - 1} + \frac{D}{x + 2} + \frac{E}{x - 2}.$$

$$\mathbf{8.5.10} \quad \frac{20x}{(x - 1)^2(x^2 + 1)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{Cx + D}{x^2 + 1}.$$

$$\mathbf{8.5.11} \quad \frac{1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

$$\mathbf{8.5.12} \quad \frac{2x^2 + 3}{(x^2 - 8x + 16)(x^2 + 3x + 4)} = \frac{2x^2 + 3}{(x - 4)^2(x^2 + 3x + 4)} = \frac{A}{x - 4} + \frac{B}{(x - 4)^2} + \frac{Cx + D}{x^2 + 3x + 4}.$$

$$\mathbf{8.5.13} \quad \frac{x^4 + 12x^2}{(x - 2)^2(x^2 + x + 2)^2} = \frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{Cx + D}{x^2 + x + 2} + \frac{Ex + F}{(x^2 + x + 2)^2}.$$

$$\mathbf{8.5.14} \quad \frac{6x^4 - 4x^3 + 15x^2 - 5x + 7}{(x - 2)(2x^2 + 3)^2} = \frac{A}{x - 2} + \frac{Bx + C}{2x^2 + 3} + \frac{Dx + E}{(2x^2 + 3)^2}.$$

$$8.5.15 \quad \frac{x}{(x-2)^2(x+2)^2(x^2+4)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{Ex+F}{x^2+4} + \frac{Gx+H}{(x^2+4)^2}.$$

$$8.5.16 \quad \frac{x^2+2x+6}{x^3(x^2+x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+x+1} + \frac{Fx+G}{(x^2+x+1)^2}.$$

8.5.17 $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$. Thus, $A(x-2) + B(x-1) = 5x-7$. Equating coefficients gives $A+B=5$ and $-2A-B=-7$. Solving this system yields $A=2$, $B=3$. Thus,

$$\frac{5x-7}{x^2-3x+2} = \frac{2}{x-1} + \frac{3}{x-2}.$$

8.5.18 $\frac{11x-10}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$. Thus, $A(x-1) + Bx = 11x-10$. Equating coefficients gives $A+B=11$, $-A=-10$. Solving this system yields $A=10$, $B=1$. Thus,

$$\frac{11x-10}{x^2-x} = \frac{10}{x} + \frac{1}{x-1}.$$

8.5.19 $\frac{6}{x^2-2x-8} = \frac{6}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$. Thus, $6 = A(x+2) + B(x-4)$. Equating coefficients gives $A+B=0$ and $2A-4B=6$. Solving this system yields $A=1$ and $B=-1$. Thus,

$$\frac{6}{x^2-2x-8} = \frac{1}{x-4} - \frac{1}{x+2}.$$

8.5.20 $\frac{x^2-4x+11}{(x-3)(x-1)(x+1)} = \frac{A}{x-3} + \frac{B}{x-1} + \frac{C}{x+1}$. Thus, $x^2-4x+11 = A(x-1)(x+1) + B(x-3)(x+1) + C(x-3)(x-1)$. Letting $x=1$ gives $B=-2$, letting $x=-1$ gives $C=2$, and letting $x=3$ gives $A=1$. Thus,

$$\frac{x^2-4x+11}{(x-3)(x-1)(x+1)} = \frac{1}{x-3} + \frac{-2}{x-1} + \frac{2}{x+1}.$$

8.5.21 By long division, $\frac{2x^2+5x+6}{x^2+3x+2} = 2 + \frac{-x+2}{(x+1)(x+2)}$. We use a partial fraction decomposition on $\frac{-x+2}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$. Therefore $A(x+2) + B(x+1) = -x+2$, so $A+B=-1$ and $2A+B=2$. So $A=3$ and $B=-4$. Thus

$$\frac{2x^2+5x+6}{x^2+3x+2} = 2 + \frac{3}{x+1} - \frac{4}{x+2}.$$

8.5.22 By long division, we can write $\frac{x^4+2x^3+x}{x^2-1}$ as $x^2+2x+1 + \frac{3x+1}{x^2-1} = (x+1)^2 + \frac{3x+1}{(x-1)(x+1)}$. We use a partial fraction decomposition on $\frac{3x+1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$. We have $A(x+1) + B(x-1) = 3x+1$, so $A+B=3$ and $A-B=1$. Therefore, $A=2$ and $B=1$. We have

$$\frac{x^4+2x^3+x}{x^2-1} = (x+1)^2 + \frac{2}{x-1} + \frac{1}{x+1}.$$

8.5.23 If we write $\frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$, we have $3 = A(x+2) + B(x-1)$. Letting $x=-2$ yields $B=-1$ and letting $x=1$ yields $A=1$. Thus, the original integral is equal to

$$\int \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx = \ln|x-1| - \ln|x+2| + C = \ln \left| \frac{x-1}{x+2} \right| + C.$$

8.5.24 If we write $\frac{8}{(x-2)(x+6)} = \frac{A}{x-2} + \frac{B}{x+6}$, we have $8 = A(x+6) + B(x-2)$. Letting $x = -6$ yields $B = -1$ and letting $x = 2$ yields $A = 1$. Thus the original integral is equal to

$$\int \left(\frac{1}{x-2} - \frac{1}{x+6} \right) dx = \ln|x-2| - \ln|x+6| + C.$$

8.5.25 If we write $\frac{6}{x^2-1} = \frac{6}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$, then we have $6 = A(x+1) + B(x-1)$. Letting $x = -1$ yields $B = -3$ and letting $x = 1$ yields $A = 3$. Thus, the original integral is equal to

$$\int \left(\frac{3}{x-1} - \frac{3}{x+1} \right) dx = 3(\ln|x-1| - \ln|x+1|) + C = 3 \ln \left| \frac{x-1}{x+1} \right| + C.$$

8.5.26 If we write $\frac{1}{t^2-9} = \frac{A}{t-3} + \frac{B}{t+3}$, then we have $1 = A(t+3) + B(t-3)$. Letting $t = -3$ yields $B = -1/6$ and letting $t = 3$ yields $A = 1/6$. Thus the original integral is equal to

$$\int_0^1 \left(\frac{1/6}{t-3} - \frac{1/6}{t+3} \right) dt = \frac{1}{6} (\ln|t-3| - \ln|t+3|) \Big|_0^1 = -\frac{\ln 2}{6}.$$

8.5.27 $\frac{8x-5}{(x-1)(3x-2)} = \frac{A}{x-1} + \frac{B}{3x-2}$. Thus $8x-5 = A(3x-2) + B(x-1) = (3A+B)x - 2A - B$. So $3A+B=8$ and $-2A-B=-5$. Solving gives $A=3$ and $B=-1$. We have

$$\int \frac{8x-5}{3x^2-5x+2} dx = \int \left(\frac{3}{x-1} - \frac{1}{3x-2} \right) dx = 3 \ln|x-1| - \frac{1}{3} \ln|3x-2| + C.$$

8.5.28 $\frac{7x-2}{3x^2-2x} = \frac{7x-2}{x(3x-2)} = \frac{A}{x} + \frac{B}{3x-2}$. Then $7x-2 = A(3x-2) + Bx = (3A+B)x - 2A$. So $A=1$ and $B=4$. We have

$$\int_1^2 \frac{7x-2}{3x^2-2x} dx = \int_1^2 \left(\frac{1}{x} + \frac{4}{3x-2} \right) dx = \left(\ln x + \frac{4}{3} \ln(3x-2) \right) \Big|_1^2 = \ln 2 + \frac{4}{3} \ln 4 = \ln 2 + \frac{8}{3} \ln 2 = \frac{11 \ln 2}{3}.$$

8.5.29 If we write $\frac{5x}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$, then we have $5x = A(x+2) + B(x-3)$. Letting $x = -2$ yields $B = 2$ and letting $x = 3$ yields $A = 3$. Thus the original integral is equal to

$$\int_{-1}^2 \left(\frac{3}{x-3} + \frac{2}{x+2} \right) dx = (3 \ln|x-3| + 2 \ln|x+2|) \Big|_{-1}^2 = \ln(16) - \ln(64) = -\ln 4.$$

8.5.30 If we write $\frac{21x^2}{x^3-x^2-12x} = \frac{21x^2}{x(x-4)(x+3)} = \frac{21x}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}$, then we have $21x = A(x+3) + B(x-4)$. Letting $x = 4$ yields $A = 12$. Letting $x = -3$ yields $B = 9$. Thus, the original integral is equal to

$$\int \left(\frac{12}{x-4} + \frac{9}{x+3} \right) dx = 12 \ln|x-4| + 9 \ln|x+3| + C.$$

8.5.31 Let $\frac{6x^2}{x^4-5x^2+4} = \frac{6x^2}{(x-2)(x+2)(x-1)(x+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-1} + \frac{D}{x+1}$. Then

$$6x^2 = A(x+2)(x-1)(x+1) + B(x-2)(x-1)(x+1) + C(x-2)(x+2)(x+1) + D(x-2)(x+2)(x-1).$$

Letting $x = 2$ gives $A = 2$, letting $x = -2$ gives $B = -2$, letting $x = 1$ gives $C = -1$, and letting $x = -1$ gives $D = 1$. Thus, the original integral is equal to

$$\int \left(\frac{2}{x-2} - \frac{2}{x+2} - \frac{1}{x-1} + \frac{1}{x+1} \right) dx = \ln \left| \frac{(x-2)^2(x+1)}{(x+2)^2(x-1)} \right| + C.$$

8.5.32 Let $\frac{4x-2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$. Thus,

$$4x-2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1).$$

Letting $x = 0$ gives $A = 2$, letting $x = 1$ gives $B = 1$, and letting $x = -1$ gives $C = -3$. Thus, the original integral is equal to

$$\int \left(\frac{2}{x} + \frac{1}{x-1} - \frac{3}{x+1} \right) dx = \ln \left| \frac{x^2(x-1)}{(x+1)^3} \right| + C.$$

8.5.33 After performing long division, we have that the original integrand is equal to $3 + \frac{13x-12}{(x-1)(x-2)}$,

and if we write $\frac{13x-12}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$, then $13x-12 = A(x-2) + B(x-1)$. Letting $x = 1$ yields $A = -1$ and letting $x = 2$ yields $B = 14$. Thus the original integral is equal to

$$3x - \int \frac{1}{x-1} dx + 14 \int \frac{1}{x-2} dx = 3x - \ln|x-1| + 14 \ln|x-2| + C.$$

8.5.34 After performing long division, we have that the original integrand is equal to $2z-1 + \frac{7z+1}{(z+3)(z-2)}$.

If we write $\frac{7z+1}{(z+3)(z-2)} = \frac{A}{z+3} + \frac{B}{z-2}$, then $7z+1 = A(z-2) + B(z+3)$. Letting $z = 2$ yields $B = 3$ and letting $z = -3$ yields $A = 4$. Thus, the original integral is equal to

$$\int \left(2z-1 + \frac{4}{z+3} + \frac{3}{z-2} \right) dz = z^2 - z + 4 \ln|z+3| + 3 \ln|z-2| + C.$$

8.5.35 Let $\frac{x^2+12x-4}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$. Then $x^2+12x-4 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2)$.

Letting $x = 0$ gives $A = 1$, letting $x = 2$ gives $B = 3$, and letting $x = -2$ gives $C = -3$. Thus, the original integral is equal to $\int \left(\frac{1}{x} + \frac{3}{x-2} - \frac{3}{x+2} \right) dx = \ln \left| \frac{x(x-2)^3}{(x+2)^3} \right| + C$.

8.5.36 Let $\frac{z^2+20z-15}{z(z+5)(z-1)} = \frac{A}{z} + \frac{B}{z+5} + \frac{C}{z-1}$. Then $z^2+20z-15 = A(z+5)(z-1) + Bz(z-1) + Cz(z+5)$.

Letting $z = 0$ gives $A = 3$, letting $z = -5$ gives $B = -3$, and letting $z = 1$ gives $C = 1$. Thus, the original integral is equal to $\int \left(\frac{3}{z} - \frac{3}{z+5} + \frac{1}{z-1} \right) dz = \ln \left| \frac{z^3(z-1)}{(z+5)^3} \right| + C$.

8.5.37 If we write $\frac{1}{x^4-10x^2+9} = \frac{1}{(x-1)(x+1)(x-3)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-3} + \frac{D}{x+3}$ then $1 = A(x+1)(x-3)(x+3) + B(x-1)(x-3)(x+3) + C(x-1)(x+1)(x+3) + D(x-1)(x+1)(x-3)$. Letting $x = -1$ yields $B = 1/16$. Letting $x = 3$ yields $C = 1/48$. Letting $x = -3$ yields $D = -1/48$, and letting $x = 1$ yields $A = -1/16$. Thus the original integral is equal to

$$\begin{aligned} & \int \left(-\frac{1/16}{x-1} + \frac{1/16}{x+1} + \frac{1/48}{x-3} - \frac{1/48}{x+3} \right) dx \\ &= -\frac{1}{16} \ln|x-1| + \frac{1}{16} \ln|x+1| + \frac{1}{48} \ln|x-3| - \frac{1}{48} \ln|x+3| + C \\ &= \ln \left| \frac{(x+1)^3(x-3)}{(x-1)^3(x+3)} \right|^{1/48} + C. \end{aligned}$$

8.5.38 If we write $\frac{2}{x^2-4x-32} = \frac{A}{x-8} + \frac{B}{x+4}$, then we have $2 = A(x+4) + B(x-8)$. Letting $x = -4$ yields $B = -1/6$ and letting $x = 8$ yields $A = 1/6$. Thus, the original integral is equal to

$$\int_0^5 \left(\frac{1/6}{x-8} - \frac{1/6}{x+4} \right) dx = \frac{1}{6} (\ln|x-8| - \ln|x+4|) \Big|_0^5 = \frac{1}{6} (\ln(1/3) - \ln 2) = -\frac{\ln 6}{6}.$$

8.5.39 If we write $\frac{81}{x^3 - 9x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-9}$, then $81 = Ax(x-9) + B(x-9) + C(x^2)$. Letting $x = 0$ yields $B = -9$. Letting $x = 9$ yields $C = 1$. If we let $x = 10$, then we have $81 = 10A + B + 100C = 10A - 9 + 100$, so $A = -1$. Thus, the original integral is equal to $\int \left(-\frac{1}{x} - \frac{9}{x^2} + \frac{1}{x-9} \right) dx = \ln \left| \frac{(x-9)}{x} \right| + \frac{9}{x} + C$.

8.5.40 If we write $\frac{16x^2}{(x-6)(x+2)^2} = \frac{A}{x-6} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$, then we have $16x^2 = A(x+2)^2 + B(x-6)(x+2) + C(x-6)$. Letting $x = 6$ yields $A = 9$. Letting $x = -2$ yields $C = -8$. Letting $x = 0$ gives $0 = 36 - 12B + 48$, so $B = 7$. Thus the original integral is equal to

$$\int \left(\frac{9}{x-6} + \frac{7}{x+2} - \frac{8}{(x+2)^2} \right) dx = \ln |(x+2)^7(x-6)^9| + \frac{8}{x+2} + C.$$

8.5.41 If we write $\frac{x}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$, then we have $x = A(x+3) + B$. Letting $x = -3$ yields $B = -3$, and then letting $x = -2$ yields $A = 1$. Thus the original integral is equal to

$$\int_{-1}^1 \left(\frac{1}{x+3} - \frac{3}{(x+3)^2} \right) dx = \left(\ln |x+3| + \frac{3}{x+3} \right) \Big|_{-1}^1 = \ln 4 + \frac{3}{4} - \left(\ln 2 + \frac{3}{2} \right) = \ln 2 - \frac{3}{4}.$$

8.5.42 If we write $\frac{1}{x^3 - 2x^2 - 4x + 8} = \frac{1}{(x+2)(x-2)^2} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$, then $1 = A(x-2)^2 + B(x+2)(x-2) + C(x+2)$. Letting $x = 2$ yields $C = 1/4$. Letting $x = -2$ yields $A = 1/16$. Letting $x = 3$ yields $1 = \frac{1}{16} + 5B + 5 \cdot \frac{1}{4}$, so $B = -\frac{1}{16}$. Thus, the original integral is equal to

$$\int \left(\frac{1/16}{x+2} - \frac{1/16}{x-2} + \frac{1/4}{(x-2)^2} \right) dx = \frac{1}{16} (\ln |x+2| - \ln |x-2|) - \frac{1}{4(x-2)} + C.$$

8.5.43 If we write $\frac{2}{x^3 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$, then $2 = Ax(x+1) + B(x+1) + Cx^2$. Letting $x = 0$ yields $B = 2$, and letting $x = -1$ yields $C = 2$. Then letting $x = 1$ yields $A = -2$. So the original integral is equal to

$$\int \left(-\frac{2}{x} + \frac{2}{x^2} + \frac{2}{x+1} \right) dx = 2 (\ln |x+1| - \ln |x|) - \frac{2}{x} + C.$$

8.5.44 If we write $\frac{2}{t^3(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \frac{D}{t+1}$ then we have $2 = At^2(t+1) + Bt(t+1) + C(t+1) + Dt^3$. Letting $t = 0$ yields $C = 2$, and letting $t = -1$ reveals that $D = -2$. Now if we let $t = 1$ we have that $2 = 2A + 2B + 4 - 2$, so $A = -B$. Letting $t = 2$ yields the equation $12 = 12A + 6B = 12A - 6A = 6A$, so $A = 2$ and $B = -2$. So the original integral is equal to

$$\int_1^2 \left(\frac{2}{t} - \frac{2}{t^2} + \frac{2}{t^3} - \frac{2}{t+1} \right) dt = \left(2 (\ln |t| - \ln |t+1|) + \frac{2}{t} - \frac{1}{t^2} \right) \Big|_1^2 = \ln(16/9) - 1/4.$$

8.5.45 If we write $\frac{x-5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$, then we have $x-5 = Ax(x+1) + B(x+1) + Cx^2$. Letting $x = 0$ yields $B = -5$, and letting $x = -1$ yields $C = -6$. Then letting $x = 1$ yields $-4 = 2A - 10 - 6$, so $A = 6$. The original integral is thus equal to

$$\int \left(\frac{6}{x} - \frac{5}{x^2} - \frac{6}{x+1} \right) dx = 6 (\ln |x| - \ln |x+1|) + \frac{5}{x} + C.$$

8.5.46 Let $\frac{x^2}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$. Then $x^2 = A(x-2)^2 + B(x-2) + C$. Letting $x = 2$ gives $C = 4$. Letting $x = 0$ gives $0 = 4A - 2B + 4$, and letting $x = 1$ gives $1 = A - B + 4$. Solving the system of two linear equations results in $A = 1$ and $B = 4$. The original integral is therefore equal to

$$\int \left(\frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{4}{(x-2)^3} \right) dx = \ln |x-2| - \frac{4}{x-2} - \frac{2}{(x-2)^2} + C.$$

8.5.47 By long division, we can write the integrand as $x + \frac{2x}{(x-5)^2}$. We write

$$\frac{2x}{(x-5)^2} = \frac{A}{x-5} + \frac{B}{(x-5)^2},$$

so that $A(x-5) + B = 2x$. Then $A = 2$ and $B = 10$. We have

$$\int \frac{x^3 - 10x^2 + 27x}{x^2 - 10x + 25} dx = \int \left(x + \frac{2}{x-5} + \frac{10}{(x-5)^2} \right) dx = \frac{x^2}{2} + 2 \ln|x-5| - \frac{10}{x-5} + C.$$

8.5.48 By long division, we can write the integrand as $1 + \frac{2x^2 - x + 2}{x(x-1)^2}$. We write

$$\frac{2x^2 - x + 2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

So

$$2x^2 - x + 2 = A(x-1)^2 + Bx(x-1) + Cx.$$

Letting $x = 0$ gives $A = 2$. Letting $x = 1$ gives $C = 3$. Letting $x = 2$ then gives $8 = 2 + 2B + 6$, so $B = 0$. We have

$$\int \frac{x^3 + 2}{x^3 - 2x^2 + x} dx = \int \left(1 + \frac{2}{x} + \frac{3}{(x-1)^2} \right) dx = x + 2 \ln|x| - \frac{3}{x-1} + C.$$

8.5.49 Let $\frac{x^2 - 4}{x^3 - 2x^2 + x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$. Then $x^2 - 4 = A(x-1)^2 + Bx(x-1) + Cx$. Letting $x = 0$ gives $A = -4$. Letting $x = 1$ gives $C = -3$. Letting $x = 2$ gives $0 = -4 + 2B - 6$, so $B = 5$. The given integral is thus equal to

$$\int \left(-\frac{4}{x} + \frac{5}{x-1} - \frac{3}{(x-1)^2} \right) dx = \ln \left| \frac{(x-1)^5}{x^4} \right| + \frac{3}{x-1} + C.$$

8.5.50 Let $\frac{8(x^2 + 4)}{x(x^2 + 8)} = \frac{Ax + B}{x^2 + 8} + \frac{C}{x}$. Then $8(x^2 + 4) = Ax^2 + Bx + C(x^2 + 8)$. Letting $x = 0$ gives $C = 4$. Letting $x = 1$ gives $40 = A + B + 36$, so $A + B = 4$. Letting $x = -1$ gives $A - B = 4$, so $A = 4$ and $B = 0$. The original integral is thus equal to

$$\int \left(\frac{4x}{x^2 + 8} + \frac{4}{x} \right) dx = 2 \ln(x^2 + 8) + 4 \ln|x| + C = \ln((x^2 + 8)^2 x^4) + C.$$

8.5.51 Let $\frac{x^2 + x + 2}{(x+1)(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x+1}$. Then $x^2 + x + 2 = (Ax + B)(x+1) + C(x^2 + 1)$. Letting $x = -1$ gives $C = 1$. Letting $x = 0$ gives $2 = B + 1$, so $B = 1$. Letting $x = 1$ gives $4 = 2A + 2 + 2$, so $A = 0$. The original integral is therefore equal to

$$\int \left(\frac{1}{x^2 + 1} + \frac{1}{x+1} \right) dx = \tan^{-1}(x) + \ln|x+1| + C.$$

8.5.52 Let $\frac{x^2 + 3x + 2}{x(x^2 + 2x + 2)} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{C}{x}$. Then $x^2 + 3x + 2 = Ax^2 + Bx + C(x^2 + 2x + 2)$. Letting $x = 0$ gives $C = 1$. Letting $x = 1$ gives $6 = A + B + 5$, so $A + B = 1$. Letting $x = -1$ gives $0 = A - B + 1$. Solving the system of linear equations gives $A = 0$ and $B = 1$. The original integral is therefore equal to

$$\begin{aligned} \int \left(\frac{1}{x^2 + 2x + 2} + \frac{1}{x} \right) dx &= \int \left(\frac{1}{(x^2 + 2x + 1) + 1} + \frac{1}{x} \right) dx \\ &= \int \left(\frac{1}{(x+1)^2 + 1} + \frac{1}{x} \right) dx = \tan^{-1}(x+1) + \ln|x| + C. \end{aligned}$$

8.5.53 Let $\frac{2x^2 + 5x + 5}{(x+1)(x^2 + 2x + 2)} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{C}{x+1}$. Then $2x^2 + 5x + 5 = (Ax + B)(x+1) + C(x^2 + 2x + 2)$. Letting $x = -1$ gives $C = 2$. Letting $x = 0$ gives $5 = B + 4$, so $B = 1$. Letting $x = 1$ gives $12 = 2A + 2 + 2(5)$, so $A = 0$. The original integral is therefore equal to

$$\int \left(\frac{1}{x^2 + 2x + 2} + \frac{2}{x+1} \right) dx = \int \left(\frac{1}{(x+1)^2 + 1} + \frac{2}{x+1} \right) dx = \tan^{-1}(x+1) + \ln((x+1)^2) + C.$$

8.5.54 If we write $\frac{z+1}{z(z^2+4)} = \frac{A}{z} + \frac{Bz+C}{z^2+4}$, then we have that $z+1 = A(z^2+4) + (Bz+C)z$. Letting $z = 0$ yields $A = 1/4$, and we have $z+1 = (1/4+B)z^2 + Cz + 1$, so equating coefficients gives $B = -1/4$ and $C = 1$. So the original integral is equal to $\int \left(\frac{1}{4z} - \frac{z}{4(z^2+4)} + \frac{1}{z^2+4} \right) dz$. The middle term can be handled via the substitution $u = z^2 + 4$, and the last term is recognizable as the derivative of $\frac{1}{2} \tan^{-1}(z/2)$. Thus the original integral is equal to

$$\frac{1}{4} \ln|z| - \frac{1}{8} \ln(z^2 + 4) + \frac{1}{2} \tan^{-1} \frac{z}{2} + C.$$

8.5.55 If we write $\frac{20x}{(x-1)(x^2+4x+5)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+5}$, then $20x = A(x^2+4x+5) + (Bx+C)(x-1)$. Letting $x = 1$ yields $A = 2$. Letting $x = 0$ yields $0 = 10 - C$, so $C = 10$. Letting $x = 2$ yields $40 = 34 + 2B + 10$, so $B = -2$. The original integral is thus

$$\begin{aligned} \int \left(\frac{2}{x-1} - \frac{2x-10}{x^2+4x+5} \right) dx &= \int \left(\frac{2}{x-1} - \frac{(2x+4)-14}{x^2+4x+5} \right) dx \\ &= \int \left(\frac{2}{x-1} - \frac{(2x+4)}{x^2+4x+5} + \frac{14}{(x+2)^2+1} \right) dx \\ &= \ln \left| \frac{(x-1)^2}{x^2+4x+5} \right| + 14 \tan^{-1}(x+2) + C. \end{aligned}$$

8.5.56 Note that this rational function is already in decomposition form, so any attempt to decompose it further will be futile. Instead we write the given integral as the sum $\int \frac{2x}{x^2+4} dx + \int \frac{1}{x^2+4} dx$. For the first integral, let $u = x^2 + 4$ so that $du = 2x dx$. It is then equal to $\int \frac{1}{u} du = \ln|x^2+4| + C$. The second integral can be written as $\frac{1}{4} \int \frac{1}{(x/2)^2 + 1} dx = \frac{1}{2} \tan^{-1} \frac{x}{2} + D$. So the original integral is equal to

$$\ln|x^2+4| + \frac{1}{2} \tan^{-1} \frac{x}{2} + E.$$

8.5.57 $\frac{x^3+5x}{(x^2+3)^2} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2}$. Then

$$x^3 + 5x = (Ax+B)(x^2+3) + Cx+D = Ax^3 + Bx^2 + (3A+C)x + 3B+D.$$

Then $A = 1$, $B = 0$, $3A + C = 5$, and $3B + D = 0$. So $C = 2$ and $D = 0$. We have

$$\int \frac{x^3+5x}{(x^2+3)^2} dx = \int \left(\frac{x}{x^2+3} + \frac{2x}{(x^2+3)^2} \right) dx = \frac{1}{2} \ln|x^2+3| - \frac{1}{x^2+3} + C.$$

8.5.58 After performing long division, we have that the original integrand is equal to $x - \frac{9x^2-1}{x(x^2+9)}$. If we write $\frac{9x^2-1}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$, we have $9x^2-1 = A(x^2+9) + (Bx+C)x$. Letting $x = 0$ yields $A = -1/9$,

and letting $x = 1$ yields $8 = 10(-1/9) + B + C$. Letting $x = -1$ yields $8 = 10(-1/9) + B - C$. Thus $C = 0$ and therefore $B = 82/9$. Therefore the original integral is equal to

$$\frac{x^2}{2} + \frac{1}{9} \ln|x| - \frac{82}{9} \int \frac{x}{x^2+9} dx = \frac{x^2}{2} + \frac{1}{9} \ln|x| - \frac{41}{9} \ln|x^2+9| + C.$$

8.5.59 $\frac{x^3+6x^2+12x+6}{(x^2+6x+10)^2} = \frac{Ax+B}{x^2+6x+10} + \frac{Cx+D}{(x^2+6x+10)^2}$. Then

$$x^3+6x^2+12x+6 = (Ax+B)(x^2+6x+10) + Cx+D = Ax^3 + (B+6A)x^2 + (10A+6B+C)x + 10B+D.$$

Then $A = 1$, $B = 0$, $C = 2$, and $D = 6$. The given integral is then

$$\int \left(\frac{x}{x^2+6x+10} + \frac{2x+6}{(x^2+6x+10)^2} \right) dx = \int \frac{x}{(x+3)^2+1} dx - \frac{1}{x^2+6x+10} + C.$$

Note that

$$\int \frac{x}{(x+3)^2+1} dx = \int \left(\frac{x+3}{(x+3)^2+1} - \frac{3}{(x+3)^2+1} \right) dx = \frac{1}{2} \ln(x^2+6x+10) - 3 \tan^{-1}(x+3) + C.$$

So the given integral is equal to

$$\frac{1}{2} \ln(x^2+6x+10) - 3 \tan^{-1}(x+3) - \frac{1}{x^2+6x+10} + C.$$

8.5.60 If we write $\frac{1}{(y^2+1)(y^2+2)} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{y^2+2}$ then

$$1 = (Ay+B)(y^2+2) + (Cy+D)(y^2+1) = (A+C)y^3 + (B+D)y^2 + (2A+C)y + 2B+D.$$

Equating coefficients gives us the equations $A+C=0$, $B+D=0$, $2A+C=0$, and $2B+D=1$. Solving this system yields $A=C=0$, $B=1$ and $D=-1$. Thus the original integral is equal to

$$\int \left(\frac{1}{y^2+1} - \frac{1}{y^2+2} \right) dy = \tan^{-1} y - \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{y}{\sqrt{2}} \right) + C.$$

8.5.61 If we write $\frac{2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$, then

$$2 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$$

Letting $x = 0$ yields $A = 2$. Expanding the right-hand side yields $2 = (2+B)x^4 + Cx^3 + (4+B+D)x^2 + (C+E)x + 2$. Equating coefficients gives us the equations $2+B=0$, $C=0$, $4+B+D=0$, and $C+E=0$, from which we can deduce that $B=-2$, $C=0$, $D=-2$ and $E=0$. The original integral is thus equal to

$$\int \left(\frac{2}{x} - \frac{2x}{x^2+1} - \frac{2x}{(x^2+1)^2} \right) dx = 2 \ln|x| - \ln|x^2+1| + \frac{1}{x^2+1} + C.$$

8.5.62 If we write $\frac{1}{(x+1)(x^2+2x+2)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2} + \frac{Dx+E}{(x^2+2x+2)^2}$, then

$$1 = A(x^2+2x+2)^2 + (Bx+C)(x+1)(x^2+2x+2) + (Dx+E)(x+1).$$

Letting $x = -1$ yields $A = 1$. Then expanding the polynomial on the right-hand side gives

$$1 = (1+B)x^4 + (4+3B+C)x^3 + (8+4B+3C+D)x^2 + (8+2B+4C+D+E)x + (4+E+2C).$$

Equating coefficients and then solving for the unknowns yields $B = -1$, $C = -1$, $D = -1$, and $E = -1$. The original integral is thus equal to

$$\int \left(\frac{1}{x+1} - \frac{x+1}{x^2+2x+2} - \frac{x+1}{(x^2+2x+2)^2} \right) dx = \ln|x+1| - \frac{1}{2} \ln|x^2+2x+2| + \frac{1}{2(x^2+2x+2)} + C.$$

8.5.63 $\frac{9x^2 + x + 21}{(3x^2 + 7)^2} = \frac{Ax + B}{3x^2 + 7} + \frac{Cx + D}{(3x^2 + 7)^2}$. Then

$$9x^2 + x + 21 = (Ax + B)(3x^2 + 7) + Cx + D = 3Ax^3 + 3Bx^2 + (7A + C)x + 7B + D.$$

So $A = 0$, $B = 3$, $C = 1$, and $D = 0$. We have

$$\begin{aligned} \int \frac{9x^2 + x + 21}{(3x^2 + 7)^2} dx &= \int \left(\frac{3}{3x^2 + 7} + \frac{x}{(3x^2 + 7)^2} \right) dx = \int \frac{1}{x^2 + (7/3)} dx + \int \frac{x}{(3x^2 + 7)^2} dx \\ &= \sqrt{\frac{3}{7}} \tan^{-1} \left(\sqrt{\frac{3}{7}} x \right) - \frac{1}{6(3x^2 + 7)} + C. \end{aligned}$$

8.5.64 $\frac{9x^5 + 6x^3}{(3x^2 + 1)^3} = \frac{Ax + B}{3x^2 + 1} + \frac{Cx + D}{(3x^2 + 1)^2} + \frac{Ex + F}{(3x^2 + 1)^3}$. Then

$$9x^5 + 6x^3 = (Ax + B)(3x^2 + 1)^2 + (Cx + D)(3x^2 + 1) + Ex + F.$$

This can be written as

$$9x^5 + 6x^3 = 9Ax^5 + 9Bx^4 + (6A + 3C)x^3 + (6B + 3D)x^2 + (A + C + E)x + B + D + F.$$

Therefore, $A = 1$, $B = C = D = 0$, $E = -1$, and $F = 0$. We have

$$\int \frac{9x^5 + 6x^3}{(3x^2 + 1)^3} dx = \int \left(\frac{x}{3x^2 + 1} - \frac{x}{(3x^2 + 1)^3} \right) dx = \frac{1}{6} \ln(3x^2 + 1) + \frac{1}{12(3x^2 + 1)^2} + C.$$

8.5.65

- False. Because the given integrand is improper, the first step would be to use long division to write the integrand as the sum of a polynomial and a proper rational function.
- False. This is easy to evaluate via the substitution $u = 3x^2 + x$.
- False. The discriminant of the denominator is $b^2 - 4ac = 169 - 168 = 1 > 0$, so the denominator factors into linear factors of the real numbers.
- True. The discriminant of the denominator is $b^2 - 4ac = 169 - 172 = -3 < 0$, so the given quadratic expression is irreducible.

8.5.66 We are seeking $\int_0^1 \frac{x - x^2}{(x + 1)(x^2 + 1)} dx$. We can write the integrand as $\frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$, and then

$$x - x^2 = A(x^2 + 1) + (Bx + C)(x + 1) = (A + B)x^2 + (C + B)x + A + C.$$

Thus $A + B = -1$, $C + B = 1$, and $A + C = 0$. Solving gives $A = -1$, $B = 0$, and $C = 1$. Thus we have

$$\int_0^1 \frac{x - x^2}{(x + 1)(x^2 + 1)} dx = \int_0^1 \left(-\frac{1}{x + 1} + \frac{1}{x^2 + 1} \right) dx = (-\ln|x + 1| + \tan^{-1} x) \Big|_0^1 = \frac{\pi}{4} - \ln 2.$$

8.5.67 We are seeking $\int_{-2}^2 \frac{10}{x^2 - 2x - 24} dx = \int_{-2}^2 \frac{10}{(x - 6)(x + 4)} dx$. If we write

$$\frac{10}{x^2 - 2x - 24} = \frac{10}{(x - 6)(x + 4)} = \frac{A}{x - 6} + \frac{B}{x + 4},$$

then $10 = A(x + 4) + B(x - 6)$. Letting $x = -4$ gives $B = -1$ and letting $x = 6$ gives $A = 1$. Thus the area in question is given by

$$-\int_{-2}^2 \left(-\frac{1}{x + 4} + \frac{1}{x - 6} \right) dx = (\ln(x + 4) - \ln|x - 6|) \Big|_{-2}^2 = \ln 6 - \ln 4 - (\ln 2 - \ln 8) = \ln 6.$$

8.5.68 The curves intersect when $\frac{3x^2 + 2x + 1}{x(x^2 + x + 1)} = \frac{2}{x}$, or when $3x^2 + 2x + 1 = 2x^2 + 2x + 2$, or $x^2 - 1 = 0$, so $x = -1$ and $x = 1$, so in the first quadrant the bounded area lies between $x = 1$ and $x = 2$. The area is given by

$$\int_1^2 \left(\frac{3x^2 + 2x + 1}{x(x^2 + x + 1)} - \frac{2}{x} \right) dx = \int_1^2 \frac{x^2 - 1}{x(x^2 + x + 1)} dx.$$

We decompose this integrand via partial fractions. Let $\frac{x^2 - 1}{x(x^2 + x + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1}$. Then $x^2 - 1 = A(x^2 + x + 1) + (Bx + C)x = (A + B)x^2 + (A + C)x + A$, so $A = -1$, $B = 2$, and $C = 1$. The area is then given by

$$\int_1^2 \left(-\frac{1}{x} + \frac{2x + 1}{x^2 + x + 1} \right) dx = \left(\ln|x^2 + x + 1| - \ln|x| \right) \Big|_1^2 = \ln 7 - \ln 2 - \ln 3 = \ln \frac{7}{6}.$$

8.5.69 Using disks, we have $\frac{V}{\pi} = \int_0^4 \frac{x^2}{(x+1)^2} dx$. Now we can perform long division to rewrite this integral as $\int_0^4 \left(1 - \frac{2x+2}{x^2+2x+1} + \frac{1}{(x+1)^2} \right) dx$. Thus we have $V = \pi \left(x - \ln|x^2 + 2x + 1| - \frac{1}{x+1} \Big|_0^4 \right) = \pi \left(4 - \ln(25) - \frac{1}{5} - (-1) \right) = \pi \left(\frac{24}{5} - \ln(25) \right)$.

8.5.70 Using shells, the volume is given by $V = 2\pi \int_1^2 \frac{x}{x^2(x^2 + 2)^2} dx = 2\pi \int_1^2 \frac{1}{x(x^2 + 2)^2} dx$. If we write

$$\frac{1}{x(x^2 + 2)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2} + \frac{Dx + E}{(x^2 + 2)^2},$$

then

$$1 = A(x^2 + 2)^2 + (Bx + C)(x)(x^2 + 2) + (Dx + E)x = (A + B)x^4 + Cx^3 + (D + 2B + 4A)x^2 + 2Cx + 4A.$$

This leads to $A = \frac{1}{4}$, $B = -\frac{1}{4}$, $C = 0$, $D = -\frac{1}{2}$, and $E = 0$. The volume is then

$$\begin{aligned} \frac{2\pi}{4} \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2 + 2} - \frac{2x}{(x^2 + 2)^2} \right) dx &= \frac{\pi}{2} \left(\ln x - \frac{1}{2} \ln(x^2 + 2) + \frac{1}{x^2 + 2} \right) \Big|_1^2 \\ &= \frac{\pi}{2} \left(\ln 2 - \frac{1}{2} \ln 6 + \frac{1}{6} - \left(0 - \frac{1}{2} \ln 3 + \frac{1}{3} \right) \right) \\ &= \frac{\pi}{2} \left(\frac{1}{2} \ln 2 - \frac{1}{6} \right) = \frac{\pi(3 \ln 2 - 1)}{12}. \end{aligned}$$

8.5.71 Using disks, we have $\frac{V}{\pi} = \int_1^2 \frac{1}{x(3-x)} dx$. If we write

$$\frac{1}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x},$$

then we have $1 = A(3-x) + Bx$. Letting $x = 0$ yields $A = 1/3$ and letting $x = 3$ yields $B = 1/3$. Thus we have

$$V = \frac{\pi}{3} \int_1^2 \left(\frac{1}{x} - \frac{1}{x-3} \right) dx = \frac{\pi}{3} \left(\ln x - \ln|x-3| \right) \Big|_1^2 = \frac{\pi}{3} \cdot 2 \ln 2.$$

8.5.72 Using shells, we have $V = 2\pi \int_5^{10} \frac{x(3x^2 + 25)}{x^2(x^2 + 25)} dx = 2\pi \int_5^{10} \frac{3x^2 + 25}{x(x^2 + 25)} dx$. Write

$$\frac{3x^2 + 25}{x(x^2 + 25)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 25}.$$

Then

$$3x^2 + 25 = A(x^2 + 25) + (Bx + C)x = (A + B)x^2 + Cx + 25A,$$

so $A = 1$, $B = 2$, and $C = 0$. We have

$$V = 2\pi \int_5^{10} \left(\frac{1}{x} + \frac{2x}{x^2 + 25} \right) dx = 2\pi (\ln x + \ln(x^2 + 25)) \Big|_5^{10} = 2\pi \ln \frac{1250}{250} = 2\pi \ln 5.$$

8.5.73 If we write $\frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$, then $1 = A(x + 1) + B(x - 1)$, so $A = 1/2$ and $B = -1/2$. Thus we have

$$\frac{1}{2} \int \frac{1}{x - 1} - \frac{1}{x + 1} dx = \frac{1}{2} (\ln |x - 1| - \ln |x + 1|) + C.$$

Now let $x = \sec \theta$, so that $dx = \sec \theta \tan \theta d\theta$. Then the original integral is equal to $\int \csc \theta d\theta = -\ln |\csc \theta + \cot \theta| + C = -\ln \left| \frac{x}{\sqrt{x^2 - 1}} + \frac{1}{\sqrt{x^2 - 1}} \right| + C = -\ln \left(\frac{|x + 1|}{\sqrt{x^2 - 1}} \right) + C = -\ln \left(\sqrt{\left| \frac{x + 1}{x - 1} \right|} \right) + C = \ln \left(\sqrt{\left| \frac{x - 1}{x + 1} \right|} \right) + C$. The two answers are equivalent.

8.5.74 $\frac{3x^2 + 2x + 1}{(x + 1)^3(x^2 + x + 1)^2} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3} + \frac{Dx + E}{x^2 + x + 1} + \frac{Fx + G}{(x^2 + x + 1)^2}$. Using a computer algebra system, we find that $A = -3$, $B = 0$, $C = 2$, $D = 3$, $E = 0$, $F = 1$, and $G = 2$. Thus

$$\frac{3x^2 + 2x + 1}{(x + 1)^3(x^2 + x + 1)^2} = -\frac{3}{x + 1} + \frac{2}{(x + 1)^3} + \frac{3x}{x^2 + x + 1} + \frac{x + 2}{(x^2 + x + 1)^2}.$$

8.5.75 $\frac{x^4 + 3x^2 + 1}{x(x^2 + 1)^2(x^2 + x + 4)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} + \frac{Fx + G}{x^2 + x + 4} + \frac{Hx + I}{(x^2 + x + 4)^2}$. Using a computer algebra system, we find that $A = 1/16$, $B = -1/100$, $C = -1/10$, $D = 2/25$, $E = 3/50$, $F = -21/400$, $G = 19/400$, $H = -1/5$, and $I = -1/20$. Thus

$$\frac{x^4 + 3x^2 + 1}{x(x^2 + 1)^2(x^2 + x + 4)^2} = \frac{1}{16x} - \frac{x + 10}{100(x^2 + 1)} + \frac{4x + 3}{50(x^2 + 1)^2} + \frac{-21x + 19}{400(x^2 + x + 4)} - \frac{4x + 1}{20(x^2 + x + 4)^2}.$$

8.5.76 Let $u = \sin x$ so that $du = \cos x dx$. Then we have $\int \frac{1}{u(u - 2)(u + 2)} du$. If we write

$\frac{1}{u(u - 2)(u + 2)} = \frac{A}{u} + \frac{B}{u - 2} + \frac{C}{u + 2}$, then we have $1 = A(u - 2)(u + 2) + Bu(u + 2) + Cu(u - 2)$. Letting $u = 2$ yields $B = 1/8$ and letting $u = -2$ yields $C = 1/8$. Letting $u = 0$ yields $A = -1/4$. Thus we have

$$\begin{aligned} \int \left(-\frac{1}{4u} + \frac{1/8}{u - 2} + \frac{1/8}{u + 2} \right) du &= -\frac{1}{4} \ln |u| + \frac{1}{8} (\ln |u - 2| + \ln |u + 2|) + C \\ &= -\frac{1}{4} \ln |\sin x| + \frac{1}{8} \ln |(\sin x - 2)(\sin x + 2)| + C \\ &= -\frac{1}{4} \ln |\sin x| + \frac{1}{8} \ln (4 - \sin^2 x) + C. \end{aligned}$$

8.5.77 Let $u = e^x$ so that $du = e^x dx$. Then the original integral is equal to $\int \frac{1}{(u-1)(u+2)} du$. If we write $\frac{1}{(u-1)(u+2)} = \frac{A}{u-1} + \frac{B}{u+2}$, then $1 = A(u+2) + B(u-1)$. Letting $u = -2$ yields $B = -1/3$ and letting $u = 1$ yields $A = 1/3$. Thus we have

$$\int \left(\frac{1/3}{u-1} - \frac{1/3}{u+2} \right) du = \frac{1}{3} (\ln |u-1| - \ln |u+2|) + C = \frac{1}{3} \ln \left| \frac{e^x - 1}{e^x + 2} \right| + C.$$

8.5.78 Let $u = \sqrt{y}$, so that $du = \frac{1}{2\sqrt{y}} dy$. So $dy = 2u du$. Substituting gives $\int \frac{2u}{u^2(\sqrt{a}-u)} du = \int \frac{2}{u(\sqrt{a}-u)} du$. Write $\frac{2}{u(\sqrt{a}-u)} = \frac{A}{u} + \frac{B}{\sqrt{a}-u}$. Then $2 = A(\sqrt{a}-u) + Bu$. Letting $u = 0$ gives $A = \frac{2}{\sqrt{a}}$, and letting $u = \sqrt{a}$ gives $B = \frac{2}{\sqrt{a}}$. Our integral is therefore equal to

$$\int \left(\frac{2/\sqrt{a}}{u} - \frac{2/\sqrt{a}}{u-\sqrt{a}} \right) du = \frac{2}{\sqrt{a}} \ln \left| \frac{u}{u-\sqrt{a}} \right| + C = \frac{2}{\sqrt{a}} \ln \left| \frac{\sqrt{y}}{\sqrt{y}-\sqrt{a}} \right| + C.$$

8.5.79 This can be written as

$$\int \frac{dt}{\cos t(1+\sin t)} = \int \frac{\cos t}{\cos^2 t(1+\sin t)} dt = \int \frac{\cos t}{(1-\sin^2 t)(1+\sin t)} dt.$$

Now let $u = \sin t$ so that $du = \cos t dt$. Substituting gives $\int \frac{du}{(1-u^2)(1+u)} = \int \frac{du}{(1+u)^2(1-u)}$. If we write $\frac{1}{(1+u)^2(1-u)} = \frac{A}{1+u} + \frac{B}{(1+u)^2} + \frac{C}{1-u}$ then $1 = A(1+u)(1-u) + B(1-u) + C(1+u)^2$. Letting $u = -1$ yields $B = 1/2$. Letting $u = 1$ yields $C = 1/4$. Letting $u = 0$ then yields $A = 1/4$. Then our integral is equal to

$$\int \left(\frac{1/4}{1+u} + \frac{1/2}{(1+u)^2} + \frac{1/4}{1-u} \right) du = \frac{1}{4} \left(\frac{-2}{1+u} + \ln \left| \frac{1+u}{1-u} \right| \right) + C = \frac{1}{4} \left(\frac{-2}{1+\sin t} + \ln \left(\frac{1+\sin t}{1-\sin t} \right) \right) + C.$$

8.5.80 Let $u = \sqrt{e^x + 1}$, so that $u^2 = e^x + 1$ and $2u du = e^x dx = (u^2 - 1) dx$. Then the original integral is equal to $\int \frac{2u^2}{u^2 - 1} du = \int 2 + \frac{2}{u^2 - 1} du$. If we write $\frac{2}{u^2 - 1} = \frac{A}{u+1} + \frac{B}{u-1}$, then $2 = A(u-1) + B(u+1)$. Letting $u = 1$ yields $B = 1$ and letting $u = -1$ yields $A = -1$. Thus we have $\int \left(2 - \frac{1}{u+1} + \frac{1}{u-1} \right) du = 2u + \ln |u-1| - \ln |u+1| + C = 2\sqrt{e^x + 1} + \ln |\sqrt{e^x + 1} - 1| - \ln |\sqrt{e^x + 1} + 1| + C$.

8.5.81 Let $u = e^x$. Then $du = e^x dx$. We have $\int \frac{e^{3x} + e^{2x} + e^x}{(e^{2x} + 1)^2} dx = \int \frac{u^2 + u + 1}{(u^2 + 1)^2} du$.

Let $\frac{u^2 + u + 1}{(u^2 + 1)^2} = \frac{Au + B}{u^2 + 1} + \frac{Cu + D}{(u^2 + 1)^2}$. Then $u^2 + u + 1 = (Au + B)(u^2 + 1) + Cu + D = Au^3 + Bu^2 + (A + C)u + B + D$. Then $A = 0$, $B = 1$, $C = 1$, and $D = 0$. So we have

$$\int \frac{u^2 + u + 1}{(u^2 + 1)^2} du = \int \left(\frac{1}{u^2 + 1} + \frac{u}{(u^2 + 1)^2} \right) du = \tan^{-1} u - \frac{1}{2(u^2 + 1)} + C = \tan^{-1} e^x - \frac{1}{2(e^{2x} + 1)} + C.$$

8.5.82 If we let $u^2 = 2x + 1$, then $2u du = 2 dx$. Substituting yields

$$\begin{aligned} \int \frac{2u}{u(u-1)(u+1)} du &= 2 \int \frac{1}{(u-1)(u+1)} du = \int \frac{1}{u-1} - \frac{1}{u+1} du \\ &= \ln |u-1| - \ln |u+1| + C = \ln \left| \frac{\sqrt{1+2x}-1}{\sqrt{1+2x}+1} \right| + C. \end{aligned}$$

8.5.83 Let $u = e^x$, so that $du = e^x dx$. Then $\int \frac{1}{1+e^x} \cdot \frac{e^x}{e^x} dx = \int \frac{1}{u(1+u)} du$. If we write $\frac{1}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u}$, then $1 = A(1+u) + Bu$. Letting $u = 0$ yields $A = 1$ and letting $u = -1$ yields $B = -1$, so the original integral is equal to $\int \left(\frac{1}{u} - \frac{1}{1+u} \right) du = \ln|u| - \ln|1+u| + C = x - \ln(1+e^x) + C$.

8.5.84 Because $\frac{x^2}{(x-4)(x+5)}$ is not proper, it is not of the proper form to be decomposed via partial fractions.

8.5.85

a. $\sec x = \frac{1}{\cos x} \cdot \frac{\cos x}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x}$.

b. $\int \sec x dx = \int \frac{\cos x}{1 - \sin^2 x} dx$. Let $u = \sin x$ so that $du = \cos x dx$. Then our integral becomes $\int \frac{du}{1-u^2}$. If we write $\frac{1}{1-u^2} = \frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$, we have $1 = A(1+u) + B(1-u)$, so $A = \frac{1}{2}$ and $B = \frac{1}{2}$. We have

$$\int \frac{du}{1-u^2} = \frac{1}{2} \int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du = \frac{1}{2} (\ln|1+u| - \ln|1-u|) + C = \frac{1}{2} \ln \left| \frac{1+\sin x}{1-\sin x} \right| + C.$$

8.5.86 Differentiating the right-hand side, we have

$$\frac{5}{x-1} - \frac{3}{x+1} + \frac{3}{(x+1)^2} + \frac{1}{2} \cdot \frac{1}{1+(x^2/4)} \cdot \frac{1}{2} = \frac{5}{x-1} - \frac{3}{x+1} + \frac{3}{(x+1)^2} + \frac{1}{x^2+4}.$$

This is the partial fraction decomposition of the integrand.

8.5.87 If $x = 2 \tan^{-1} u$, then $dx = \frac{2}{1+u^2} du$.

If $u = \tan(x/2)$, then $u^2 + 1 = \tan^2(x/2) + 1 = \sec^2(x/2) = \frac{1}{\cos^2(x/2)}$. Thus, $\frac{1}{u^2+1} = \cos^2(x/2) = \frac{1+\cos x}{2}$, so $\cos x = \frac{2}{u^2+1} - 1 = \frac{1-u^2}{u^2+1} = \frac{1-u^2}{1+u^2}$.

Also, $\sin x = 2 \sin(x/2) \cos(x/2) = 2 \tan(x/2) \cos^2(x/2) = \frac{2u}{1+u^2}$.

8.5.88 Using the substitution $x = 2 \tan^{-1} u$ yields

$$\begin{aligned} \int \frac{1}{2+\cos x} dx &= \int \frac{2}{1+u^2} \cdot \frac{1}{2+\frac{1-u^2}{1+u^2}} du = 2 \int \frac{1}{3+u^2} du = \frac{2 \tan^{-1} \left(\frac{u}{\sqrt{3}} \right)}{\sqrt{3}} + C \\ &= \frac{2 \tan^{-1} \left(\frac{\tan(\frac{x}{2})}{\sqrt{3}} \right)}{\sqrt{3}} + C. \end{aligned}$$

8.5.89 Using the substitution $x = 2 \tan^{-1} u$ yields

$$\int \frac{1}{1-\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int u^{-2} du = -\frac{1}{u} + C = -\cot(x/2) + C.$$

8.5.90 Using the substitution $x = 2 \tan^{-1} u$ yields

$$\begin{aligned} &\int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du \\ &= \int \frac{2}{2+2u} du = \int \frac{1}{1+u} du = \ln|1+u| + C = \ln|1+\tan(x/2)| + C. \end{aligned}$$

8.5.91 Using the substitution $\theta = 2 \tan^{-1} u$ yields $\int_0^1 \frac{1}{\frac{1-u^2}{1+u^2} + \frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int_0^1 \frac{2}{1+2u-u^2} du = -2 \int_0^1 \frac{1}{u^2-2u+1-2} du = -2 \int_0^1 \frac{1}{(u-1)^2-2} du$. If we factor the denominator as the difference of squares we have $(u-1)^2-2 = (u-1-\sqrt{2})(u-1+\sqrt{2})$, and using a partial fractions decomposition yields

$$\frac{1}{\sqrt{2}} \int_0^1 \left(\frac{1}{u+\sqrt{2}-1} - \frac{1}{u-\sqrt{2}-1} \right) du = \frac{1}{\sqrt{2}} \ln \left| \frac{u+\sqrt{2}-1}{u-\sqrt{2}-1} \right| \Big|_0^1 = \frac{1}{\sqrt{2}} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right).$$

8.5.92 Using the substitution $\theta = 2 \tan^{-1} u$ yields

$$\int_0^{1/\sqrt{3}} \frac{\frac{2u}{1+u^2}}{1-\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int_0^{1/\sqrt{3}} \frac{4u}{(1+u^2)(u^2-2u+1)} du = \int_0^{1/\sqrt{3}} \frac{4u}{(u^2+1)(u-1)^2} du.$$

Decomposing this integral via partial fractions yields

$$\int_0^{1/\sqrt{3}} \left(\frac{2}{(u-1)^2} - \frac{2}{u^2+1} \right) du = \left(\frac{-2}{u-1} - 2 \tan^{-1} u \right) \Big|_0^{1/\sqrt{3}} = \frac{-2}{\frac{1}{\sqrt{3}}-1} - \frac{\pi}{3} - 2 = 1 + \sqrt{3} - \frac{\pi}{3}.$$

8.5.93 $s_A(t) = \int v_A(t) dt = 88 \int \frac{t}{t+1} dt = 88 \int \left(1 - \frac{1}{1+t} \right) dt = 88t - 88 \ln(t+1) + C$. Because $s_A(0) = 0$, we see that $C = 0$, so $s_A(t) = 88t - 88 \ln(1+t)$.

$s_B(t) = \int v_B(t) dt = 88 \int \frac{t^2}{(t+1)^2} dt = 88 \int \left(1 - \frac{2t+2}{t^2+2t+1} + \frac{1}{(t+1)^2} \right) dt = 88(t - \ln(t^2+2t+1) - \frac{1}{t+1}) + D$. Because $s_B(0) = 0$, we see that $D = 88$, so $s_B(t) = 88(t - \ln(t^2+2t+1) - \frac{1}{t+1} + 1)$.

$s_C(t) = \int v_C(t) dt = 88 \int \frac{t^2}{t^2+1} dt = 88 \int \left(1 - \frac{1}{t^2+1} \right) dt = 88(t - \tan^{-1}(t)) + E$. Because $s_C(0) = 0$, we have that $E = 0$. Thus $s_C(t) = 88t - 88 \tan^{-1}(t)$.

a. $s_A(1) = 88(1 - \ln(2)) \approx 27$. $s_B(1) = 88(1 - \ln 4 - (1/2) + 1) \approx 88(3/2 - \ln 4) \approx 10$. $s_C(1) = 88(1 - \tan^{-1}(1)) \approx 18.9$. So car A travels farthest.

b. $s_A(5) = 88(5 - \ln 6) \approx 282$. $s_B(5) = 88(5 - \ln 36 - (1/6) + 1) \approx 198$. $s_C(5) = 88(5 - \tan^{-1}(5)) \approx 319$. So car C travels farthest.

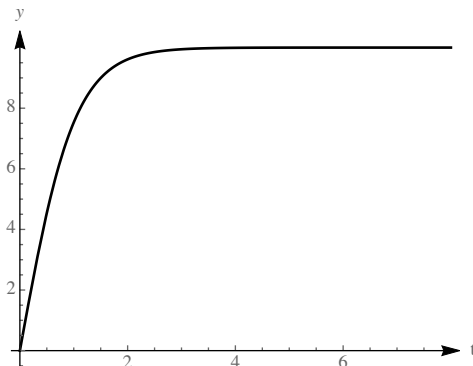
c. See the development above.

d. Ultimately car C gains the lead. This can be seen by the fact that car C's velocity function is greater than that of the other cars.

8.5.94

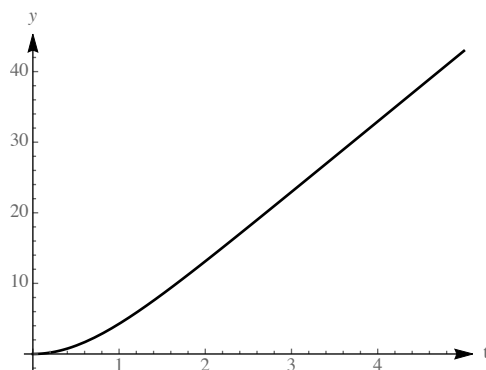
a. $v(0) = V \cdot \frac{1-1}{1+1} = 0$. $\lim_{t \rightarrow \infty} \frac{1-(1/e^{kt})}{1+(1/e^{kt})} = 1$, when $k = 2g/V > 0$ (which it is.) Thus $\lim_{t \rightarrow \infty} v(t) = V \cdot 1 = V$.

b.



c. $s(t) = \int v(t) dt = V \int \frac{e^{kt} - 1}{e^{kt} + 1} dt = V \int \left(\frac{e^{kt}}{e^{kt} + 1} - \frac{1}{e^{kt} + 1} \right) dt$. Let $u = e^{kt} + 1$ and $w = e^{kt}$. Then we have $V \left(\frac{1}{k} \int \frac{1}{u} du - \frac{1}{k} \int \frac{1}{w(w+1)} dw \right) = \frac{V}{k} \ln |e^{kt} + 1| - \frac{V}{k} \int \left(\frac{1}{w} - \frac{1}{w+1} \right) dw = \frac{V}{k} \ln |e^{kt} + 1| - \frac{V}{k} \ln \left(\frac{e^{kt}}{e^{kt} + 1} \right) + C = \frac{V}{k} (2 \ln(e^{kt} + 1) - kt) + C$. Now because $s(0) = 0$, we must have $C = -\frac{V}{k} \ln 4$. Thus, $s(t) = \frac{V}{k} (\ln(e^{kt} + 1)^2 - kt - \ln 4) = \frac{V}{k} \left(\ln \left(\frac{e^{kt} + 1}{2} \right)^2 - kt \right) = \frac{V}{k} 2 \ln \left(\frac{e^{kt} + 1}{2} \right) - Vt = \frac{V^2}{g} \left(\ln \frac{1 + e^{-kt}}{2} + \ln e^{kt} \right) - Vt = \frac{V^2}{g} \ln \frac{1 + e^{-kt}}{2} + \frac{V^2}{g} kt - Vt = \frac{V^2}{g} \ln \frac{1 + e^{-kt}}{2} + Vt$.

d.



8.5.95 First note that the numerator of the given integrand can be written as $x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$, and this quantity when divided by $x^2 + 1$ yields $x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$. Thus the given integral is equal to $\left(\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x \right) \Big|_0^1 - (4 \tan^{-1}(x)) \Big|_0^1 = \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi = \frac{22}{7} - \pi$. Because the given integrand is positive on the interval $(0, 1)$, we know that this integral is positive. Thus,

$$0 < \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \frac{22}{7} - \pi.$$

Adding π to both sides of this inequality yields $\pi < \frac{22}{7}$.

8.5.96 Let $u^2 = \tan x$ so that $2u \, du = \sec^2 x \, dx = 1 + u^4 \, dx$. Then we can write

$$\int \sqrt{\tan x} \, dx = \int \frac{2u^2}{1 + u^4} \, du = \int \frac{u^2}{(u^2 - \sqrt{2}u + 1)(u^2 + \sqrt{2}u + 1)} \, du.$$

If we write $\frac{u^2}{(u^2 - \sqrt{2}u + 1)(u^2 + \sqrt{2}u + 1)} = \frac{Au + B}{u^2 - \sqrt{2}u + 1} + \frac{Cu + D}{u^2 + \sqrt{2}u + 1}$ then we have $u^2 = (Au + B)(u^2 + \sqrt{2}u + 1) + (Cu + D)(u^2 - \sqrt{2}u + 1)$. Multiplying out the right-hand side, equating coefficients, and solving for the unknowns yields $A = \frac{\sqrt{2}}{4}$, $B = 0$, $C = -\frac{\sqrt{2}}{4}$, and $D = 0$. Now

$$\begin{aligned} \frac{u^2}{1 + u^4} &= \frac{\sqrt{2}}{4} \left(\frac{u}{u^2 - \sqrt{2}u + 1} - \frac{u}{u^2 + \sqrt{2}u + 1} \right) = \frac{\sqrt{2}}{8} \left(\frac{2u}{u^2 - \sqrt{2}u + 1} - \frac{2u}{u^2 + \sqrt{2}u + 1} \right) \\ &= \frac{\sqrt{2}}{8} \left(\frac{2u - \sqrt{2}}{u^2 - \sqrt{2}u + 1} + \frac{\sqrt{2}}{u^2 - \sqrt{2}u + 1} - \frac{2u + \sqrt{2}}{u^2 + \sqrt{2}u + 1} + \frac{\sqrt{2}}{u^2 + \sqrt{2}u + 1} \right). \end{aligned}$$

Then

$$\begin{aligned} 2 \int \frac{u^2}{1 + u^4} \, du &= \frac{\sqrt{2}}{4} \left(\ln |u^2 - \sqrt{2}u + 1| - \ln |u^2 + \sqrt{2}u + 1| \right) \\ &\quad + \frac{1}{2} \int \left(\frac{1}{(u^2 - \sqrt{2}u + 1)} + \frac{1}{(u^2 + \sqrt{2}u + 1)} \right) \, du. \end{aligned}$$

This last integral can be written as

$$\int \left(\frac{1}{(\sqrt{2}u + 1)^2 + 1} + \frac{1}{(1 - \sqrt{2}u)^2 + 1} \right) \, du = \frac{1}{\sqrt{2}} \left(\tan^{-1}(\sqrt{2}u + 1) - \tan^{-1}(1 - \sqrt{2}u) \right) + C.$$

Putting this all together and replacing u by $\sqrt{\tan x}$ yields

$$\begin{aligned} &\frac{\sqrt{2}}{4} \left(\ln |\tan x - \sqrt{2\tan x} + 1| - \ln |\tan x + \sqrt{2\tan x} + 1| \right) \\ &\quad + \frac{1}{\sqrt{2}} \left(\tan^{-1}(1 + \sqrt{2\tan x}) - \tan^{-1}(1 - \sqrt{2\tan x}) \right) + C. \end{aligned}$$

Thus,

$$\int_0^{\pi/4} \sqrt{\tan x} \, dx = \frac{\sqrt{2}}{4} \left(\ln(2 - \sqrt{2}) - \ln(2 + \sqrt{2}) \right) + \frac{1}{\sqrt{2}} \left(\tan^{-1}(1 + \sqrt{2}) - \tan^{-1}(1 - \sqrt{2}) \right) \approx 0.4875.$$

8.6 Integration Strategies

8.6.1 Because the integrand is a product of a polynomial and a trigonometric function, it makes sense to integrate by parts.

8.6.2 Because $\frac{d}{dx}(1 + \tan x) = \sec^2 x$, use a u -substitution of $u = 1 + \tan x$.

8.6.3 The presence of $64 - x^2$ in the integrand suggests the trigonometric substitution $x = 8 \sin \theta$.

8.6.4 One approach is to rewrite the integrand by replacing $\tan^2 x + 1$ with $\sec^2 x$ and then proceed with a u -substitution of $u = \tan x$. Another approach is to split up the fraction and simplify the integrand by writing

$$\int \frac{\tan^2 x + 1}{\tan x} \, dx = \int \left(\frac{\tan^2 x}{\tan x} + \frac{1}{\tan x} \right) \, dx = \int (\tan x + \cot x) \, dx.$$

Then use the integral formulas for $\tan x$ and $\cot x$.

8.6.5 The method of partial fractions is appropriate because the integrand is a proper rational function.

8.6.6 Using the identity $1 - \sin^2 x = \cos^2 x$, rewrite the integrand as $\frac{\cos^5 x \sin^4 x}{\cos^2 x}$, or $\cos^3 x \sin^4 x$. Then split off a factor of $\cos x$, express the remaining powers in terms of $\sin x$, and make the substitution $u = \sin x$.

8.6.7 Let $u = \cos \theta$, which implies that $du = -\sin \theta d\theta$, or $-du = \sin \theta d\theta$. The lower limit of integration becomes $u = 1$ and the upper limit becomes $u = 0$. So we have

$$\int_0^{\pi/2} \frac{\sin \theta}{1 + \cos^2 \theta} d\theta = - \int_1^0 \frac{1}{1 + u^2} = \int_0^1 \frac{1}{1 + u^2} = \tan^{-1} u \Big|_0^1 = \frac{\pi}{4}.$$

8.6.8 Using the identity $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$ with $\theta = 10x$, we have

$$\int \cos^2 10x dx = \int \frac{1 + \cos 20x}{2} dx = \frac{x}{2} + \frac{\sin 20x}{40} + C.$$

8.6.9 After completing the square of $8x - x^2$, we make a substitution of $u = x - 4$. The lower limit of integration becomes $u = 0$ and the upper limit of integration becomes $u = 2$. Integrating, we have

$$\int_4^6 \frac{dx}{\sqrt{8x - x^2}} = \int_4^6 \frac{dx}{\sqrt{16 - (x - 4)^2}} = \int_0^2 \frac{dx}{\sqrt{16 - u^2}} = \sin^{-1} \frac{u}{4} \Big|_0^2 = \frac{\pi}{6}.$$

8.6.10 Because $\sin x$ and $\cos x$ both have odd powers, we have a choice of either splitting off a factor $\sin x$ or a factor of $\cos x$. In this case, you can verify that it is easier to split off $\cos x$ by rewriting $\cos^3 x$ as $\cos^2 x \cos x$. Here is the integration process:

$$\begin{aligned} \int \sin^9 x \cos^3 x dx &= \int \sin^9 x \cos^2 x \cos x dx && \text{Split off } \cos x. \\ &= \int \sin^9 x (1 - \sin^2 x) \cos x dx && \text{Pythagorean identity} \\ &= \int (\sin^9 x - \sin^{11} x) \cos x dx && \text{Expand.} \\ &= \int (u^9 - u^{11}) du && \text{Let } u = \sin x; du = \cos x dx. \\ &= \frac{u^{10}}{10} - \frac{u^{12}}{12} + C && \text{Integrate.} \\ &= \frac{\sin^{10} x}{10} - \frac{\sin^{12} x}{12} + C. && \text{Replace } u \text{ with } \sin x. \end{aligned}$$

8.6.11 The integral is evaluated after expanding the integrand.

$$\begin{aligned} \int_0^{\pi/4} (\sec x - \cos x)^2 dx &= \int_0^{\pi/4} (\sec^2 x - 2 \underbrace{\sec x \cos x}_1 + \cos^2 x) dx && \text{Expand integrand.} \\ &= \int_0^{\pi/4} \left(\sec^2 x - 2 + \frac{1 + \cos 2x}{2} \right) dx && \cos^2 x = \frac{1 + \cos 2x}{2} \\ &= \int_0^{\pi/4} \left(\sec^2 x - \frac{3}{2} + \frac{1}{2} \cos 2x \right) dx && \text{Simplify integrand.} \\ &= \left(\tan x - \frac{3}{2}x + \frac{1}{4} \sin 2x \right) \Big|_0^{\pi/4} = \frac{5}{4} - \frac{3\pi}{8} && \text{Evaluate.} \end{aligned}$$

8.6.12 Letting $u = e^x$, it follows that $du = e^x dx$. So we have

$$\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + C = \sin^{-1} e^x + C.$$

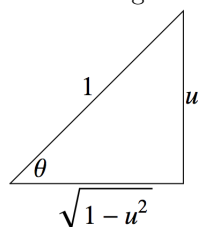
8.6.13 Let $u = e^x$, which implies that $du = e^x dx$ and $dx = \frac{du}{e^x} = \frac{du}{u}$. The integrand is then expressed in terms of u :

$$\int \frac{dx}{e^x \sqrt{1 - e^{2x}}} = \int \frac{du}{u^2 \sqrt{1 - u^2}}.$$

The expression $1 - u^2$ suggests the trigonometric substitution $u = \sin \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. It follows that $du = \cos \theta d\theta$ and

$$\sqrt{1 - u^2} = \sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = |\cos \theta| = \cos \theta.$$

Using the equation $u = \sin \theta$, we create a reference triangle that helps us evaluate the integral.



Summarizing what we've done so far and then evaluating the integral, we have

$$\begin{aligned} \int \frac{dx}{e^x \sqrt{1 - e^{2x}}} &= \int \frac{du}{u^2 \sqrt{1 - u^2}} && u\text{-substitution} \\ &= \int \frac{\cos \theta}{\sin^2 \theta \cos \theta} d\theta && \text{Trigonometric substitution} \\ &= \int \csc^2 \theta d\theta && \text{Simplify.} \\ &= -\cot \theta + C && \text{Integrate.} \\ &= -\frac{\sqrt{1 - u^2}}{u} + C && \text{Use the reference triangle.} \\ &= -\frac{\sqrt{1 - e^{2x}}}{e^x} + C. && \text{Replace } u \text{ with } e^x. \end{aligned}$$

8.6.14 We start by eliminating the negative powers in the integrand by multiplying the numerator and denominator by x^3 . Then we split the integral into a sum of integrals:

$$\int \frac{x^{-2} + x^{-3}}{x^{-1} + 16x^{-3}} \cdot \frac{x^3}{x^3} dx = \int \frac{x + 1}{x^2 + 16} dx = \int \frac{x}{x^2 + 16} dx + \int \frac{1}{x^2 + 16} dx.$$

The second integral on the right equals $\frac{1}{4} \tan^{-1} \frac{x}{4} + C$. For the first integral on the right, we let $u = x^2 + 16$. It follows that $du = 2x dx$, or $\frac{1}{2} du = x dx$. Therefore

$$\int \frac{x}{x^2 + 16} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln(x^2 + 16) + C.$$

Adding the integrals, we have

$$\int \frac{x^{-2} + x^{-3}}{x^{-1} + 16x^{-3}} dx = \frac{1}{2} \ln(x^2 + 16) + \frac{1}{4} \tan^{-1} \frac{x}{4} + C.$$

8.6.15 We let $u = \sqrt{x}$, which implies that $du = \frac{dx}{2\sqrt{x}}$ or $2du = \frac{dx}{\sqrt{x}}$. The lower limit of integration is $u = 1$ and the upper limit is $u = 2$. Therefore

$$\int_1^4 \frac{2\sqrt{x}}{\sqrt{x}} dx = 2 \int_1^2 2^u du = \left. \frac{2 \cdot 2^u}{\ln 2} \right|_1^2 = \frac{4}{\ln 2}.$$

8.6.16 The denominator factors as $x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1)$. Therefore the appropriate form of the partial fraction decomposition is

$$\frac{1}{x^4 - 1} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x + 1} + \frac{D}{x - 1}.$$

Multiplying both sides of the equation by $(x^2 + 1)(x + 1)(x - 1)$ leads to

$$1 = (Ax + B)(x + 1)(x - 1) + C(x^2 + 1)(x - 1) + D(x^2 + 1)(x + 1).$$

We can use this equation to find the constants A, B, C, and D.

$$x = 1 \Rightarrow 1 = 4D \text{ or } D = \frac{1}{4}$$

$$x = -1 \Rightarrow 1 = -4C \text{ or } C = -\frac{1}{4}$$

$$x = 0 \Rightarrow 1 = -B + \frac{1}{4} + \frac{1}{4} \text{ or } B = -\frac{1}{2}$$

$$\text{Equating coefficients of } x^3 \Rightarrow 0 = A + C + D \Rightarrow A = 0$$

So the partial fraction decomposition is

$$\frac{1}{x^4 - 1} = -\frac{1}{2(x^2 + 1)} - \frac{1}{4(x + 1)} + \frac{1}{4(x - 1)}.$$

Using this decomposition, integration is straightforward:

$$\int \frac{dx}{x^4 - 1} = -\int \frac{dx}{2(x^2 + 1)} - \int \frac{dx}{4(x + 1)} + \int \frac{dx}{4(x - 1)} = -\frac{1}{2} \tan^{-1} x - \frac{1}{4} \ln |x + 1| + \frac{1}{4} \ln |x - 1| + C.$$

8.6.17 Let's evaluate the corresponding indefinite integral first. Let $z = w^2$ which implies that $dz = 2w dw$, or $\frac{dz}{2} = w dw$. We then have

$$\int w^3 e^{w^2} dw = \frac{1}{2} \int z e^z dz.$$

For the integral on the right, use integration by parts with the following choices.

$$u = z \quad dv = e^z dz$$

$$du = dz \quad v = e^z$$

Integrating by parts, we have

$$\int w^3 e^{w^2} dw = \frac{1}{2} \int z e^z dz = \frac{1}{2} \left(z e^z - \int e^z dz \right) = \frac{e^z}{2} (z - 1) + C$$

After replacing z with w^2 , we evaluate the definite integral:

$$\int_1^2 w^3 e^{w^2} dx = \left. \left(\frac{1}{2} e^{w^2} (w^2 - 1) \right) \right|_1^2 = \frac{3e^4}{2}.$$

8.6.18 Letting $u = x - 5$, it follows that $du = dx$ and $x = u + 5$. The lower limit of integration is $u = 0$ and the upper limit is $u = 1$.

$$\int_5^6 x(x-5)^{10} dx = \int_0^1 (u+5)u^{10} du = \int_0^1 (u^{11} + 5u^{10}) du = \left(\frac{u^{12}}{12} + \frac{5u^{11}}{11} \right) \Big|_0^1 = \frac{71}{132}.$$

8.6.19 The integral has an odd power of $\sin x$, so we start by splitting off a factor of $\sin x$.

$$\begin{aligned} \int_0^{\pi/2} \sin^7 x \, dx &= \int_0^{\pi/2} (\sin^2 x)^3 \sin x \, dx && \text{Split off } \sin x. \\ &= \int_0^{\pi/2} (1 - \cos^2 x)^3 \sin x \, dx && \sin^2 x = 1 - \cos^2 x \\ &= \int_1^0 -(1 - u^2)^3 \, du && \text{Let } u = \cos x; \, du = -\sin x \, dx. \\ & && x = 0 \Rightarrow u = 1; \, x = \frac{\pi}{2} \Rightarrow u = 0 \\ &= \int_0^1 (-u^6 + 3u^4 - 3u^2 + 1) \, du && \text{Expand.} \\ &= \left(-\frac{u^7}{7} + \frac{3}{5}u^5 - u^3 + u \right) \Big|_0^1 && \text{Integrate.} \\ &= \frac{16}{35}. && \text{Evaluate.} \end{aligned}$$

8.6.20 Let $u = \sqrt{t}$ which implies that $du = \frac{dt}{2\sqrt{t}}$, $t = u^2$, and $\frac{dt}{\sqrt{t}} = 2du$. The lower limit of integration becomes $u = 1$ and the upper limit of integration becomes $u = \sqrt{3}$. Changing variables, we have

$$\int_1^3 \frac{dt}{\sqrt{t}(t+1)} = 2 \int_1^{\sqrt{3}} \frac{du}{u^2+1} = 2 \tan^{-1} u \Big|_1^{\sqrt{3}} = 2 \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \frac{\pi}{6}.$$

8.6.21 We use integration by parts with the following choices.

$$u = \ln 3x \quad dv = x^9 \, dx$$

$$du = \frac{1}{x} dx \quad v = \frac{x^{10}}{10}$$

Integrating by parts, we have

$$\int x^9 \ln 3x \, dx = \frac{x^{10}}{10} \ln 3x - \frac{1}{10} \int x^9 \, dx = \frac{x^{10}}{10} \ln 3x - \frac{x^{10}}{100} + C.$$

8.6.22 The denominator consists of two distinct linear terms $x - a$ and $x - b$ because $a \neq b$. An appropriate form of the partial fraction decomposition is

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

Multiplying both sides of this equation by $(x-a)(x-b)$ gives us

$$1 = A(x-b) + B(x-a).$$

Choosing $x = b$, we have $1 = B(b - a)$ or $B = 1/(b - a)$ and if $x = a$, $1 = A(a - b)$ or $A = 1/(a - b)$. Substituting these values for A and B in the partial fraction decomposition, we carry out integration.

$$\begin{aligned} \int \frac{dx}{(x-a)(x-b)} &= \frac{1}{a-b} \int \frac{dx}{x-a} + \frac{1}{b-a} \int \frac{dx}{x-b} && \text{Partial fractions} \\ &= \frac{1}{a-b} \ln|x-a| + \frac{1}{b-a} \ln|x-b| + C && \text{Integrate.} \\ &= \frac{\ln|x-a| - \ln|x-b|}{a-b} + C. && \text{Simplify.} \end{aligned}$$

8.6.23 Letting $u = \cos x$ and $du = -\sin x dx$, or $-du = \sin x dx$, we have

$$\int \frac{\sin x}{\cos^2 x + \cos x} dx = - \int \frac{du}{u^2 + u} = - \int \frac{du}{u(u+1)}.$$

To evaluate the integral on the right, we first find the constants A and B in the partial fraction decomposition

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}.$$

Multiplying both sides of this equation by $u(u+1)$, we have

$$1 = A(u+1) + Bu = (A+B)u + A.$$

It follows that $A = 1$ and $B = -1$. Using these constants in the partial fraction decomposition, integration is straightforward.

$$\begin{aligned} \int \frac{\sin x}{\cos^2 x + \cos x} dx &= - \int \frac{du}{u(u+1)} && u = \cos x; du = -\sin x dx \\ &= - \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du && \text{Partial fractions} \\ &= -(\ln|u| - \ln|u+1|) + C && \text{Integrate.} \\ &= \ln(\cos x + 1) - \ln|\cos x| + C. && \text{Replace } u \text{ with } \cos x. \end{aligned}$$

8.6.24 The degree of the polynomial in the numerator is greater than the degree of the polynomial in the denominator, so we perform long division to obtain

$$\frac{3w^5 + 2w^4 - 12w^3 - 12w - 32}{w^3 - 4w} = 3w^2 + 2w + \frac{8w^2 - 12w - 32}{w^3 - 4w}.$$

The first part of the right side of the equation is easy to integrate, so we focus on the fractional part. Factoring the denominator and then setting up the partial fraction decomposition, we have

$$\frac{8w^2 - 12w - 32}{w(w+2)(w-2)} = \frac{A}{w} + \frac{B}{w+2} + \frac{C}{w-2}.$$

We multiply both sides by $w(w+2)(w-2)$:

$$8w^2 - 12w - 32 = A(w+2)(w-2) + Bw(w-2) + Cw(w+2).$$

Now we find the values of unknown constants:

$$w = 0 \Rightarrow -32 = -4A \text{ or } A = 8$$

$$w = -2 \Rightarrow 24 = 8B \text{ or } B = 3$$

$$w = 2 \Rightarrow -24 = 8C \text{ or } C = -3$$

Substituting these values into decomposition and evaluating the integral, we have

$$\begin{aligned} \int \frac{3w^5 + 2w^4 - 12w^3 - 12w - 32}{w^3 - 4w} dw &= \int \left(3w^2 + 2w + \frac{8}{w} + \frac{3}{w+2} - \frac{3}{w-2} \right) dw \\ &= w^3 + w^2 + 8 \ln |w| + 3 \ln |w+2| - 3 \ln |w-2| + C. \end{aligned}$$

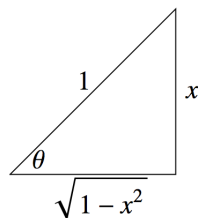
8.6.25 One approach for evaluating this integral is to use a trigonometric substitution. The expression $1-x^2$ suggests the substitution $x = \sin \theta$ with $-\pi/2 < \theta < \pi/2$. With this substitution, $dx = \cos \theta d\theta$ and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = |\cos \theta| = \cos \theta.$$

Substituting these values into the integral and simplifying, we have

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \csc \theta d\theta = -\ln |\csc \theta + \cot \theta| + C.$$

Given that $\sin \theta = x$, a geometric description of the relationship between x and θ is given by the following reference triangle.



With the help of the reference triangle, we continue:

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\ln |\csc \theta + \cot \theta| + C = -\ln \left| \frac{1}{x} + \frac{\sqrt{1-x^2}}{x} \right| + C = \ln \left| \frac{x}{1+\sqrt{1-x^2}} \right| + C.$$

8.6.26 Let $u = \ln x$ which implies that $du = \frac{dx}{x}$. Using this substitution, the lower limit of integration is $u = -1$ and the upper limit of integration is $u = 0$. After making these substitutions and then completing the square, we evaluate the integral.

$$\int_{1/e}^1 \frac{dx}{x(\ln^2 x + 2 \ln x + 2)} = \int_{-1}^0 \frac{du}{u^2 + 2u + 2} = \int_{-1}^0 \frac{du}{(u+1)^2 + 1} = \tan^{-1}(u+1) \Big|_{-1}^0 = \frac{\pi}{4}$$

8.6.27 Because we have an even power of $\sin \frac{x}{2}$, we use the half-angle formula for $\sin^2 \frac{x}{2}$ to rewrite the integrand:

$$\int \sin^4 \frac{x}{2} dx = \int \underbrace{\left(\frac{1 - \cos x}{2} \right)^2}_{\sin^2 \frac{x}{2}} dx = \frac{1}{4} \int (1 - 2 \cos x + \cos^2 x) dx.$$

Using the half-angle formula for $\cos^2 x$, the evaluation may be completed:

$$\begin{aligned}\int \sin^4 \frac{x}{2} dx &= \frac{1}{4} \int \left(1 - 2 \cos x + \underbrace{\frac{1 + \cos 2x}{2}}_{\cos^2 x} \right) dx = \frac{1}{4} \int \left(\frac{3}{2} - 2 \cos x + \frac{\cos 2x}{2} \right) dx \\ &= \frac{3x}{8} - \frac{1}{2} \sin x + \frac{1}{16} \sin 2x + C.\end{aligned}$$

8.6.28 The denominator factors as $(x^2 + 1)^2$, which is repeated irreducible quadratic factor. So the form of the partial fraction decomposition is

$$\frac{3x^2 + 2x + 3}{x^4 + 2x^2 + 1} = \frac{3x^2 + 2x + 3}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}.$$

Multiplying by $(x^2 + 1)^2$, we have

$$3x^2 + 2x + 3 = (Ax + B)(x^2 + 1) + Cx + D = Ax^3 + Bx^2 + (A + C)x + B + D.$$

Comparing coefficients of equal powers, starting with the highest power, we find that $A = 0$, $B = 3$, $C = 2$, and $D = 0$. Now we use the partial fraction decomposition, with these constant values, to integrate:

$$\begin{aligned}\int \frac{3x^2 + 2x + 3}{x^4 + 2x^2 + 1} dx &= \int \left(\frac{3}{x^2 + 1} + \frac{2x}{(x^2 + 1)^2} \right) dx && \text{Partial fractions} \\ &= \int \frac{3}{x^2 + 1} dx + \int \frac{2x}{(x^2 + 1)^2} dx && \text{Split into 2 integrals.} \\ &= 3 \tan^{-1} x + \int \frac{du}{u^2} && \text{Evaluate first integral;} \\ & && \text{Second integral: } u = x^2 + 1, du = 2x dx \\ &= 3 \tan^{-1} x - \frac{1}{x^2 + 1} + C. && \text{Evaluate and replace } u \text{ with } x^2 + 1.\end{aligned}$$

8.6.29 Before integrating, let's write $\cot x$ in terms of $\sin x$ and $\cos x$ and then simplify the integrand by multiplying the numerator and denominator by $\sin x$.

$$\frac{2 \cos x + \overbrace{\frac{\cot x}{\cos x}}^{\cot x}}{1 + \sin x} = \frac{2 \sin x \cos x + \cos x}{\sin x + \sin^2 x}.$$

The integral is evaluated with the help of a u -substitution.

$$\begin{aligned}\int \frac{2 \cos x + \cot x}{1 + \sin x} dx &= \int \frac{2 \sin x \cos x + \cos x}{\sin x + \sin^2 x} dx && \text{Write in terms of } \sin x \text{ and } \cos x; \text{ simplify.} \\ &= \int \frac{du}{u} && \text{Let } u = \sin x + \sin^2 x; du = (\cos x + 2 \sin x \cos x) dx. \\ &= \ln |u| + C && \text{Integrate.} \\ &= \ln |\sin x + \sin^2 x| + C. && \text{Substitute } \sin x + \sin^2 x \text{ for } u.\end{aligned}$$

8.6.30 Using the trigonometric substitution $v = 5 \sin \theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, it follows that $dv = 5 \cos \theta d\theta$ and

$$\sqrt{25 - v^2} = \sqrt{25 - 25 \sin^2 \theta} = |5 \cos \theta| = 5 \cos \theta.$$

The lower limit $v = \frac{5}{2}$ implies that $\sin \theta = \frac{1}{2}$ and it follows that $\theta = \frac{\pi}{6}$. Similarly, if $v = \frac{5\sqrt{3}}{2}$ then $\sin \theta = \frac{\sqrt{3}}{2}$ and so $\theta = \frac{\pi}{3}$.

$$\begin{aligned} \int_{5/2}^{5\sqrt{3}/2} \frac{dv}{v^2 \sqrt{25 - v^2}} &= \int_{\pi/6}^{\pi/3} \frac{5 \cos \theta}{(25 \sin^2 \theta)(5 \cos \theta)} d\theta = \frac{1}{25} \int_{\pi/6}^{\pi/3} \csc^2 \theta d\theta = -\frac{1}{25} \cot \theta \Big|_{\pi/6}^{\pi/3} \\ &= -\frac{1}{25} \left(\frac{1}{\sqrt{3}} - \sqrt{3} \right) = \frac{2}{25\sqrt{3}}. \end{aligned}$$

8.6.31 Removing a factor of 3 from the square root, we have

$$\int \sqrt{36 - 9x^2} dx = \int \sqrt{9(4 - x^2)} dx = 3 \int \sqrt{4 - x^2} dx.$$

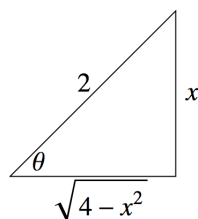
The expression $4 - x^2$ suggests the substitution $x = 2 \sin \theta$, for $\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. It follows that $dx = 2 \cos \theta d\theta$ and

$$\sqrt{4 - x^2} = \sqrt{4 - 4 \sin^2 \theta} = \sqrt{4(1 - \sin^2 \theta)} = 2|\cos \theta| = 2 \cos \theta.$$

Making these substitutions, we have

$$\begin{aligned} \int \sqrt{36 - 9x^2} dx &= 3 \int \sqrt{4 - x^2} dx && \text{Factor.} \\ &= 12 \int \cos^2 \theta d\theta && \text{Trigonometric substitution} \\ &= 6 \int (1 + \cos 2\theta) d\theta && \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \\ &= 6\theta + 3 \sin 2\theta + C && \text{Integrate.} \\ &= 6\theta + 6 \sin \theta \cos \theta + C. && \sin 2\theta = 2 \sin \theta \cos \theta \end{aligned}$$

From the equation $\sin \theta = \frac{x}{2}$ we have $\theta = \sin^{-1} \frac{x}{2}$. We use the following reference triangle to observe that $\cos \theta = \frac{\sqrt{4-x^2}}{2}$.



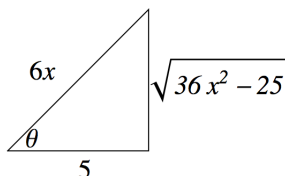
Therefore

$$\int \sqrt{36 - 9x^2} dx = 6 \underbrace{\sin^{-1} \frac{x}{2}}_{\theta} + 6 \underbrace{\left(\frac{x}{2} \right)}_{\sin \theta} \underbrace{\left(\frac{\sqrt{4-x^2}}{2} \right)}_{\cos \theta} + C = 6 \sin^{-1} \frac{x}{2} + \frac{3}{2} x \sqrt{4-x^2} + C.$$

8.6.32 We factor a 6 out of the square root, then proceed with a trigonometric substitution of $x = \frac{5}{6} \sec \theta$, where $0 < \theta < \frac{\pi}{2}$.

$$\begin{aligned}
\int \frac{dx}{\sqrt{36x^2 - 25}} &= \int \frac{dx}{6\sqrt{x^2 - \frac{25}{36}}} && \text{Factor.} \\
&= \int \frac{\frac{5}{6} \sec \theta \tan \theta}{6\sqrt{\frac{25}{36} \sec^2 \theta - \frac{25}{36}}} d\theta && \text{Trigonometric substitution} \\
&= \int \frac{\frac{5}{6} \sec \theta \tan \theta}{5 \tan \theta} d\theta && \sqrt{\frac{25}{36} \sec^2 \theta - \frac{25}{36}} = \frac{5}{6} |\tan \theta| = \frac{5}{6} \tan \theta \\
&= \frac{1}{6} \int \sec \theta d\theta && \text{Simplify.} \\
&= \frac{1}{6} \ln(\sec \theta + \tan \theta) + C && \text{Integrate.}
\end{aligned}$$

From our trigonometric substitution, we know that $\sec \theta = \frac{6x}{5}$ which is represented by the following reference triangle.



Using this triangle, we complete the integral:

$$\begin{aligned}
\int \frac{dx}{\sqrt{36x^2 - 25}} &= \frac{1}{6} \ln(\sec \theta + \tan \theta) + C \\
&= \frac{1}{6} \ln \left(\frac{6x}{5} + \frac{\sqrt{36x^2 - 25}}{5} \right) + C \\
&= \frac{1}{6} \ln (6x + \sqrt{36x^2 - 25}) - \frac{\ln 5}{6} + C \\
&= \frac{1}{6} \ln (6x + \sqrt{36x^2 - 25}) + C.
\end{aligned}$$

8.6.33 Letting $u = e^x$, we have $e^{2x} = (e^x)^2 = u^2$ and $du = e^x dx$. Provided that $a \neq 0$,

$$\int \frac{e^x}{a^2 + e^{2x}} dx = \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C = \frac{1}{a} \tan^{-1} \frac{e^x}{a} + C.$$

8.6.34 Let $u = \cos 3x + 1$ which implies that $du = -3 \sin 3x dx$, or $-\frac{1}{3} du = \sin 3x dx$.

$$\int_0^{\pi/9} \frac{\sin 3x}{\cos 3x + 1} dx = -\frac{1}{3} \int_2^{3/2} \frac{du}{u} = -\frac{1}{3} \left(\ln u \right) \Big|_2^{3/2} = \frac{\ln 2 - \ln \frac{3}{2}}{3} = \frac{1}{3} \ln \frac{4}{3}.$$

8.6.35 Let $u = \tan \theta$ which implies that $du = \sec^2 \theta d\theta$. The lower limit of integration becomes $u = \tan 0 = 0$ and the upper limit becomes $u = \tan \pi/4 = 1$. Changing variables, we have

$$\int_0^{\pi/4} (\tan^2 \theta + \tan \theta + 1) \sec^2 \theta d\theta = \int_0^1 (u^2 + u + 1) du = \left(\frac{u^3}{3} + \frac{u^2}{2} + u \right) \Big|_0^1 = \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$$

8.6.36 We apply integration by parts with the following choices for our variables.

$$\begin{aligned} u &= x & dv &= 10^x dx \\ du &= dx & v &= \frac{10^x}{\ln 10} \end{aligned}$$

We then have

$$\int x \cdot 10^x dx = \frac{x \cdot 10^x}{\ln 10} - \int \frac{10^x}{\ln 10} dx = \frac{x \cdot 10^x}{\ln 10} - \frac{10^x}{\ln^2 10} + C = \frac{10^x}{\ln 10} \left(x - \frac{1}{\ln 10} \right) + C.$$

8.6.37 We start by multiplying the integrand by $\frac{1 + \sin 2x}{1 + \sin 2x}$.

$$\begin{aligned} \int_0^{\pi/6} \frac{1}{1 - \sin 2x} \cdot \frac{1 + \sin 2x}{1 + \sin 2x} dx &= \int_0^{\pi/6} \frac{1 + \sin 2x}{1 - \sin^2 2x} dx && \text{Simplify integrand.} \\ &= \int_0^{\pi/6} \frac{1 + \sin 2x}{\cos^2 2x} dx && 1 - \sin^2 2x = \cos^2 2x. \\ &= \int_0^{\pi/6} (\sec^2 2x + \sec 2x \tan 2x) dx && \text{Simplify integrand.} \\ &= \frac{1}{2} (\tan 2x + \sec 2x) \Big|_0^{\pi/6} && \text{Integrate.} \\ &= \frac{1}{2} (\sqrt{3} + 2 - 1) = \frac{\sqrt{3} + 1}{2}. && \text{Evaluate.} \end{aligned}$$

8.6.38 One approach for evaluating $\int_{\pi/6}^{\pi/2} \cos x \ln(\sin x) dx$ is to use a u -substitution by letting $u = \sin x$ and to then apply the integral formula for $\ln u$. Another approach that we use here is to use integration by parts with the following variable choices.

$$\begin{aligned} u &= \ln(\sin x) & dv &= \cos x dx \\ du &= \frac{\cos x}{\sin x} dx & v &= \sin x \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} \int_{\pi/6}^{\pi/2} \cos x \ln(\sin x) dx &= \sin x \ln(\sin x) \Big|_{\pi/6}^{\pi/2} - \int_{\pi/6}^{\pi/2} \cos x dx = -\frac{1}{2} \ln \frac{1}{2} - \sin x \Big|_{\pi/6}^{\pi/2} \\ &= -\frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} = \frac{\ln 2 - 1}{2}. \end{aligned}$$

8.6.39 We integrate by parts with the following variable choices.

$$\begin{aligned} u &= \ln(\sin x) & dv &= \sin x dx \\ du &= \frac{\cos x}{\sin x} dx & v &= -\cos x \end{aligned}$$

It follows that

$$\begin{aligned}
\int \sin x \ln(\sin x) dx &= -\cos x \ln(\sin x) + \int \frac{\cos^2 x}{\sin x} dx && \text{Integration by parts} \\
&= -\cos x \ln(\sin x) + \int \frac{1 - \sin^2 x}{\sin x} dx && \cos^2 x = 1 - \sin^2 x \\
&= -\cos x \ln(\sin x) + \int (\csc x - \sin x) dx && \text{Simplify integrand.} \\
&= -\cos x \ln(\sin x) - \ln |\csc x + \cot x| + \cos x + C. && \text{Evaluate.}
\end{aligned}$$

8.6.40 We integrate by parts with the following variable choices.

$$u = \ln(\sin x) \quad dv = \sin 2x dx$$

$$du = \frac{\cos x}{\sin x} dx \quad v = -\frac{1}{2} \cos 2x$$

Then we have

$$\begin{aligned}
\int \sin 2x \ln(\sin x) dx &= -\frac{1}{2} \cos 2x \ln(\sin x) + \int \frac{\cos x \cos 2x}{2 \sin x} dx && \text{Integration by parts} \\
&= -\frac{1}{2} \cos 2x \ln(\sin x) + \int \frac{\cos x (1 - 2 \sin^2 x)}{2 \sin x} dx && 1 - 2 \sin^2 x = \cos 2x \\
&= -\frac{1}{2} \cos 2x \ln(\sin x) + \frac{1}{2} \int \cot x dx - \int \sin x \cos x dx && \text{Expand the integral.} \\
&= -\frac{1}{2} \cos 2x \ln(\sin x) + \frac{1}{2} \int \cot x dx - \int \frac{1}{2} \sin 2x dx && \sin x \cos x = \frac{1}{2} \sin 2x \\
&= -\frac{1}{2} \cos 2x \ln(\sin x) + \frac{1}{2} \ln |\sin x| + \frac{1}{4} \cos 2x + C. && \text{Evaluate.}
\end{aligned}$$

8.6.41 Split $\csc^4 x$ and use the identity $\csc^2 x = 1 + \cot^2 x$:

$$\begin{aligned}
\int \cot^{3/2} x \csc^4 x dx &= \int \cot^{3/2} x \csc^2 x \csc^2 x dx && \csc^4 x = \csc^2 x \csc^2 x \\
&= \int \cot^{3/2} x (1 + \cot^2 x) \csc^2 x dx && \csc^2 x = 1 + \cot^2 x \\
&= - \int u^{3/2} (1 + u^2) du && u = \cot x, du = -\csc^2 x dx \\
&= - \int (u^{3/2} + u^{7/2}) du && \text{Expand integrand.} \\
&= -\frac{2}{5} u^{5/2} - \frac{2}{9} u^{9/2} + C && \text{Evaluate.} \\
&= -\frac{2}{5} \cot^{5/2} x - \frac{2}{9} \cot^{9/2} x + C. && \text{Replace } u \text{ with } \cot x.
\end{aligned}$$

8.6.42 Let $u = \sin^{-1} x$, which implies that $du = \frac{dx}{\sqrt{1-x^2}}$. We also change the limits of integration.

$$\begin{aligned}
x = 0 &\Rightarrow u = \sin^{-1} 0 = 0 \\
x = \frac{1}{2} &\Rightarrow u = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}
\end{aligned}$$

Integrate:

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\pi/6} u du = \frac{u^2}{2} \Big|_0^{\pi/6} = \frac{\pi^2}{72}.$$

8.6.43 The integral $\int \frac{x^9}{\sqrt{1-x^{20}}} dx$ can be expressed as $\int \frac{x^9}{\sqrt{1-(x^{10})^2}} dx$, so we let $u = x^{10}$ which implies that $du = 10x^9 dx$, or $x^9 dx = \frac{du}{10}$. Therefore

$$\int \frac{x^9}{\sqrt{1-(x^{10})^2}} dx = \frac{1}{10} \int \frac{du}{\sqrt{1-u^2}} = \frac{\sin^{-1} u}{10} + C = \frac{\sin^{-1} x^{10}}{10} + C.$$

8.6.44 Factoring $x^3 - x^2$, an appropriate form of the partial fraction decomposition is

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Multiplying both sides by $x^2(x-1)$ leads to

$$1 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (B-A)x - B.$$

Equating coefficients of equal powers of x , we find that $A = -1$, $B = -1$, and $C = 1$. Therefore

$$\int \frac{dx}{x^3-x^2} = -\int \frac{1}{x} dx - \int \frac{dx}{x^2} + \int \frac{dx}{x-1} = -\ln|x| + \frac{1}{x} + \ln|x-1| + C.$$

8.6.45 Letting $u = 1 + e^x$, it follows that $du = e^x dx$, $e^x = u - 1$, and $dx = \frac{du}{e^x} = \frac{du}{u-1}$. The lower limit of integration is $u = 2$ and the upper limit is $u = 3$. We have

$$\int_0^{\ln 2} \frac{dx}{(1+e^x)^2} = \int_2^3 \frac{du}{u^2(u-1)}.$$

Partial fractions is used to evaluate the integral on the right. An appropriate form for the partial fraction decomposition of the integrand is

$$\frac{1}{u^2(u-1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u-1}.$$

Multiplying both sides by $u^2(u-1)$ leads to

$$1 = Au(u-1) + B(u-1) + Cu^2 = (A+C)u^2 + (B-A)u - B.$$

Equating coefficients of equal powers of u , we find that $A = -1$, $B = -1$, and $C = 1$. Therefore

$$\begin{aligned} \int_0^{\ln 2} \frac{dx}{(1+e^x)^2} &= \int_2^3 \frac{du}{u^2(u-1)} = \int_2^3 \left(-\frac{1}{u} - \frac{1}{u^2} + \frac{1}{u-1} \right) du = \left(-\ln|u| + \frac{1}{u} + \ln|u-1| \right) \Big|_2^3 \\ &= \ln \frac{2}{3} - \frac{1}{6} + \ln 2 = \ln \frac{4}{3} - \frac{1}{6}. \end{aligned}$$

8.6.46 Letting $u = e^x$, we have $du = e^x dx$, $dx = \frac{du}{e^x} = \frac{du}{u}$, and $\int \frac{dx}{e^{2x}+1} = \int \frac{du}{u(u^2+1)}$. The form of the partial fraction decomposition of $\frac{1}{u(u^2+1)}$ is

$$\frac{1}{u(u^2+1)} = \frac{A}{u} + \frac{Bu+C}{u^2+1}.$$

Multiplying both sides by $u(u^2 + 1)$, we have

$$1 = A(u^2 + 1) + (Bu + C)u = (A + B)u^2 + Cu + A.$$

Comparing coefficients of equal powers, we find that $A = 1$, $B = -1$, and $C = 0$.

Inserting these constants into our partial fraction decomposition implies that

$$\int \frac{du}{u(u^2 + 1)} = \int \frac{du}{u} - \int \frac{u}{u^2 + 1} du.$$

In the second integral, we let $v = u^2 + 1$. Then $dv = 2u du$, or $\frac{1}{2}dv = u du$. It follows that

$$\int \frac{u}{u^2 + 1} du = \frac{1}{2} \int \frac{dv}{v} = \frac{1}{2} \ln |v| + C = \frac{1}{2} \ln(u^2 + 1) + C.$$

Assembling all our pieces, we have

$$\begin{aligned} \int \frac{dx}{e^{2x} + 1} &= \int \frac{du}{u} - \int \frac{u}{u^2 + 1} du \\ &= \ln |u| - \frac{1}{2} \ln(u^2 + 1) + C \\ &= \ln e^x - \frac{1}{2} \ln(e^{2x} + 1) + C \\ &= x - \frac{1}{2} \ln(e^{2x} + 1) + C. \end{aligned}$$

8.6.47 Apply long division to obtain

$$\frac{2x^3 + x^2 - 2x - 4}{x^2 - x - 2} = 2x + 3 + \frac{5x + 2}{x^2 - x - 2}.$$

Factor $x^2 - x - 2$ and then find the partial fraction decomposition of $\frac{5x + 2}{x^2 - x - 2}$.

$$\frac{5x + 2}{(x - 2)(x + 1)} = \frac{A}{x - 2} + \frac{B}{x + 1}$$

Multiply both sides by $(x - 2)(x + 1)$.

$$5x + 2 = A(x + 1) + B(x - 2).$$

Note that $x = 2$ implies that $A = 4$ and $x = -1$ implies that $B = 1$. Therefore

$$\begin{aligned} \int \left(\frac{2x^3 + x^2 - 2x - 4}{x^2 - x - 2} \right) dx &= \int \left(2x + 3 + \frac{4}{x - 2} + \frac{1}{x + 1} \right) dx \\ &= x^2 + 3x + 4 \ln |x - 2| + \ln |x + 1| + C. \end{aligned}$$

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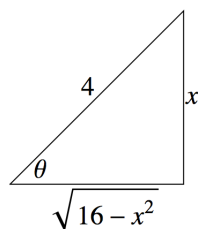
8.6.48 Let $x = 4 \sin \theta$, for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\theta \neq 0$. This substitution implies that $dx = 4 \cos \theta d\theta$ and

$$\sqrt{16 - x^2} = \sqrt{16 - 16 \sin^2 \theta} = \sqrt{16 \cos^2 \theta} = |4 \cos \theta| = 4 \cos \theta.$$

With these substitutions, we have

$$\begin{aligned}
 \int \frac{\sqrt{16-x^2}}{x^2} dx &= \int \frac{4 \cos \theta}{16 \sin^2 \theta} 4 \cos \theta d\theta \\
 &= \int \cot^2 \theta d\theta \\
 &= \int (\csc^2 \theta - 1) d\theta = -\cot \theta - \theta + C.
 \end{aligned}$$

Given that $\sin \theta = \frac{x}{4}$, we can form the following right triangle that helps us evaluate the integral.



Therefore

$$\int \frac{\sqrt{16-x^2}}{x^2} dx = -\cot \theta - \theta + C = -\frac{\sqrt{16-x^2}}{x} - \sin^{-1} \frac{x}{4} + C.$$

8.6.49 We start by splitting off factors of $\tan x$ and $\sec x$ and then we replace $\tan^2 x$ with $\sec^2 x - 1$:

$$\begin{aligned}
 \int \tan^3 x \sec^9 x dx &= \int \tan^2 x \sec^8 x \sec x \tan x dx && \text{Split off } \tan x \text{ and } \sec x. \\
 &= \int (\sec^2 x - 1) \sec^8 x \sec x \tan x dx && \tan^2 x = \sec^2 x - 1 \\
 &= \int (\sec^{10} x - \sec^8 x) \sec x \tan x dx && \text{Expand integrand.} \\
 &= \int (u^{10} - u^8) du && u = \sec x, du = \sec x \tan x dx \\
 &= \frac{u^{11}}{11} - \frac{u^9}{9} + C && \text{Integrate.} \\
 &= \frac{\sec^{11} x}{11} - \frac{\sec^9 x}{9} + C. && \text{Replace } u \text{ with } \sec x.
 \end{aligned}$$

8.6.50 With an even power of $\sec x$, we split off a factor of $\sec^2 x$ and prepare the integral for a substitution of $u = \tan x$:

$$\begin{aligned}
\int \tan^7 x \sec^4 x \, dx &= \int \tan^7 x \sec^2 x \sec^2 x \, dx && \text{Split off } \sec^2 x. \\
&= \int \tan^7 x (\tan^2 x + 1) \sec^2 x \, dx && \sec^2 x = \tan^2 x + 1 \\
&= \int u^7 (u^2 + 1) \, du && u = \tan x, \, du = \sec^2 x \, dx \\
&= \int (u^9 + u^7) \, du && \text{Expand.} \\
&= \frac{u^{10}}{10} + \frac{u^8}{8} + C && \text{Integrate.} \\
&= \frac{\tan^{10} x}{10} + \frac{\tan^8 x}{8} + C. && \text{Replace } u \text{ with } \tan x.
\end{aligned}$$

8.6.51 Rewrite $\sec^{7/4} x$ as $\sec^{3/4} x \sec x$ and let $u = \sec x$. Then $du = \sec x \tan x$ and after making this substitution, the lower limit of integration is $u = \sec 0 = 1$ and the upper limit is $u = \sec \frac{\pi}{3} = 2$. Integrating, we have

$$\int_0^{\pi/3} \tan x \sec^{7/4} x \, dx = \int_1^2 \sec^{3/4} x \sec x \tan x \, dx = \int_1^2 u^{3/4} du = \frac{4}{7} u^{7/4} \Big|_1^2 = \frac{4}{7} (2^{7/4} - 1).$$

8.6.52 We apply the method of integration by parts using the following choices.

$$\begin{aligned}
u &= t^2 & dv &= e^{3t} \, dt \\
du &= 2t \, dt & v &= \frac{e^{3t}}{3}
\end{aligned}$$

The integral then becomes

$$\int t^2 e^{3t} \, dt = \frac{t^2 e^{3t}}{3} - \frac{2}{3} \int t e^{3t} \, dt.$$

Now we apply integration by parts again to the integral $\int t e^{3t} \, dt$ with the following choices.

$$\begin{aligned}
u &= t & dv &= e^{3t} \, dt \\
du &= dt & v &= \frac{e^{3t}}{3}
\end{aligned}$$

We then have

$$\begin{aligned}
\frac{t^2 e^{3t}}{3} - \frac{2}{3} \int t e^{3t} \, dt &= \frac{t^2 e^{3t}}{3} - \frac{2}{3} \left(\frac{t e^{3t}}{3} - \frac{1}{3} \int e^{3t} \, dt \right) \\
&= \frac{t^2 e^{3t}}{3} - \frac{2t e^{3t}}{9} + \frac{2e^{3t}}{27} + C \\
&= e^{3t} \left(\frac{t^2}{3} - \frac{2t}{9} + \frac{2}{27} \right) + C.
\end{aligned}$$

8.6.53 We begin with the substitution $u = e^x$, $du = e^x \, dx$, and then split apart the integrand.

$$\int e^x \cot^3 e^x \, dx = \int \cot^3 u \, du = \int \cot^2 u \cot u \, du = \int (\csc^2 u - 1) \cot u \, du = \int \csc^2 u \cot u \, du - \int \cot u \, du$$

Using the substitution $v = \cot u$ in the first integral with $dv = -\csc^2 u \, du$ or $-dv = \csc^2 u \, du$, we find that

$$\int \csc^2 u \cot u \, du = -\int v \, dv = -\frac{v^2}{2} + C = -\frac{\cot^2 u}{2} + C.$$

Therefore

$$\int e^x \cot^3 e^x \, dx = \int \csc^2 u \cot u \, du - \int \cot u \, du = -\frac{\cot^2 u}{2} - \ln |\sin u| + C = -\frac{\cot^2 e^x}{2} - \ln |\sin e^x| + C.$$

8.6.54 The denominator of the integrand consists of a linear function and an irreducible quadratic factor. The form of the partial fraction decomposition is

$$\frac{2x^2 + 3x + 26}{(x-2)(x^2+16)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+16}.$$

Multiplying both sides of this equation by $(x-2)(x^2+16)$ leads to

$$2x^2 + 3x + 26 = A(x^2 + 16) + (Bx + C)(x - 2) = (A + B)x^2 + (C - 2B)x + (16A - 2C)$$

Equating coefficients of equal powers of x results in the equations

$$A + B = 2, C - 2B = 3, \text{ and } 16A - 2C = 26.$$

The solution to this system of equations is $A = 2$, $B = 0$, and $C = 3$. Therefore

$$\int \frac{2x^2 + 3x + 26}{(x-2)(x^2+16)} \, dx = \int \frac{2}{x-2} \, dx + \int \frac{3}{x^2+16} \, dx = 2 \ln |x-2| + \frac{3}{4} \tan^{-1} \frac{x}{4} + C.$$

8.6.55 The denominator factors as $x^3 + x = x(x^2 + 1)$ and therefore the form of the partial fraction decomposition is

$$\frac{3x^2 + 3x + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

Multiplying both sides by $x(x^2 + 1)$ leads to

$$3x^2 + 3x + 1 = A(x^2 + 1) + (Bx + C)x = (A + B)x^2 + Cx + A.$$

By equating coefficients of equal powers of x , we find that $A = 1$, $B = 2$, and $C = 3$. The original integral can be written as

$$\int \frac{3x^2 + 3x + 1}{x^3 + x} \, dx = \int \frac{dx}{x} + \int \frac{2x+3}{x^2+1} \, dx = \int \frac{dx}{x} + \int \frac{2x}{x^2+1} \, dx + \int \frac{3}{x^2+1} \, dx.$$

Using a substitution if $u = x^2 + 1$ in the second integral with $du = 2x \, dx$, we find that $\int \frac{2x}{x^2+1} \, dx = \ln(x^2 + 1) + C$. Completing the integration process we have

$$\int \frac{3x^2 + 3x + 1}{x^3 + x} \, dx = \int \frac{dx}{x} + \int \frac{2x}{x^2+1} \, dx + \int \frac{3}{x^2+1} \, dx = \ln |x| + \ln(x^2 + 1) + 3 \tan^{-1} x + C.$$

8.6.56 Using the trigonometric identity $\sin 2x = 2 \sin x \cos x$ the integral equals $\int_{\pi}^{3\pi/2} 2 \sin x \cos x e^{\sin^2 x} \, dx$.

Making the substitution $u = \sin^2 x$, we have $du = 2 \sin x \cos x \, dx$; it is also the case that $x = \pi \Rightarrow u = \sin^2 0 = 0$ and $x = 3\pi/2 \Rightarrow u = \sin^2(\frac{3\pi}{2}) = 1$.

Therefore

$$\int_{\pi}^{3\pi/2} \sin 2x e^{\sin^2 x} \, dx = \int_{\pi}^{3\pi/2} 2 \sin x \cos x e^{\sin^2 x} \, dx = \int_0^1 e^u \, du = e^u \Big|_0^1 = e - 1.$$

8.6.57 Some observations are in order before we start integrating. Notice that

$$\int \sin \sqrt{x} \, dx = \int \sqrt{x} \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx.$$

Also observe that $\int \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx = -2 \cos \sqrt{x} + C$ because $\frac{d}{dx}(-2 \cos \sqrt{x}) = \frac{\sin \sqrt{x}}{\sqrt{x}}$. These facts will help us evaluate the integral using integration by parts with the following variable choices.

$$\begin{aligned} u &= \sqrt{x} & dv &= \frac{\sin \sqrt{x}}{\sqrt{x}} \, dx \\ du &= \frac{dx}{2\sqrt{x}} & v &= -2 \cos \sqrt{x} \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} \int \sin \sqrt{x} \, dx &= -2\sqrt{x} \cos \sqrt{x} + \int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx \\ &= -2\sqrt{x} \cos \sqrt{x} + 2 \sin \sqrt{x} + C. \end{aligned}$$

8.6.58 We apply integration by parts with the following choices for our variables.

$$\begin{aligned} u &= \tan^{-1} w & dv &= w^2 \, dw \\ du &= \frac{dw}{1+w^2} & v &= \frac{w^3}{3} \end{aligned}$$

The integral then becomes

$$\int w^2 \tan^{-1} w \, dw = \frac{w^3}{3} \tan^{-1} w - \frac{1}{3} \int \frac{w^3}{1+w^2} \, dw.$$

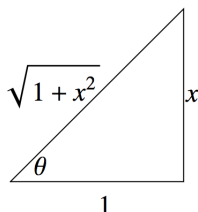
Using long division to obtain the equality $\frac{w^3}{1+w^2} = w - \frac{w}{1+w^2}$ and a u -substitution of $u = 1 + w^2$ to evaluate $\int \frac{w}{1+w^2} \, dw$, the integral is straightforward to complete:

$$\begin{aligned} \int w^2 \tan^{-1} w \, dw &= \frac{1}{3} \left(w^3 \tan^{-1} w - \int w \, dw + \int \frac{w}{1+w^2} \, dw \right) \\ &= \frac{1}{3} \left(w^3 \tan^{-1} w - \frac{w^2}{2} + \frac{1}{2} \ln(1+w^2) \right) + C. \end{aligned}$$

8.6.59 Factor the denominator and make the trigonometric substitution $x = \tan \theta$, where $-\pi/2 < \theta < \pi/2$. Then $dx = \sec^2 \theta \, d\theta$ and

$$\int \frac{dx}{x^4 + x^2} = \int \frac{dx}{x^2(x^2 + 1)} = \int \frac{\sec^2 \theta}{\underbrace{\tan^2 \theta (\tan^2 \theta + 1)}_{\sec^2 \theta}} d\theta = \int \cot^2 \theta \, d\theta = \int (\csc^2 \theta - 1) d\theta = -\cot \theta - \theta + C.$$

Because $\tan \theta = x$, we use the following reference triangle and the substitution $\theta = \tan^{-1} x$ to complete the integration.



$$\int \frac{dx}{x^4 + x^2} = -\cot \theta - \theta + C = -\frac{1}{x} - \tan^{-1} x + C.$$

8.6.60 We evaluate the indefinite integral first. Apply integration by parts with the following variable choices.

$$\begin{aligned} u &= e^{-3x} & dv &= \cos x \, dx \\ du &= -3e^{-3x} dx & v &= \sin x \end{aligned}$$

The corresponding integral then becomes

$$\int e^{-3x} \cos x \, dx = e^{-3x} \sin x + 3 \int e^{-3x} \sin x \, dx.$$

Integration by parts is applied again to the integral on the right side of the equation using the following choices.

$$\begin{aligned} u &= e^{-3x} & dv &= \sin x \, dx \\ du &= -3e^{-3x} dx & v &= -\cos x \end{aligned}$$

We then have

$$\int e^{-3x} \cos x \, dx = e^{-3x} \sin x + 3 \int e^{-3x} \sin x \, dx = e^{-3x} \sin x + 3 \left(-e^{-3x} \cos x - 3 \int e^{-3x} \cos x \, dx \right).$$

Simplifying the equation, we find that

$$\int e^{-3x} \cos x \, dx = e^{-3x} \sin x - 3e^{-3x} \cos x - 9 \int e^{-3x} \cos x \, dx.$$

The indefinite integral is then found by solving for $\int e^{-3x} \cos x \, dx$ and adding a constant of integration:

$$\int e^{-3x} \cos x \, dx = \frac{e^{-3x}}{10} (\sin x - 3 \cos x) + C.$$

Using this result, we evaluate the definite integral:

$$\int_0^{\pi/2} e^{-3x} \cos x \, dx = \frac{e^{-3x}}{10} (\sin x - 3 \cos x) \Big|_0^{\pi/2} = \frac{e^{-3\pi/2} + 3}{10}.$$

8.6.61 Letting $u = \sin^{-1} x$ we have $x = \sin u$ and $dx = \cos u \, du$. In this case, $x = 0 \Rightarrow u = \sin^{-1} 0 = 0$ and $x = \frac{\sqrt{2}}{2} \Rightarrow u = \sin^{-1} \left(\frac{\sqrt{2}}{2} \right) = \frac{\pi}{4}$. So the integral becomes

$$\int_0^{\sqrt{2}/2} e^{\sin^{-1} x} \, dx = \int_0^{\pi/4} e^u \cos u \, du.$$

Using integration by parts twice (similar to the solution in the previous problem), it can be shown that

$$\int e^u \cos u \, du = \frac{e^u}{2} (\cos u + \sin u) + C.$$

Now we summarize what we've done so far and then complete the exercise:

$$\begin{aligned}
 \int_0^{\sqrt{2}/2} e^{\sin^{-1} x} dx &= \int_0^{\pi/4} e^u \cos u \, du && u\text{-substitution} \\
 &= \frac{e^u}{2} (\cos u + \sin u) \Big|_0^{\pi/4} && \text{Integration by parts (twice)} \\
 &= \frac{e^{\pi/4}}{2} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - \frac{1}{2} && \text{Evaluate.} \\
 &= \frac{\sqrt{2}e^{\pi/4} - 1}{2}. && \text{Simplify.}
 \end{aligned}$$

8.6.62 Rearranging the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

and replacing θ with $\theta/2$, we have $\sqrt{1 + \cos \theta} = \sqrt{2 \cos^2 \frac{\theta}{2}} = \sqrt{2} \cos \frac{\theta}{2}$ (the last equality is justified because $\cos \frac{\theta}{2} \geq 0$ for $0 \leq \theta \leq \pi/2$). Therefore

$$\int_0^{\pi/2} \sqrt{1 + \cos \theta} \, d\theta = \sqrt{2} \int_0^{\pi/2} \cos \frac{\theta}{2} \, d\theta = 2\sqrt{2} \sin \frac{\theta}{2} \Big|_0^{\pi/2} = 2.$$

8.6.63 We apply integration by parts with the following choices for our variables.

$$\begin{aligned}
 u &= \ln x & dv &= x^a dx \\
 du &= \frac{1}{x} dx & v &= \frac{x^{a+1}}{a+1}
 \end{aligned}$$

The integration process is straightforward:

$$\begin{aligned}
 \int x^a \ln x \, dx &= \frac{x^{a+1} \ln x}{a+1} - \int \frac{x^a}{a+1} dx && \text{Integration by parts} \\
 &= \frac{x^{a+1} \ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} + C && \text{Evaluate.} \\
 &= \frac{x^{a+1}}{a+1} \left(\ln x - \frac{1}{a+1} \right) + C. && \text{Factor.}
 \end{aligned}$$

8.6.64 Letting $u = \ln ax$ so that $du = \frac{1}{ax} a \, dx = \frac{dx}{x}$, we have

$$\int \frac{\ln ax}{x} \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{\ln^2 ax}{2} + C.$$

8.6.65 By factoring the denominator, we can apply the formula $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$:

$$\int_0^{1/6} \frac{dx}{\sqrt{1 - 9x^2}} = \int_0^{1/6} \frac{dx}{\sqrt{9(\frac{1}{9} - x^2)}} = \frac{1}{3} \int_0^{1/6} \frac{dx}{\sqrt{(\frac{1}{3})^2 - x^2}} = \frac{1}{3} \sin^{-1}(3x) \Big|_0^{1/6} = \frac{\pi}{18}.$$

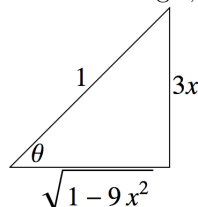
8.6.66 Although trigonometric substitution works here, it's easier to use a u -substitution of $u = 1 - 9x^2$. This implies that $du = -18x \, dx$ or $-\frac{1}{18} du = x \, dx$. The substitution may now be done:

$$\int \frac{x}{\sqrt{1 - 9x^2}} \, dx = -\frac{1}{18} \int \frac{du}{\sqrt{u}} = -\frac{1}{9} \sqrt{u} + C = -\frac{1}{9} \sqrt{1 - 9x^2} + C.$$

8.6.67 $\sqrt{1-9x^2} = \sqrt{1-(3x)^2}$ suggests the change of variables $3x = \sin \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. This substitution implies that $3 dx = \cos \theta d\theta$ and

$$\sqrt{1-9x^2} = \sqrt{1-(3x)^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = |\cos \theta| = \cos \theta.$$

Using these substitutions and the following reference triangle, the integral is evaluated as follows.



$$\begin{aligned} \int \frac{x^2}{\sqrt{1-(3x)^2}} dx &= \frac{1}{27} \int \frac{\sin^2 \theta}{\cos \theta} \cdot \cos \theta d\theta && \text{Trigonometric substitution} \\ &= \frac{1}{27} \int \sin^2 \theta d\theta && \text{Simplify.} \\ &= \frac{1}{54} \int (1 - \cos 2\theta) d\theta && \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \\ &= \frac{1}{54} \left(\theta - \frac{1}{2} \sin 2\theta \right) + C && \text{Evaluate.} \\ &= \frac{1}{54} (\theta - \sin \theta \cos \theta) + C && \frac{1}{2} \sin 2\theta = \sin \theta \cos \theta \\ &= \frac{1}{54} \left(\sin^{-1} 3x - 3x \sqrt{1-9x^2} \right) + C && \text{Reference triangle} \end{aligned}$$

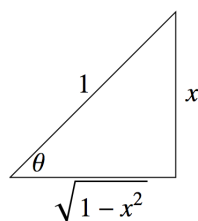
8.6.68 We begin by making a u -substitution and then we complete the square in the denominator.

$$\begin{aligned} \int \frac{e^x}{e^{2x} + 2e^x + 17} dx &= \int \frac{du}{u^2 + 2u + 17} && u = e^x, du = e^x dx \\ &= \int \frac{du}{(u+1)^2 + 16} && \text{Complete the square.} \\ &= \frac{1}{4} \tan^{-1} \frac{u+1}{4} + C && \text{Evaluate.} \\ &= \frac{1}{4} \tan^{-1} \frac{e^x + 1}{4} + C && \text{Replace } u \text{ with } e^x. \end{aligned}$$

8.6.69 The presence of the expression $1-x^2$ in the integrand suggests a trigonometric substitution of $x = \sin \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. For this choice, $dx = \cos \theta d\theta$, $1-x^2 = 1-\sin^2 \theta = \cos^2 \theta$ and $\sqrt{1-x^2} = |\cos \theta| = \cos \theta$. This trigonometric substitution gets the integration process started.

$$\begin{aligned}
\int \frac{dx}{1-x^2+\sqrt{1-x^2}} &= \int \frac{\cos \theta}{\cos^2 \theta + \cos \theta} d\theta && \text{Trigonometric substitution} \\
&= \int \frac{d\theta}{\cos \theta + 1} && \text{Simplify.} \\
&= \int \frac{1}{\cos \theta + 1} \cdot \frac{1 - \cos \theta}{1 - \cos \theta} d\theta && \text{Multiply by } \frac{1 - \cos \theta}{1 - \cos \theta}. \\
&= \int \frac{1 - \cos \theta}{\sin^2 \theta} d\theta && \text{Simplify.} \\
&= \int (\csc^2 \theta - \csc \theta \cot \theta) d\theta && \frac{1 - \cos \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} - \frac{\cos \theta}{\sin^2 \theta} = \csc^2 \theta - \csc \theta \cot \theta \\
&= -\cot \theta + \csc \theta + C && \text{Evaluate.}
\end{aligned}$$

Given that $\sin \theta = x$, a geometric description of the relationship between x and θ is given by the following reference triangle.



Therefore

$$\int \frac{dx}{1-x^2+\sqrt{1-x^2}} = -\cot \theta + \csc \theta = -\frac{\sqrt{1-x^2}}{x} + \frac{1}{x} + C = \frac{1-\sqrt{1-x^2}}{x} + C.$$

8.6.70 We use integration by parts with the following choices for the variables.

$$\begin{aligned}
u &= \ln(x^2 + a^2) & dv &= dx \\
du &= \frac{2x}{x^2+a^2} dx & v &= x
\end{aligned}$$

Applying integration by parts, we have

$$\int \ln(x^2 + a^2) dx = x \ln(x^2 + a^2) - 2 \int \frac{x^2}{x^2 + a^2} dx.$$

Using long division, it can be shown that $\frac{x^2}{x^2 + a^2} = 1 - \frac{a^2}{x^2 + a^2}$. Therefore

$$\begin{aligned}
\int \ln(x^2 + a^2) dx &= x \ln(x^2 + a^2) - 2 \int \left(1 - \frac{a^2}{x^2 + a^2}\right) dx \\
&= x \ln(x^2 + a^2) - 2 \int dx + 2a^2 \int \frac{dx}{x^2 + a^2} \\
&= x \ln(x^2 + a^2) - 2x + 2a \tan^{-1} \frac{x}{a} + C.
\end{aligned}$$

8.6.71 We start by multiplying the numerator and the denominator of the integrand by $(1 - \cos x)$:

$$\begin{aligned}
\frac{1 - \cos x}{1 + \cos x} \cdot \frac{1 - \cos x}{1 - \cos x} &= \frac{1 - 2 \cos x + \cos^2 x}{1 - \cos^2 x} && \text{Multiply by } \frac{1 - \cos x}{1 - \cos x}. \\
&= \frac{1 - 2 \cos x + \cos^2 x}{\sin^2 x} && \text{Trigonometric Identity.} \\
&= \csc^2 x - 2 \cot x \csc x + \cot^2 x && \text{Simplify.} \\
&= \csc^2 x - 2 \cot x \csc x + \csc^2 x - 1 && \text{Replace } \cot^2 x \text{ with } \csc^2 x - 1. \\
&= 2 \csc^2 x - 2 \cot x \csc x - 1 && \text{Simplify.}
\end{aligned}$$

With the transformed integrand, the integral is easier to evaluate:

$$\int \frac{1 - \cos x}{1 + \cos x} dx = \int (2 \csc^2 x - 2 \cot x \csc x - 1) dx = -2 \cot x + 2 \csc x - x + C.$$

8.6.72 We use integration by parts with the following choices for the variables.

$$\begin{aligned}
u &= x^2 & dv &= \sinh x \, dx \\
du &= 2x \, dx & v &= \cosh x \\
\int x^2 \sinh x \, dx &= x^2 \cosh x - \int 2x \cosh x \, dx
\end{aligned}$$

We apply integration by parts again with the following choices for the variables.

$$\begin{aligned}
u &= 2x & dv &= \cosh x \, dx \\
du &= 2 \, dx & v &= \sinh x \\
\int x^2 \sinh x \, dx &= x^2 \cosh x - \int 2x \cosh x \, dx \\
&= x^2 \cosh x - 2x \sinh x + \int 2 \sinh x \, dx + C \\
&= x^2 \cosh x - 2x \sinh x + 2 \cosh x + C.
\end{aligned}$$

8.6.73 Letting $u = 1 + \sqrt{x}$ it follows that $du = \frac{dx}{2\sqrt{x}}$. Rearranging these equations, we have $\sqrt{x} = u - 1$ and $dx = 2\sqrt{x} du = 2(u - 1)du$. After the substitution is made, the lower limit of integration is $u = 4$ and the upper limit is $u = 5$. Therefore

$$\begin{aligned}
\int_9^{16} \sqrt{1 + \sqrt{x}} \, dx &= 2 \int_4^5 \sqrt{u}(u - 1) du \\
&= 2 \int_4^5 (u^{3/2} - u^{1/2}) du \\
&= 2 \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) \Big|_4^5 \\
&= 4u^{3/2} \left(\frac{u}{5} - \frac{1}{3} \right) \Big|_4^5 \\
&= \frac{4u^{3/2}}{15} (3u - 5) \Big|_4^5 \\
&= \frac{40\sqrt{5}}{3} - \frac{224}{15}.
\end{aligned}$$

8.6.74 If $u = e^x - 1$, then $du = e^x dx$, $e^x = u + 1$, and $e^{2x} = (u + 1)^2$. Therefore

$$\begin{aligned}
 \int \frac{e^{3x}}{e^x - 1} dx &= \int \frac{e^{2x} e^x}{e^x - 1} dx \\
 &= \int \frac{(u + 1)^2}{u} du \\
 &= \int \left(u + 2 + \frac{1}{u} \right) du \\
 &= \frac{u^2}{2} + 2u + \ln |u| + C \\
 &= \frac{(e^x - 1)^2}{2} + 2(e^x - 1) + \ln |e^x - 1| + C \\
 &= \frac{e^{2x}}{2} - e^x + \frac{1}{2} + 2e^x - 2 + \ln |e^x - 1| + C \\
 &= \frac{e^{2x}}{2} + e^x + \ln |e^x - 1| + C.
 \end{aligned}$$

8.6.75 Letting $u = \tan^{-1} \sqrt{x}$, it follows that $du = \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} dx = \frac{dx}{2(\sqrt{x} + x^{3/2})}$. Then $2du = \frac{dx}{\sqrt{x} + x^{3/2}}$. The lower limit of integration is $u = \tan^{-1} 1 = \frac{\pi}{4}$ and the upper limit is $u = \tan^{-1} \sqrt{3} = \frac{\pi}{3}$. Therefore

$$\int_1^3 \frac{\tan^{-1} \sqrt{x}}{x^{1/2} + x^{3/2}} dx = \int_{\pi/4}^{\pi/3} 2u du = u^2 \Big|_{\pi/4}^{\pi/3} = \frac{7\pi^2}{144}.$$

8.6.76 By completing the square of the denominator of the integrand and then making a u -substitution we have

$$\begin{aligned}
 \int \frac{x}{x^2 + 6x + 18} dx &= \int \frac{x}{(x + 3)^2 + 9} dx && \text{Complete the square.} \\
 &= \int \frac{u - 3}{u^2 + 9} du && u = x + 3, du = dx, x = u - 3 \\
 &= \int \frac{u}{u^2 + 9} du - \int \frac{3}{u^2 + 9} du && \text{Split into two integrals.}
 \end{aligned}$$

The second integral equals $\tan^{-1} \frac{u}{3}$; the first integral is evaluated by making a substitution of $w = u^2 + 9$, which implies that $dw = 2u du$ or $\frac{1}{2}dw = u du$:

$$\int \frac{u}{u^2 + 9} du = \frac{1}{2} \int \frac{dw}{w} = \frac{1}{2} \ln |w| + C = \frac{1}{2} \ln(u^2 + 9) + C.$$

Therefore,

$$\int \frac{x}{x^2 + 6x + 18} dx = \frac{1}{2} \ln(u^2 + 9) - \tan^{-1} \frac{u}{3} + C = \frac{1}{2} \ln((x + 3)^2 + 9) - \tan^{-1} \frac{x + 3}{3} + C.$$

8.6.77 We use integration by parts with the following choices for the variables:

$$u = \cos^{-1} x \quad dv = dx$$

$$du = -\frac{dx}{\sqrt{1-x^2}} \quad v = x$$

Now we apply integration by parts followed by integration by substitution.

$$\begin{aligned} \int \cos^{-1} x \, dx &= x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} dx && \text{Integration by parts} \\ &= x \cos^{-1} x - \frac{1}{2} \int u^{-1/2} du && u = 1 - x^2; \, du = -2x \, dx \\ &= x \cos^{-1} x - u^{1/2} + C && \text{Integrate.} \\ &= x \cos^{-1} x - \sqrt{1-x^2} + C. && \text{Replace } u \text{ with } 1 - x^2. \end{aligned}$$

8.6.78 Start with integration by parts, using the following variable choices.

$$u = (\cos^{-1} x)^2 \quad dv = dx$$

$$du = -\frac{2 \cos^{-1} x}{\sqrt{1-x^2}} dx \quad v = x$$

It follows that

$$\int (\cos^{-1} x)^2 \, dx = x(\cos^{-1} x)^2 + 2 \int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} \, dx.$$

Apply integration by parts again to the integral on the right using the following choices for the variables.

$$u = \cos^{-1} x \quad dv = \frac{x}{\sqrt{1-x^2}} dx$$

$$du = -\frac{dx}{\sqrt{1-x^2}} \quad v = -\sqrt{1-x^2}$$

The second integral becomes

$$\int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \cos^{-1} x - \int dx = -\sqrt{1-x^2} \cos^{-1} x - x + C.$$

Assembling our work, we have

$$\begin{aligned} \int (\cos^{-1} x)^2 \, dx &= x(\cos^{-1} x)^2 + 2 \int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx \\ &= x(\cos^{-1} x)^2 + 2 \left(-\sqrt{1-x^2} \cos^{-1} x - x \right) + C \\ &= x(\cos^{-1} x)^2 - 2\sqrt{1-x^2} \cos^{-1} x - 2x + C. \end{aligned}$$

8.6.79 We apply integration by parts with the following choices for the variables.

$$u = \sin^{-1} x \quad dv = \frac{dx}{x^2}$$

$$du = \frac{dx}{\sqrt{1-x^2}} \quad v = -\frac{1}{x}$$

Using these substitutions with integration by parts, along with the answer to Exercise 25, we evaluate the integral.

$$\int \frac{\sin^{-1} x}{x^2} dx = -\frac{\sin^{-1} x}{x} + \int \frac{dx}{x\sqrt{1-x^2}} = -\frac{\sin^{-1} x}{x} + \ln \left| \frac{x}{1+\sqrt{1-x^2}} \right| + C.$$

8.6.80 Start by completing the square of $-4x - x^2$ and then make a u -substitution of $u = x + 2$. The lower limit of integration becomes 0 and the upper limit becomes 1.

$$\int_{-2}^{-1} \sqrt{-4x - x^2} dx = \int_{-2}^{-1} \sqrt{4 - (x+2)^2} dx = \int_0^1 \sqrt{4 - u^2} du$$

Using the trigonometric substitution $u = 2 \sin \theta$, it follows that $du = 2 \cos \theta d\theta$. With this substitution, $u = 0$ implies that $\theta = 0$ and $u = 1$ implies $\theta = \frac{\pi}{6}$. So we have

$$\int_{-2}^{-1} \sqrt{-4x - x^2} dx = \int_0^1 \sqrt{4 - u^2} du = \int_0^{\pi/6} 4 \cos^2 \theta d\theta.$$

We evaluate the integral with the help of the identity $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$.

$$\int_{-2}^{-1} \sqrt{-4x - x^2} dx = \int_0^{\pi/6} 2(1 + \cos 2\theta) d\theta = (2\theta + \sin 2\theta) \Big|_0^{\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

8.6.81 Factoring the denominator, we have

$$x^5 + 2x^3 + x = x(x^4 + 2x^2 + 1) = x(x^2 + 1)^2.$$

So the partial fraction decomposition of the integrand has the form

$$\frac{x^4 + 2x^3 + 5x^2 + 2x + 1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}.$$

Multiplying by $x(x^2 + 1)^2$, we have

$$x^4 + 2x^3 + 5x^2 + 2x + 1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x.$$

To find the constants, we expand the right hand side of equation and group equal powers of x :

$$x^4 + 2x^3 + 5x^2 + 2x + 1 = (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A.$$

Comparing coefficients of equal powers, we see that $A = 1$, $B = 0$, $C = 2$, $D = 3$, and $E = 0$. Using these values, the integral is evaluated with the help of a u -substitution in the third integral.

$$\begin{aligned}
\int \frac{x^4 + 2x^3 + 5x^2 + 2x + 1}{x(x^2 + 1)^2} dx &= \int \frac{dx}{x} + \int \frac{2}{x^2 + 1} dx + \int \frac{3x}{(x^2 + 1)^2} dx && \text{Partial fractions.} \\
&= \int \frac{dx}{x} + \int \frac{2}{x^2 + 1} dx + \frac{3}{2} \int \frac{du}{u^2} && u = x^2 + 1, \frac{1}{2} du = x dx \\
&= \ln |x| + 2 \tan^{-1} x - \frac{3}{2u} + C && \text{Evaluate.} \\
&= \ln |x| + 2 \tan^{-1} x - \frac{3}{2(x^2 + 1)} + C. && \text{Replace } u \text{ with } x^2 + 1.
\end{aligned}$$

8.6.82 Letting $u = \tan x$ it follows that $du = \sec^2 x dx$, and

$$dx = \frac{du}{\sec^2 x} = \frac{du}{\tan^2 x + 1} = \frac{du}{u^2 + 1}.$$

With this change of variable,

$$\int \frac{dx}{1 + \tan x} = \int \frac{du}{(u + 1)(u^2 + 1)}.$$

Because the denominator on the right side of the equation contains a linear term and an irreducible quadratic factor, the form of the partial fraction decomposition is

$$\frac{1}{(u + 1)(u^2 + 1)} = \frac{A}{u + 1} + \frac{Bu + C}{u^2 + 1}$$

Multiplying by $(u + 1)(u^2 + 1)$ leads to

$$1 = A(u^2 + 1) + (Bu + C)(u + 1) = (A + B)u^2 + (B + C)u + A + C.$$

Comparing coefficients of equal powers, we obtain three equations with three unknowns:

$$A + B = 0, B + C = 0, \text{ and } A + C = 1.$$

Solving this system gives

$$A = \frac{1}{2}, B = -\frac{1}{2}, \text{ and } C = \frac{1}{2}.$$

Substituting these values into the partial fraction decomposition, the integral can be evaluated as follows:

$$\begin{aligned}
\int \frac{dx}{1 + \tan x} &= \int \frac{du}{(u + 1)(u^2 + 1)} && u\text{-substitution} \\
&= \frac{1}{2} \int \frac{du}{u + 1} - \frac{1}{2} \int \frac{u}{u^2 + 1} du + \frac{1}{2} \int \frac{du}{u^2 + 1} && \text{Partial fractions} \\
&= \frac{1}{2} \ln |u + 1| - \frac{1}{4} \ln(u^2 + 1) + \frac{1}{2} \tan^{-1} u + C && \text{Integrate.} \\
&= \frac{1}{2} \ln |\tan x + 1| - \frac{1}{4} \ln(\tan^2 x + 1) + \frac{1}{2} \tan^{-1}(\tan x) + C && \text{Replace } u \text{ with } \tan x. \\
&= \frac{1}{2} \ln |\tan x + 1| - \frac{1}{4} \ln(\sec^2 x) + \frac{x}{2} + C. && 1 + \tan^2 x = \sec^2 x
\end{aligned}$$

8.6.83 We simplify the integrand first using the substitution $u = e^x$, so that $du = e^x dx$ and

$$\int e^x \sin^{998}(e^x) \cos^3(e^x) dx = \int \sin^{998} u \cos^3 u du.$$

We then continue the integration process, by first splitting $\cos^3 u$.

$$\begin{aligned} \int e^x \sin^{998}(e^x) \cos^3(e^x) dx &= \int \sin^{998} u \cos^3 u du && u = e^x, du = e^x dx \\ &= \int \sin^{998} u \cos^2 u \cos u du && \text{Split } \cos^3 u. \\ &= \int \sin^{998} u (1 - \sin^2 u) \cos u du && \cos^2 u = 1 - \sin^2 u \\ &= \int (\sin^{998} u - \sin^{1000} u) \cos u du && \text{Expand.} \\ &= \int (w^{998} - w^{1000}) dw && w = \sin u, dw = \cos u du \\ &= \frac{w^{999}}{999} - \frac{w^{1001}}{1001} + C && \text{Integrate.} \\ &= \frac{\sin^{999} u}{999} - \frac{\sin^{1001} u}{1001} + C && \text{Replace } w \text{ with } \sin u. \\ &= \frac{\sin^{999}(e^x)}{999} - \frac{\sin^{1001}(e^x)}{1001} + C && \text{Replace } u \text{ with } e^x. \end{aligned}$$

8.6.84 Let $u = \tan \theta$ which implies that $du = \sec^2 \theta d\theta$.

$$\begin{aligned} \int \frac{\tan \theta + \tan^3 \theta}{(1 + \tan \theta)^{50}} d\theta &= \int \frac{\tan \theta (1 + \tan^2 \theta)}{(1 + \tan \theta)^{50}} d\theta \\ &= \int \frac{\tan \theta \sec^2 \theta}{(1 + \tan \theta)^{50}} d\theta = \int \frac{u}{(1 + u)^{50}} du. \end{aligned}$$

Now let $w = u + 1$ so that $dw = du$ and $u = w - 1$. We then have

$$\begin{aligned} \int \frac{w-1}{w^{50}} dw &= \int (w^{-49} - w^{-50}) dw \\ &= -\frac{w^{-48}}{48} + \frac{w^{-49}}{49} + C = \frac{(u+1)^{-49}}{49} - \frac{(u+1)^{-48}}{48} + C \\ &= \frac{1}{49(\tan \theta + 1)^{49}} - \frac{1}{48(\tan \theta + 1)^{48}} + C. \end{aligned}$$

8.6.85

- True. For example, the method of partial fractions or a trigonometric substitution can be used to evaluate the integral.
- True. See Example 4.
- False. It is easiest to use the substitution $u = \tan 3x$.
- False. The integral $\int \frac{u^2}{u-1} du$ would imply that $du = dx$, which is not the case here.

8.6.86 We need to compute

$$A = \int_0^2 \left(\frac{2}{x^2 - 2x + 2} - \frac{x}{x^2 - 2x + 2} \right) dx = \int_0^2 \frac{2-x}{(x-1)^2 + 1} dx.$$

Let $u = x - 1$ so that $x = u + 1$ and $du = dx$. Then we have

$$\int_{-1}^1 \frac{1-u}{u^2+1} du = \int_{-1}^1 \frac{1}{u^2+1} du - \int_{-1}^1 \frac{u}{u^2+1} du.$$

The first integral is that of an even function while the second is that of an odd function, so the sum is equal to

$$2 \int_0^1 \frac{1}{u^2+1} du = 2 \tan^{-1} u \Big|_0^1 = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}.$$

8.6.87 The surface area is given by $S = 2\pi \int_0^{\pi/2} \sin x \sqrt{1 + \cos^2 x} dx$. Let $u = \cos x$ so that $du = -\sin x dx$.

Then $S = -2\pi \int_1^0 \sqrt{1+u^2} du = 2\pi \int_0^1 \sqrt{1+u^2} du$. Now let $u = \tan \theta$ with $du = \sec^2 \theta d\theta$. Substituting gives

$$2\pi \int_0^{\pi/4} \sec^3 \theta d\theta = \pi (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \Big|_0^{\pi/4} = \pi (\sqrt{2} + \ln(1 + \sqrt{2})).$$

The last evaluated integral came from the reduction formula for $\int \sec^3 \theta d\theta$ together with the integral formula for $\int \sec \theta d\theta$.

8.6.88 We use the Shell method.

$$V = \int_0^1 2\pi x \cdot \frac{x}{\sqrt{4-x^2}} dx = 2\pi \int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx.$$

Let $x = 2 \sin \theta$ so that $dx = 2 \cos \theta d\theta$ and $\sqrt{4-x^2} = 2 \cos \theta$. We then have

$$\begin{aligned} V &= 4\pi \int_0^{\pi/6} 2 \sin^2 \theta d\theta = 4\pi \int_0^{\pi/6} (1 - \cos 2\theta) d\theta \\ &= 4\pi \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/6} = 4\pi \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) = 2\pi \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) = \frac{\pi}{3} (2\pi - 3\sqrt{3}). \end{aligned}$$

8.6.89 We use the Disk method.

$$\begin{aligned} V &= \int_0^{\pi/4} \pi \frac{dx}{(1 - \sin x)^2} = \pi \int_0^{\pi/4} \frac{(1 + \sin x)^2}{(1 - \sin x)^2 (1 + \sin x)^2} dx = \pi \int_0^{\pi/4} \frac{1 + 2 \sin x + \sin^2 x}{(1 - \sin^2 x)^2} dx \\ &= \pi \int_0^{\pi/4} \frac{1 + 2 \sin x + \sin^2 x}{\cos^4 x} dx \\ &= \pi \int_0^{\pi/4} \sec^4 x dx + \pi \int_0^{\pi/4} 2 \tan x \sec^3 x dx + \pi \int_0^{\pi/4} \tan^2 x \sec^2 x dx. \end{aligned}$$

For the first integral in the sum, write the integrand as $\sec^4 x = \sec^2 x \sec^2 x = (1 + \tan^2 x) \sec^2 x$. Then let $u = \tan x$ so that $du = \sec^2 x dx$. The first integral is equal to

$$\pi \int_0^1 (1 + u^2) du = \pi \left(u + \frac{u^3}{3} \right) \Big|_0^1 = \pi \left(1 + \frac{1}{3} \right) = \frac{4\pi}{3}.$$

For the second integral in the above sum, let $u = \sec x$ so that $du = \sec x \tan x dx$. We have

$$\pi \int_1^{\sqrt{2}} 2u^2 du = \frac{2\pi u^3}{3} \Big|_1^{\sqrt{2}} = \frac{4\sqrt{2}\pi}{3} - \frac{2\pi}{3}.$$

For the third integral in the sum, let $u = \tan x$ so that $du = \sec^2 x dx$. Then that last integral is equal to

$$\pi \int_0^1 u^2 du = \frac{\pi u^3}{3} \Big|_0^1 = \frac{\pi}{3}.$$

Adding these three results together gives

$$V = \pi \left(\frac{4}{3} + \frac{4\sqrt{2}}{3} - \frac{2}{3} + \frac{1}{3} \right) = \pi \left(\frac{4\sqrt{2}}{3} + 1 \right) = \frac{\pi}{3} (4\sqrt{2} + 3).$$

8.6.90 The work is given by

$$W = \int_0^2 1000(9.8)\pi \sqrt{y^2 + 4}(2 - y) dy = 9800\pi \left(2 \int_0^2 \sqrt{y^2 + 4} dy - \int_0^2 y \sqrt{y^2 + 4} dy \right).$$

For the first integral we let $y = 2 \tan \theta$ so that $dy = 2 \sec^2 \theta d\theta$ and $\sqrt{y^2 + 4} = 2 \sec \theta$. For the second integral let $u = y^2 + 4$ so that $du = 2y dy$. Then

$$\begin{aligned} W &= 9800\pi \left(2 \int_0^{\pi/4} 4 \sec^3 \theta d\theta - \frac{1}{2} \int_4^8 u^{1/2} du \right) \\ &= 9800\pi \left(8 \left(\frac{\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|}{2} \right) \Big|_0^{\pi/4} - \frac{1}{3} u^{3/2} \Big|_4^8 \right) \\ &= 9800\pi \left(4(\sqrt{2} + \ln |1 + \sqrt{2}|) - \frac{1}{3} (16\sqrt{2} - 8) \right) \\ &= \frac{9800\pi}{3} (12\sqrt{2} + 12 \ln(1 + \sqrt{2}) - 16\sqrt{2} + 8) = \frac{9800\pi}{3} (12 \ln(1 + \sqrt{2}) + 8 - 4\sqrt{2}) \\ &= \frac{39200\pi}{3} (3 \ln(1 + \sqrt{2}) + 2 - \sqrt{2}). \end{aligned}$$

8.6.91 The work is given by

$$W = \int_0^{\pi/3} 1000(9.8)\pi \sec^2 y \left(\frac{\pi}{3} - y \right) dy = 9800\pi \left(\int_0^{\pi/3} \frac{\pi}{3} \sec^2 y dy - \int_0^{\pi/3} y \sec^2 y dy \right).$$

For the second integral, we integrate by parts. Let $u = y$ and $dv = \sec^2 y dy$. Then $du = dy$ and $v = \tan y$. Then

$$\int \sec^2 y dy = y \tan y - \int \tan y dy = y \tan y + \ln |\cos y| + C.$$

Then

$$W = 9800\pi \left(\frac{\pi}{3} \tan y \Big|_0^{\pi/3} - (y \tan y + \ln |\cos y|) \Big|_0^{\pi/3} \right) = 9800\pi \left(\frac{\pi}{3} \sqrt{3} - \left(\frac{\pi}{3} \sqrt{3} + \ln \frac{1}{2} \right) \right) = 9800\pi \ln 2.$$

8.6.92 We evaluate the indefinite integral first. Starting with a substitution of $w = y^3$, it follows that $dw = 3y^2 dy$ and $w^2 = y^6$. This substitution gives us

$$\int y^8 e^{y^3} dy = \frac{1}{3} \int w^2 e^w dw.$$

Apply integration by parts to the integral on the right with the following variable choices.

$$\begin{aligned}u &= w^2 & dv &= e^w dw \\ du &= 2w dw & v &= e^w\end{aligned}$$

Then we have

$$\int y^8 e^{y^3} dy = \frac{1}{3} \int w^2 e^w dw = \frac{1}{3} \left(w^2 e^w - 2 \int w e^w dw \right).$$

We apply integration by parts to $\int w e^w dw$ with the following variable choices.

$$\begin{aligned}u &= w & dv &= e^w dw \\ du &= dw & v &= e^w\end{aligned}$$

After evaluating the last integral, we replace w with y^3 and factor.

$$\begin{aligned}\int y^8 e^{y^3} dy &= \frac{1}{3} \left(w^2 e^w - 2 \int w e^w dw \right) \\ &= \frac{1}{3} \left(w^2 e^w - 2 \left(w e^w - \int e^w dw \right) \right) \\ &= \frac{1}{3} (w^2 e^w - 2w e^w + 2e^w) + C \\ &= \frac{e^{y^3}}{3} (y^6 - 2y^3 + 2) + C.\end{aligned}$$

Evaluating the definite integral, we have

$$\int_1^{\sqrt[3]{2}} y^8 e^{y^3} dy = \frac{e^{y^3}}{3} (y^6 - 2y^3 + 2) \Big|_1^{\sqrt[3]{2}} = \frac{2e^2}{3} - \frac{e}{3} = \frac{e(2e-1)}{3}.$$

8.6.93 Start with a substitution of $u = e^x$. Then $du = e^x dx$, which implies that $dx = \frac{du}{e^x} = \frac{du}{u}$. Making this substitution, we have

$$\int \frac{68}{e^{2x} + 2e^x + 17} dx = \int \frac{68}{u(u^2 + 2u + 17)} du.$$

The denominator has one simple linear factor u and one irreducible quadratic factor $u^2 + 2u + 17$. Therefore the partial fraction decomposition has the form

$$\frac{68}{u(u^2 + 2u + 17)} = \frac{A}{u} + \frac{Bu + C}{u^2 + 2u + 17}.$$

Multiplying both sides by $u(u^2 + 2u + 17)$, we have

$$68 = A(u^2 + 2u + 17) + (Bu + C)u = (A + B)u^2 + (2A + C)u + 17A.$$

By equating coefficients of equal powers of x , we find that $A = 4$, $B = -4$, and $C = -8$. Using these values, we have

$$\int \frac{68}{u(u^2 + 2u + 17)} du = \int \left(\frac{4}{u} + \frac{-4u - 8}{u^2 + 2u + 17} \right) du = \int \frac{4}{u} du - \int \frac{4u + 8}{u^2 + 2u + 17} du.$$

The first integral on the right equals $4 \ln |u| + C$. For the second integral on the right, we use a substitution of $w = u^2 + 2u + 17$, which implies that $dw = (2u + 2)du$. We write the numerator in the second integral as $4u + 8 = 2(2u + 2) + 4$ and split the integral:

$$\begin{aligned}\int \frac{68}{u(u^2 + 2u + 17)} du &= \int \frac{4}{u} du - 2 \int \frac{2u + 2}{\underbrace{u^2 + 2u + 17}_{w=u^2+2u+17}} du - 4 \int \frac{du}{\underbrace{u^2 + 2u + 17}_{(u+1)^2+16}} \\ &= 4 \ln |u| - 2 \ln |u^2 + 2u + 17| - \tan^{-1} \left(\frac{u+1}{4} \right) + C.\end{aligned}$$

Replacing u with e^x and then simplifying, we have

$$\begin{aligned}\int \frac{68}{e^{2x} + 2e^x + 17} dx &= 4 \ln e^x - 2 \ln |e^{2x} + 2e^x + 17| - \tan^{-1} \left(\frac{e^x + 1}{4} \right) + C \\ &= 4x - 2 \ln(e^{2x} + 2e^x + 17) - \tan^{-1} \left(\frac{e^x + 1}{4} \right) + C.\end{aligned}$$

8.6.94 Factoring the denominator, we find that

$$t^3 + 1 = (t + 1)(t^2 - t + 1).$$

The partial fraction decomposition is

$$\frac{1}{(t + 1)(t^2 - t + 1)} = \frac{A}{t + 1} + \frac{Bt + C}{t^2 - t + 1},$$

which leads to the equation

$$1 = A(t^2 - t + 1) + (Bt + C)(t + 1) = (A + B)t^2 + (-A + B + C)t + (A + C).$$

Comparing coefficients of equal powers, we have

$$A + B = 0, -A + B + C = 0, A + C = 1.$$

The solution to this system is $A = 1/3$, $B = -1/3$, and $C = 2/3$. Using these constants, the partial fraction decomposition is

$$\frac{1}{(t + 1)(t^2 - t + 1)} = \frac{1}{3(t + 1)} + \frac{-t + 2}{3(t^2 - t + 1)}.$$

By splitting up the second fraction on the right, we can use integration by substitution and trigonometric substitution.

$$\int \frac{1}{(t + 1)(t^2 - t + 1)} dt = \int \frac{1}{3(t + 1)} dt + \int \frac{-t + \frac{1}{2}}{3(t^2 - t + 1)} dt + \int \frac{\frac{3}{2}}{3(t^2 - t + 1)} dt.$$

Let $u = t^2 - t + 1$ in the second integral so that $du = (2t - 1)dt$ or $-\frac{1}{2}du = (-t + \frac{1}{2})dt$:

$$\int \frac{-t + \frac{1}{2}}{3(t^2 - t + 1)} dt = -\frac{1}{6} \int \frac{du}{u} = -\frac{1}{6} \ln |u| + C = -\frac{1}{6} \ln |t^2 - t + 1| + C$$

For the third integral, we complete a square first:

$$\int \frac{\frac{3}{2}}{3(t^2 - t + 1)} dt = \frac{1}{2} \int \frac{dt}{t^2 - t + 1} = \frac{1}{2} \int \frac{dt}{(t - \frac{1}{2})^2 + \frac{3}{4}} = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2t - 1}{\sqrt{3}} \right) + C.$$

Assembling all the pieces, we have

$$\begin{aligned}\int \frac{dt}{t^3 + 1} &= \int \frac{1}{3(t + 1)} dt + \int \frac{-t + \frac{1}{2}}{3(t^2 - t + 1)} dt + \int \frac{\frac{3}{2}}{3(t^2 - t + 1)} dt \\ &= \frac{1}{3} \ln |t + 1| - \frac{1}{6} \ln |t^2 - t + 1| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2t - 1}{\sqrt{3}} \right) + C.\end{aligned}$$

8.6.95 Let $u = \tan x$. Then $du = \sec^2 x \, dx$, and

$$dx = \frac{du}{\sec^2 x} = \frac{du}{\tan^2 x + 1} = \frac{du}{u^2 + 1}.$$

With this change of variable, the integral becomes

$$\int \frac{dx}{1 - \tan^2 x} = \int \frac{du}{(1 - u^2)(u^2 + 1)} = \int \frac{du}{(1 + u)(1 - u)(u^2 + 1)}.$$

The form of the partial fraction decomposition of the integrand on the right is

$$\frac{1}{(1 + u)(1 - u)(u^2 + 1)} = \frac{A}{1 + u} + \frac{B}{1 - u} + \frac{Cu + D}{u^2 + 1}.$$

Multiplying both sides by $(1 + u)(1 - u)(u^2 + 1)$ and then regrouping terms, we have

$$\begin{aligned} 1 &= A(1 - u)(u^2 + 1) + B(1 + u)(u^2 + 1) + (Cu + D)(1 - u^2) \\ &= (-A + B - C)u^3 + (A + B - D)u^2 + (-A + B + C)u + A + B + D. \end{aligned}$$

By comparing coefficients of equal powers, we obtain the system of equations

$$-A + B - C = 0, A + B - D = 0, -A + B + C = 0, A + B + D = 1.$$

You can verify that the values satisfying this system are $A = 1/4$, $B = 1/4$, $C = 0$, $D = 1/2$.

The partial fraction decomposition makes the integration process straightforward:

$$\begin{aligned} \int \frac{dx}{1 - \tan^2 x} &= \int \frac{du}{(1 - u^2)(u^2 + 1)} && u\text{-substitution} \\ &= \frac{1}{4} \int \frac{du}{1 + u} + \frac{1}{4} \int \frac{du}{1 - u} + \frac{1}{2} \int \frac{du}{u^2 + 1} && \text{Partial fractions} \\ &= \frac{1}{4} \ln |u + 1| - \frac{1}{4} \ln |u - 1| + \frac{1}{2} \tan^{-1} u + C && \text{Integrate.} \\ &= \frac{1}{4} \ln |\tan x + 1| - \frac{1}{4} \ln |\tan x - 1| + \frac{1}{2} \tan^{-1}(\tan x) + C && \text{Replace } u \text{ with } \tan x. \\ &= \frac{1}{4} \ln |\tan x + 1| - \frac{1}{4} \ln |\tan x - 1| + \frac{x}{2} + C && \text{Simplify.} \end{aligned}$$

8.6.96 Let $w = x^{1/3}$. Then $x = w^3$ and $dx = 3w^2 dw$. Changing variables, we have

$$\int e^{(x^{1/3})} dx = \int 3w^2 e^w dw.$$

Using integration by parts, with the following choices for the variables.

$$u = 3w^2 \quad dv = e^w dw$$

$$du = 6w \, dw \quad v = e^w$$

Then we have

$$\int e^{(x^{1/3})} dx = \int 3w^2 e^w dw = 3w^2 e^w - \int 6we^w dw.$$

We apply integration by parts again to the integral on the right.

$$u = 6w \quad dv = e^w dw$$

$$du = 6 dw \quad v = e^w$$

Using these substitutions, we have

$$\int e^{(x^{1/3})} dx = 3w^2 e^w - (6we^w - \int 6e^w dw) = 3w^2 e^w - 6we^w + 6e^w + C.$$

Factoring and then replacing w with $x^{1/3}$, we obtain

$$\int e^{(x^{1/3})} dx = 3w^2 e^w - 6we^w + 6e^w = 3e^w(w^2 - 2w + 2) = 3e^{(x^{1/3})}(x^{2/3} - 2x^{1/3} + 2) + C.$$

8.6.97 We integrate by parts with the following variable assignments.

$$u = \tan^{-1} \sqrt[3]{x} \quad dv = dx$$

$$du = \frac{dx}{3x^{2/3}(1+x^{2/3})} \quad v = x$$

Integrating by parts, we have

$$\int \tan^{-1} \sqrt[3]{x} dx = x \tan^{-1} \sqrt[3]{x} - \int \frac{x^{1/3}}{3(1+x^{2/3})} dx.$$

For the integral on the right, we make a substitution of $v = x^{1/3}$ which implies that $dv = \frac{dx}{3x^{2/3}}$, and $dx = 3v^2 dv$. With this change of variable, we have

$$\int \frac{x^{1/3}}{3(1+x^{2/3})} dx = \int \frac{v}{3(1+v^2)} 3v^2 dv = \int \frac{v^3}{1+v^2} dv.$$

It follows by long division that $\frac{v^3}{1+v^2} = v - \frac{v}{1+v^2}$. Letting $w = 1+v^2$, it follows that $dw = 2v dv$ and

$$\begin{aligned} \int \frac{v^3}{1+v^2} dv &= \int v dv - \int \frac{v}{1+v^2} dv \\ &= \int v dv - \frac{1}{2} \int \frac{dw}{w} \\ &= \frac{v^2}{2} - \frac{1}{2} \ln |w| + C \\ &= \frac{v^2}{2} - \frac{1}{2} \ln(v^2 + 1) + C. \end{aligned}$$

Summarizing and then completing our work, we have

$$\begin{aligned}
 \int \tan^{-1} \sqrt[3]{x} dx &= x \tan^{-1} \sqrt[3]{x} - \int \frac{x^{1/3}}{3(1+x^{2/3})} dx && \text{Integration by parts} \\
 &= x \tan^{-1} \sqrt[3]{x} - \int \frac{v^3}{1+v^2} dv && v = x^{1/3}, dv = \frac{dx}{3x^{2/3}} \\
 &= x \tan^{-1} \sqrt[3]{x} - \left(\int v dv - \int \frac{v}{1+v^2} dv \right) && \frac{v^3}{1+v^2} = v - \frac{v}{1+v^2} \\
 &= x \tan^{-1} \sqrt[3]{x} - \left(\frac{v^2}{2} - \frac{1}{2} \ln(1+v^2) \right) + C && \text{Integrate.} \\
 &= x \tan^{-1} \sqrt[3]{x} - \frac{x^{2/3}}{2} + \frac{1}{2} \ln(1+x^{2/3}) + C. && \text{Replace } v \text{ with } x^{1/3}; \text{ simplify.}
 \end{aligned}$$

8.6.98 To evaluate this integral, we use the trick of multiplying by 1 by introducing a factor of $\frac{\sqrt{\sin x}}{\sqrt{\sin x}}$ in the integral and then make a substitution of $w = \sqrt{\sin x}$ with $dw = \frac{\cos x}{2\sqrt{\sin x}} dx$.

$$\int e^{\sqrt{\sin x}} \cos x dx = \int \frac{\sqrt{\sin x}}{\sqrt{\sin x}} e^{\sqrt{\sin x}} \cos x dx = \int 2we^w dw$$

Using integration by parts with the following substitutions,

$$u = 2w \quad dv = e^w dw$$

$$du = 2dw \quad v = e^w$$

we have

$$\int e^{\sqrt{\sin x}} \cos x dx = \int 2we^w dw = 2we^w - \int 2e^w dw = 2we^w - 2e^w + C.$$

Finally, we factor and replace w with $\sqrt{\sin x}$:

$$\int e^{\sqrt{\sin x}} \cos x dx = 2e^w(w-1) + C = 2e^{\sqrt{\sin x}}(\sqrt{\sin x} - 1) + C.$$

8.6.99 The surface area is given by

$$S = \int_0^{\pi/4} 2\pi \tan x \sqrt{1 + \sec^4 x} dx = 2\pi \int_0^{\pi/4} \frac{\sqrt{1 + \sec^4 x} \tan x \sec x}{\sec x} dx.$$

Let $u = \sec x$ so that $du = \sec x \tan x dx$. Substituting gives $2\pi \int_1^{\sqrt{2}} \frac{\sqrt{1+u^4}}{u} du$. Now let $u^2 = \tan \theta$ so that $2u du = \sec^2 \theta d\theta$. Substituting gives

$$2\pi \int_{\pi/4}^{\tan^{-1} 2} \frac{\sec \theta}{\sqrt{\tan \theta}} \cdot \frac{\sec^2 \theta}{2\sqrt{\tan \theta}} d\theta = \pi \int_{\pi/4}^{\tan^{-1} 2} \frac{\sec^3 \theta}{\tan \theta} \cdot \frac{\tan \theta}{\tan \theta} d\theta = \pi \int_{\pi/4}^{\tan^{-1} 2} \frac{\sec^2 \theta \sec \theta \tan \theta}{\sec^2 \theta - 1} d\theta.$$

Now let $w = \sec \theta$ so that $dw = \sec \theta \tan \theta d\theta$. Then

$$S = \pi \int_{\sqrt{2}}^{\sqrt{5}} \frac{w^2}{w^2 - 1} dw.$$

Using long division and then a partial fraction decomposition, we can write this integrand as

$$\frac{w^2}{w^2 - 1} = 1 - \frac{1/2}{w+1} + \frac{1/2}{w-1}.$$

Therefore

$$\begin{aligned} S &= \pi \int_{\sqrt{2}}^{\sqrt{5}} \left(1 - \frac{1}{2(w+1)} + \frac{1}{2(w-1)} \right) dw = \pi \left(w - \frac{1}{2} \ln(w+1) + \frac{1}{2} \ln(w-1) \right) \Big|_{\sqrt{2}}^{\sqrt{5}} \\ &= \pi \left(\sqrt{5} - \sqrt{2} + \frac{1}{2} \ln \frac{\sqrt{5}-1}{\sqrt{5}+1} - \frac{1}{2} \ln \frac{\sqrt{2}-1}{\sqrt{2}+1} \right) \approx 3.83908. \end{aligned}$$

8.7 Other Methods of Integration

8.7.1 The power rule, substitution, integration by parts, and partial fraction decomposition are examples of analytical methods.

8.7.2 Many computer algebra systems can give exact answers, although sometimes they don't express the answers in elementary terms, and many don't add the arbitrary constant.

8.7.3 The computer algebra system may use a different algorithm than whoever prepared the table, so the results may look different – however, they should differ by a constant.

8.7.4 Using a reduction formula is an analytical method, however, when reduced, the final integral may need to be attacked by a numerical method.

8.7.5 Let $u = e^x$. Then $du = e^x dx$. Substituting yields

$$\int \cos^3 u \, du = -\frac{1}{3} \sin^3 u + \sin u + C = -\frac{1}{3} \sin^3(e^x) + \sin(e^x) + C.$$

8.7.6 Let $u = \sin x$. Then $du = \cos x \, dx$. Substituting gives

$$\int \sqrt{100 - u^2} \, du = \frac{u}{2} \sqrt{100 - u^2} + 50 \sin^{-1} \left(\frac{u}{10} \right) + C = \frac{\sin x}{2} \sqrt{100 - \sin^2 x} + 50 \sin^{-1} \left(\frac{\sin x}{10} \right) + C.$$

8.7.7 Using table entry 17, we have $\int \cos^{-1}(x) \, dx = x \cos^{-1} x - \sqrt{1 - x^2} + C$.

8.7.8 Using table entry 48, we have $\int \sin(3x) \cos(2x) \, dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$.

8.7.9 Using table entry 77 for $\int \frac{1}{\sqrt{x^2 + a^2}} \, dx$, we see that $\int \frac{1}{\sqrt{x^2 + 16}} \, dx = \ln(x + \sqrt{x^2 + 16}) + C$.

8.7.10 Using table entry 69 for $\int \frac{1}{\sqrt{x^2 - a^2}} \, dx$, we see that $\int \frac{1}{\sqrt{x^2 - 25}} \, dx = \ln|x + \sqrt{x^2 - 25}| + C$.

8.7.11 Using table entry 90 for $\int \frac{x}{ax + b} \, dx$, we see that $\int \frac{3u}{2u + 7} \, du = \frac{3u}{2} - \frac{21}{4} \ln|2u + 7| + C$.

8.7.12 Using table entry 96 for $\int \frac{1}{x(ax + b)} \, dx$, we have $\int \frac{1}{y(2y + 9)} \, dy = \frac{1}{9} \ln \left| \frac{y}{2y + 9} \right| + C$.

8.7.13 Using table entry 47, we have $\int \frac{dx}{1 - \cos 4x} = -\frac{1}{4} \cot(2x) + C$.

8.7.14 Using table entry 62 for $\int \frac{1}{x\sqrt{a^2 - x^2}} \, dx$ we have that $\int \frac{1}{x\sqrt{81 - x^2}} \, dx = \frac{-1}{9} \ln \left| \frac{9 + \sqrt{81 - x^2}}{x} \right| + C$.

8.7.15 Using table entry 94 for $\int \frac{x}{\sqrt{ax+b}} dx$, we have that $\int \frac{x dx}{\sqrt{4x+1}} = \frac{1}{24}(4x-2)\sqrt{4x+1} + C = \frac{1}{12}(2x-1)\sqrt{4x+1} + C$.

8.7.16 Using table entry 93 for $\int x\sqrt{ax+b} dx$ we have that $\int t\sqrt{4t+12} dt = \frac{2}{240}(12t-24)(4t+12)^{3/2} + C = \frac{1}{10}(t-2)(4t+12)^{3/2} + C = \frac{4}{5}(t-2)(t+3)^{3/2} + C$.

8.7.17 Using table entry 69 for $\int \frac{1}{\sqrt{x^2-a^2}} dx$ we have $\int \frac{1}{\sqrt{9x^2-100}} du = \frac{1}{3} \int \frac{1}{\sqrt{u^2-100}} du$ where $u = 3x$. This is then equal to $\frac{1}{3} \ln|u + \sqrt{u^2-100}| + C = \frac{1}{3} \ln|3x + \sqrt{9x^2-100}| + C$. This can be written as $\frac{1}{3} \ln|x + \sqrt{x^2-(100/9)}| + C$.

8.7.18 $\int \frac{1}{225-16x^2} dx = \frac{1}{16} \int \frac{1}{(225/16)-x^2} dx$. Using table entry 67 for $\int \frac{1}{a^2-x^2} dx$ we have

$$\begin{aligned} \frac{1}{16} \int \frac{1}{(15/4)^2 - x^2} dx &= \frac{1}{16} \cdot \frac{2}{15} \ln \left| \frac{x + (15/4)}{x - (15/4)} \right| + C \\ &= \frac{1}{120} \ln \left| \frac{4x + 15}{4x - 15} \right| + C. \end{aligned}$$

8.7.19 $\int \frac{e^x}{\sqrt{e^{2x}+4}} dx = \int \frac{1}{\sqrt{u^2+4}} du$ where $u = e^x$. Then using table entry 77, we have $\ln(u + \sqrt{u^2+4}) + C = \ln(e^x + \sqrt{e^{2x}+4}) + C$.

8.7.20 $\int \frac{\sqrt{\ln^2 x + 4}}{x} dx = \int \sqrt{u^2+4} du$ where $u = \ln x$. Then using table entry 76 we have $\frac{u}{2}\sqrt{u^2+4} + \frac{4}{2} \ln(u + \sqrt{u^2+4}) + C = \frac{\ln x}{2} \sqrt{\ln^2 x + 4} + 2 \ln(\ln x + \sqrt{\ln^2 x + 4}) + C$.

8.7.21 $\int \frac{\cos x}{\sin^2 x + 2 \sin x} dx = \int \frac{1}{u^2 + 2u} du = \int \frac{1}{(u+1)^2 - 1} du$ where $u = \sin x$. Then using table entry 74, we have

$$-\frac{1}{2} \ln \left| \frac{u+2}{u} \right| + C = -\frac{1}{2} \ln \left| \frac{\sin x + 2}{\sin x} \right| + C.$$

8.7.22 Let $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}} dx$. Substituting yields $2 \int \cos^{-1} u du$. Then using table entry 17, we have $2 \int \cos^{-1}(u) du = 2(u \cos^{-1}(u) - \sqrt{1-u^2}) + C = 2\sqrt{x} \cos^{-1} \sqrt{x} - 2\sqrt{1-x} + C$.

8.7.23 Let $u = \ln x$, so that $du = \frac{1}{x} dx$. Substituting yields $\int u \sin^{-1}(u) du = \frac{2u^2-1}{4} \sin^{-1}(u) + \frac{u\sqrt{1-u^2}}{4} + C = \frac{2\ln^2 x - 1}{4} \sin^{-1}(\ln x) + \frac{\ln x \sqrt{1-\ln^2 x}}{4} + C$, where we used table entry 105.

8.7.24 Let $u = 1 + 4e^t$, so that $du = 4e^t dt$. Substituting yields $\int \frac{4e^t}{4e^t \sqrt{1+4e^t}} dt = \int \frac{1}{(u-1)\sqrt{u}} du$. Now let $v = \sqrt{u}$, so that $dv = \frac{1}{2\sqrt{u}} du$. Then we have $\int \frac{2}{v^2-1} dv = \ln \left| \frac{v-1}{v+1} \right| + C = \ln \left| \frac{v^2-1}{(v+1)^2} \right| + C = \ln \left| \frac{4e^t}{(\sqrt{1+4e^t}+1)^2} \right| + C$, where we used table entry 74.

8.7.25 Using table entry 84 for $\int \frac{1}{(a^2 + x^2)^{3/2}} dx$, we have $\int \frac{1}{(16 + 9x^2)^{3/2}} dx = \frac{1}{3} \int \frac{1}{(16 + u^2)^{3/2}} du = \frac{1}{3} \cdot \frac{3x}{16\sqrt{16 + 9x^2}} + C = \frac{x}{16\sqrt{16 + 9x^2}} + C$.

8.7.26 Using table entry 68 for $\int \sqrt{x^2 - a^2} dx$, we have $\int \sqrt{4x^2 - 9} dx = \frac{1}{2} \int \sqrt{u^2 - 9} du = \frac{x}{2} \sqrt{4x^2 - 9} - \frac{9}{4} \ln|2x + \sqrt{4x^2 - 9}| + C$.

8.7.27 Using table entry 62 for $\int \frac{1}{x\sqrt{a^2 - x^2}} dx$, we have $\int \frac{1}{x\sqrt{(12)^2 - x^2}} dx = \frac{-1}{12} \ln \left| \frac{12 + \sqrt{144 - x^2}}{x} \right| + C$.

8.7.28 Using table entry 85 for $\int \frac{dx}{x(x^2 + a^2)}$, we have $\int \frac{dv}{v(v^2 + 8)} = \frac{1}{16} \ln \left(\frac{v^2}{v^2 + 8} \right) + C$.

8.7.29 Using table entry 103 for $\int \ln^n x dx$ we have that $\int \ln^2 x dx = x \ln^2 x - 2 \int \ln x dx = x \ln^2 x - 2(x \ln x - x) + C = 2x + x \ln^2 x - 2x \ln x + C$.

8.7.30 Using table entry 102 (twice) for $\int x^n e^{ax} dx$ we have that $\int x^2 e^{5x} dx = \frac{1}{5} x^2 e^{5x} - \frac{2}{5} \int x e^{5x} dx = \frac{1}{5} x^2 e^{5x} - \frac{2}{25} \left(x e^{5x} - \frac{1}{5} e^{5x} \right) + C$.

8.7.31 Note that $\sqrt{x^2 + 10x} = \sqrt{x^2 + 10x + 25 - 25} = \sqrt{(x + 5)^2 - 5^2}$. Thus, the given integral is equal to $\int \sqrt{u^2 - 5^2} du$ where $u = x + 5$. Using table entry 68 we have that this is equal to $\frac{u}{2} \sqrt{u^2 - 25} - \frac{25}{2} \ln|u + \sqrt{u^2 - 25}| + C = \frac{x + 5}{2} \sqrt{(x + 5)^2 - 25} - \frac{25}{2} \ln|x + 5 + \sqrt{(x + 5)^2 - 25}| + C$.

8.7.32 Note that $\sqrt{x^2 - 8x} = \sqrt{x^2 - 8x + 16 - 16} = \sqrt{(x - 4)^2 - 4^2}$. Thus, the given integral is equal to $\int \sqrt{u^2 - 4^2} du$ where $u = x - 4$. Using table entry 68 we have that this is equal to $\frac{u}{2} \sqrt{u^2 - 16} - 8 \ln|u + \sqrt{u^2 - 16}| + C = \frac{x - 4}{2} \sqrt{(x - 4)^2 - 16} - 8 \ln|x - 4 + \sqrt{(x - 4)^2 - 16}| + C$.

8.7.33 $\int \frac{1}{x^2 + 2x + 10} dx = \int \frac{1}{(x + 1)^2 + 9} dx$. Using table entry 14 this is equal to $\frac{1}{3} \tan^{-1} \left(\frac{x + 1}{3} \right) + C$.

8.7.34 $\int \sqrt{x^2 - 4x + 8} dx = \int \sqrt{(x - 2)^2 + 4} dx$. Using table entry 76, this is equal to

$$\frac{x - 2}{2} \sqrt{(x - 2)^2 + 4} + 2 \ln(x - 2 + \sqrt{(x - 2)^2 + 4}) + C.$$

8.7.35 $\int \frac{1}{x(x^{10} + 1)} dx = \int \frac{10x^9}{10x^{10}(x^{10} + 1)} dx$. Let $u = x^{10} + 1$ so that $du = 10x^9 dx$. Substituting yields $\int \frac{1}{10} \cdot \frac{1}{(u - 1)u} du$. Using table entry 96 for $\int \frac{dx}{x(ax + b)}$ we have

$$\frac{1}{10} \ln \left| \frac{u - 1}{u} \right| + C = \frac{1}{10} \ln \left| \frac{x^{10}}{x^{10} + 1} \right| + C.$$

8.7.36 $\int \frac{1}{t(t^8 - 256)} dt = \int \frac{8t^7}{8t^8(t^8 - 256)} dt$. Let $u = t^8 - 256$ so that $du = 8t^7 dt$. Substituting yields $\frac{1}{8} \int \frac{1}{u(u + 256)} du$. Using table entry 96 for $\int \frac{dt}{t(at + b)}$ we have

$$\frac{1}{2048} \ln \left| \frac{u}{u + 256} \right| + C = \frac{1}{2048} \ln \left| \frac{t^8 - 256}{t^8} \right| + C.$$

8.7.37 $\int \frac{1}{\sqrt{x^2 - 6x}} dx = \int \frac{1}{\sqrt{(x-3)^2 - 9}} dx$. Using table entry 69, this is equal to $\ln|x - 3 + \sqrt{(x-3)^2 - 9}| + C$. This can also be written as $2\ln(\sqrt{x-6} + \sqrt{x}) + C$.

8.7.38 $\int \frac{1}{\sqrt{x^2 + 10x}} dx = \int \frac{1}{\sqrt{(x+5)^2 - 25}} dx$. Using table entry 69, this is equal to $\ln|x + 5 + \sqrt{(x+5)^2 - 25}| + C$.

8.7.39 Let $u = x^3$, so that $du = 3x^2 dx$. Substituting yields

$$\frac{1}{3} \int \frac{\tan^{-1}(u)}{u^2} du = -\frac{1}{3} \left(\frac{1}{u} \tan^{-1}(u) - \int \frac{1}{u(1+u^2)} du \right) = -\frac{1}{3} \left(\frac{\tan^{-1}(x^3)}{x^3} \right) + \frac{1}{3} \int \frac{1}{u(1+u^2)} du,$$

where we used table entry 18. Now let $w = u^2 + 1$ so that $dw = 2u du$. This last integral is thus equal to

$$\frac{1}{3} \int \frac{2u du}{2u^2(1+u^2)} = \frac{1}{6} \int \frac{dw}{(w-1)w} = -\frac{1}{6} \ln \left| \frac{w}{w-1} \right| + C = -\frac{1}{6} \ln \left| \frac{u^2+1}{u^2} \right| + C = -\frac{1}{6} \ln \left| \frac{x^6+1}{x^6} \right| + C,$$

where we used table entry 96. Thus the original integral is equal to $-\frac{1}{3} \left(\frac{\tan^{-1}(x^3)}{x^3} \right) - \frac{1}{6} \ln \left| \frac{x^6+1}{x^6} \right| + C$.

This can be written as $-\frac{\tan^{-1}(x^3)}{3x^3} + \ln \left| \frac{x}{(x^6+1)^{1/6}} \right| + C$.

8.7.40 Let $u = e^t$ so that $du = e^t dt$. Substituting yields $\int \frac{u^2}{\sqrt{4+u^2}} du$. Using table entry 82, this yields $-2\ln(u + \sqrt{4+u^2}) + \frac{u\sqrt{4+u^2}}{2} + C$, so the value of the integral is $-2\ln(e^t + \sqrt{4+e^{2t}}) + \frac{e^t\sqrt{4+e^{2t}}}{2} + C$.

8.7.41 The integral which gives the length of the curve is $\frac{1}{2} \int_0^8 \sqrt{4+x^2} dx$. Using table entry 76, we have

$$\frac{1}{2} \left(\frac{x}{2} \sqrt{4+x^2} + \frac{4}{2} \ln(x + \sqrt{4+x^2}) \right) \Big|_0^8 = 4\sqrt{17} + \ln(8 + 2\sqrt{17}) - \ln 2 = 4\sqrt{17} + \ln(4 + \sqrt{17}) \approx 18.59.$$

8.7.42 The integral which gives the length of the curve is

$$\frac{1}{2} \int_0^2 \sqrt{4+9x} dx = \frac{1}{18} \left(\frac{2}{3} \cdot (4+9x)^{3/2} \right) \Big|_0^2 = \frac{1}{27} (\sqrt{22^3} - 8) \approx 3.53.$$

We used table entry 86.

8.7.43 The integral which gives the length of the curve is $\int_0^{\ln 2} \sqrt{1+e^{2x}} dx = \int_1^2 \frac{\sqrt{1+u^2}}{u} du$ where $u = e^x$. This is equal to

$$\left(\sqrt{1+u^2} - \ln \left| \frac{1+\sqrt{1+u^2}}{u} \right| \right) \Big|_1^2 = \sqrt{5} - \sqrt{2} + \ln(1+\sqrt{2}) - \ln \left(\frac{1+\sqrt{5}}{2} \right) \approx 1.22.$$

Note that we used table entry 83.

8.7.44 The volume is given by $\pi \int_1^e x^4 \ln x \, dx$. We use table entry 100. This gives

$$\pi \left(\left(\frac{x^5 \ln x}{5} \right) - \frac{x^5}{25} \right) \Big|_1^e = \frac{\pi(4e^5 + 1)}{25}.$$

8.7.45 Using the method of shells, we have $\frac{V}{2\pi} = \int_0^{12} \frac{x}{\sqrt{x+4}} \, dx = \frac{2}{3}(x-8)\sqrt{x+4} \Big|_0^{12} = \frac{32}{3} + \frac{32}{3} = \frac{64}{3}$. Thus $V = \frac{128\pi}{3}$. We used table entry 94 to compute the integral.

8.7.46 The area is given by

$$\int_0^3 \frac{1}{\sqrt{x^2 - 2x + 2}} \, dx = \int_0^3 \frac{1}{\sqrt{(x-1)^2 + 1}} \, dx = \int_{-1}^2 \frac{1}{\sqrt{u^2 + 1}} \, du = \ln(u + \sqrt{u^2 + 1}) \Big|_{-1}^2 = \ln \frac{2 + \sqrt{5}}{\sqrt{2} - 1}.$$

We used table entry 77 to compute the integral.

8.7.47 Using the method of disks, we have $\frac{V}{\pi} = \int_0^{\pi/2} \sin^2 y \, dy = \frac{y}{2} - \frac{\sin 2y}{4} \Big|_0^{\pi/2} = \frac{\pi}{4}$. Thus $V = \frac{\pi^2}{4}$. We used table entry 31 to compute the antiderivative.

8.7.48 The average value of f is

$$\frac{1}{2} \int_{-1}^1 \frac{2}{x^2 + 1} \, dx = \tan^{-1}(x) \Big|_{-1}^1 = \frac{\pi}{2} \approx 1.57.$$

The average value of g is

$$\frac{1}{2} \int_{-1}^1 \frac{7}{4\sqrt{x^2 + 1}} \, dx = \frac{7}{8} \int_{-1}^1 \frac{1}{\sqrt{x^2 + 1}} \, dx = \frac{7}{8} \left(\ln(x + \sqrt{x^2 + 1}) \Big|_{-1}^1 \right) = \frac{7}{8} \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \approx 1.542.$$

Thus, the average value of f is greater than the average value of g . We used table entries 14 and 77.

$$\mathbf{8.7.49} \quad \int \frac{x}{\sqrt{2x+3}} \, dx = \frac{1}{3}(x-3)\sqrt{3+2x} + C.$$

$$\mathbf{8.7.50} \quad \int \sqrt{4x^2 + 36} \, dx = 2 \left(\frac{1}{2}x\sqrt{x^2 + 9} + \frac{9}{2} \sinh^{-1} \left(\frac{x}{3} \right) \right) + C = \left(x\sqrt{x^2 + 9} + 9 \sinh^{-1} \left(\frac{x}{3} \right) \right) + C$$

$$\mathbf{8.7.51} \quad \int \tan^2 3x \, dx = \frac{1}{3} \tan 3x - x + C.$$

$$\mathbf{8.7.52} \quad \int_0^{\pi/2} \cos^6 x \, dx = \frac{5\pi}{32} \approx 0.491.$$

$$\mathbf{8.7.53} \quad \int_0^4 (9 + x^2)^{3/2} \, dx = \frac{1540 + 243 \ln 3}{8} \approx 225.870.$$

$$\mathbf{8.7.54} \quad \int (a^2 - t^2)^{-2} \, dt = \frac{1}{4} \frac{\ln(a+t)}{a^3} - \frac{1}{4} \frac{\ln(t-a)}{a^3} - \frac{1}{4a^2(t-a)} - \frac{1}{4a^2(t+a)} + C.$$

$$\mathbf{8.7.55} \quad \int \frac{(x^2 - a^2)^{3/2}}{x} \, dx = \frac{1}{3}(x^2 - a^2)^{3/2} - a^2 \sqrt{x^2 - a^2} + a^3 \cos^{-1}(a/x) + C.$$

$$\mathbf{8.7.56} \quad \int \frac{1}{x(a^2 - x^2)^2} \, dx = \frac{\frac{a^2}{a^2 - x^2} + 2 \ln(x) - \ln(x^2 - a^2)}{2a^4} + C.$$

$$\mathbf{8.7.57} \quad \int_0^{\pi/2} \frac{1}{1 + \tan^2 t} dt = \frac{\pi}{4} \approx 0.785.$$

$$\mathbf{8.7.58} \quad \int_0^{2\pi} \frac{1}{(4 + 2 \sin t)^2} dt = \frac{\pi\sqrt{3}}{9} \approx 0.605.$$

$$\mathbf{8.7.59} \quad \int (a^2 - x^2)^{3/2} dx = \frac{-x}{8}(2x^2 - 5a^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} \sin^{-1}(x/a) + C.$$

$$\mathbf{8.7.60} \quad \int (y^2 + a^2)^{-5/2} dy = \frac{y(3a^2 + 2y^2)}{3(a^2 + y^2)^{3/2}a^4} + C.$$

$$\mathbf{8.7.61} \quad \int_0^1 \ln x \ln(1 + x) dx = 2 - \frac{\pi^2}{12} - \ln 4 \approx -0.209.$$

$$\mathbf{8.7.62} \quad \int_0^1 x^{31} e^{x^2} dx = 653,837,184,000 - 240,533,257,867e \approx 0.0802.$$

$$\mathbf{8.7.63} \quad \int_1^2 \frac{x^{19}}{\sqrt{x^4 - 1}} dx = \frac{27,456\sqrt{15}}{7} \approx 15,190.9.$$

8.7.64 Let $u = x + 1$. Then $du = dx$ and $x = u - 1$. We have

$$\begin{aligned} \int x(x+1)^8 dx &= \int (u-1)u^8 du = \int (u^9 - u^8) du = \frac{u^{10}}{10} - \frac{u^9}{9} + C \\ &= \frac{(x+1)^{10}}{10} - \frac{(x+1)^9}{9} + C = (x+1)^9 \left(\frac{x+1}{10} - \frac{1}{9} \right) + C \\ &= \frac{1}{90}(x+1)^9(9x-1) + C. \end{aligned}$$

$$\begin{aligned} \mathbf{8.7.65} \quad \int x^3 e^{2x} dx &= \frac{1}{2}x^3 e^{2x} - \frac{3}{2} \int x^2 e^{2x} dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{2} \left(\frac{1}{2}x^2 e^{2x} - \int x e^{2x} dx \right) = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \\ &\frac{3}{2} \left(\frac{1}{2}x e^{2x} - \frac{1}{2} \int e^{2x} dx \right) = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + C. \end{aligned}$$

8.7.66

$$\begin{aligned} \int p^2 e^{-3p} dp &= \frac{-1}{3}p^2 e^{-3p} + \frac{2}{3} \int p e^{-3p} dp = -\frac{1}{3}p^2 e^{-3p} + \frac{2}{3} \left(-\frac{1}{3}p e^{-3p} + \frac{1}{3} \int e^{-3p} dp \right) \\ &= -\frac{1}{3}p^2 e^{-3p} - \frac{2}{9}p e^{-3p} - \frac{2}{27}e^{-3p} + C. \end{aligned}$$

$$\begin{aligned} \mathbf{8.7.67} \quad \text{Let } u = 3y. \text{ Then } \int \tan^4 3y dy &= \frac{1}{3} \int \tan^4 u du = \frac{1}{3} \left(\frac{1}{3} \tan^3 u - \int \tan^2 u du \right) = \frac{1}{9} \tan^3 u - \\ &\frac{1}{3} \left(\tan u - \int du \right) = \frac{1}{9} \tan^3 3y - \frac{1}{3} \tan 3y + y + C. \end{aligned}$$

8.7.68 Let $u = 4t$. Then

$$\int \sec^4 4t dt = \frac{1}{4} \int \sec^4 u du = \frac{1}{4} \left(\frac{1}{3} \tan u \sec^2 u + \frac{2}{3} \int \sec^2 u du \right) = \frac{1}{12} \tan 4t \sec^2 4t + \frac{1}{6} \tan 4t + C.$$

8.7.69 By inspection, the curves intersect at $x = 1$ where they both have value $\sqrt{2}$. The area we are seeking is given by $\int_0^1 \left(\frac{2}{\sqrt{\sqrt{x}+1}} - \sqrt{\sqrt{x}+x} \right) dx$. Using a computer algebra system, we find that

$$\int_0^1 \left(\frac{2}{\sqrt{\sqrt{x}+1}} - \sqrt{\sqrt{x}+x} \right) dx = \frac{1}{24} (128 - 78\sqrt{2} - 3 \ln(3 + 2\sqrt{2})).$$

8.7.70 Note that $f'(x) = \cos x - \sin x$. The surface area is given by

$$\int_0^\pi 2\pi(1 + \sin x + \cos x)\sqrt{1 + (\cos x - \sin x)^2} dx \approx 41.8741.$$

8.7.71 Let $u = ax + b$, so that $du = a dx$. Then we have

$$\begin{aligned} \frac{1}{a} \int \frac{ax + b - b}{ax + b} dx &= \frac{1}{a} \int \left(1 - \frac{b}{ax + b}\right) dx = \frac{1}{a} \left(x - \frac{b}{a} \int \frac{1}{u} du\right) \\ &= \frac{1}{a} \left(x - \frac{b}{a} \ln|u|\right) + C = \frac{1}{a} \left(x - \frac{b}{a} \ln|ax + b|\right) + C. \end{aligned}$$

8.7.72 Let $u = \sqrt{ax + b}$, so that $du = \frac{a}{2\sqrt{ax + b}} dx$. Then we have

$$\begin{aligned} \frac{2}{a} \int x \cdot \frac{a}{2\sqrt{ax + b}} dx &= \frac{2}{a} \int \frac{u^2 - b}{a} du = \frac{2}{a^2} \int (u^2 - b) du = \frac{2}{a^2} \left(\frac{u^3}{3} - bu\right) + C \\ &= \frac{2}{a^2} \left(\frac{(ax + b)^{3/2}}{3} - b\sqrt{ax + b}\right) + C. \end{aligned}$$

8.7.73 Let $u = ax + b$, so that $du = a dx$. Then we have

$$\begin{aligned} \frac{1}{a} \int \frac{u - b}{a} u^n du &= \frac{1}{a^2} \int (u^{n+1} - bu^n) du = \frac{1}{a^2} \left(\frac{u^{n+2}}{n+2} - \frac{bu^{n+1}}{n+1}\right) + C \\ &= \frac{1}{a^2} \left(\frac{(ax + b)^{n+2}}{n+2} - \frac{b(ax + b)^{n+1}}{n+1}\right) + C. \end{aligned}$$

8.7.74 Let $u = \sin^{-1} x$ and $dv = x^n dx$. Then $du = \frac{1}{\sqrt{1-x^2}} dx$ and $v = \frac{x^{n+1}}{n+1}$. Then the original integral is equal to

$$\frac{x^{n+1}}{n+1} \sin^{-1} x - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{\sqrt{1-x^2}} dx = \frac{1}{n+1} \left(x^{n+1} \sin^{-1} x - \int \frac{x^{n+1}}{\sqrt{1-x^2}} dx\right).$$

8.7.75

a. True. This is because these are equal for $x > 1$.

b. True. Note that $\frac{1}{9} = 0.\overline{11}$.

8.7.76 Note that $\frac{d}{dx} \left(\frac{1}{2} \cos^{-1} \sqrt{x^{-4}}\right) = \frac{1}{2} \cdot \frac{-1}{\sqrt{1-x^{-4}}} \cdot \frac{-2}{x^3} = \frac{1}{x^3} \cdot \frac{1}{\sqrt{1-(1/x^4)}} = \frac{1}{x\sqrt{x^4-1}}.$

Also, $\frac{d}{dx} \left(\frac{1}{2} \cos^{-1} x^{-2}\right) = \frac{1}{2} \cdot \frac{-1}{\sqrt{1-x^{-4}}} \cdot \frac{-2}{x^3} = \frac{1}{x\sqrt{x^4-1}}.$

And finally, $\frac{d}{dx} \left(\frac{1}{2} \tan^{-1} \sqrt{x^4-1}\right) = \frac{1}{2} \cdot \frac{1}{1+x^4-1} \cdot \frac{4x^3}{2\sqrt{x^4-1}} = \frac{1}{x\sqrt{x^4-1}}.$

Thus all three answers are correct in the sense that they are all antiderivatives of the original integrand, and thus must differ by a constant.

8.7.77 The two answers differ by a constant, namely the constant one. This can be seen as follows:

$$\begin{aligned} \frac{2 \sin(x/2)}{\cos(x/2) + \sin(x/2)} &= \frac{2 \tan(x/2)}{1 + \tan(x/2)} = 2 \frac{\sin x}{1 + \cos x} \cdot \frac{1}{1 + \frac{\sin x}{1 + \cos x}} \\ &= \frac{2 \sin x}{1 + \cos x + \sin x} = \frac{2 \sin x}{1 + \cos x + \sin x} \cdot \frac{1 - \cos x - \sin x}{1 - \cos x - \sin x} \\ &= \frac{2 \sin x(1 - \cos x - \sin x)}{-2 \sin x \cos x} = \frac{\sin x + \cos x - 1}{\cos x} = \frac{\sin x - 1}{\cos x} + 1. \end{aligned}$$

8.7.78 Note that $\frac{\ln|x-1|}{3} + \frac{\ln|x+2|}{6} - \frac{\ln|x|}{2} + C = \frac{1}{6}(2\ln|x-1| + \ln|x+2| - 3\ln|x|) + C =$
 $\frac{1}{6}(\ln(x-1)^2 + \ln|x+2| - \ln|x|^3) + C = \frac{1}{6}\ln\left(\frac{(x-1)^2|x+2|}{|x|^3}\right) + C.$

8.7.79 Let $u = 2x$. Then

$$\int x \sin^{-1} 2x \, dx = \frac{1}{4} \int u \sin^{-1} u \, du = \frac{2u^2 - 1}{16} \sin^{-1} u + \frac{u\sqrt{1-u^2}}{16} + C = \frac{8x^2 - 1}{16} \sin^{-1} 2x + \frac{x\sqrt{1-4x^2}}{8} + C.$$

8.7.80 Let $u = 10x$. Then

$$\begin{aligned} \frac{1}{25} \int u \cos^{-1} u \, du &= \frac{1}{25} \left(\frac{2u^2 - 1}{4} \cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} \right) + C \\ &= \frac{1}{25} \left(\frac{200x^2 - 1}{4} \cos^{-1} 10x - \frac{5x\sqrt{1-100x^2}}{2} \right) + C. \end{aligned}$$

8.7.81 $\int x^{-2} \tan^{-1} x \, dx = -(1/x) \tan^{-1} x + \int \frac{1}{x(1+x^2)} \, dx = \frac{-\tan^{-1} x}{x} + \ln\left(\frac{|x|}{\sqrt{1+x^2}}\right) + C.$

8.7.82 $\int \frac{\sin^{-1} ax}{x^2} \, dx = -a \left(\frac{\sin^{-1} u}{u} - \int \frac{1}{u\sqrt{1-u^2}} \, du \right)$, where $u = ax$. Then we have

$$\frac{-a \sin^{-1} u}{u} - a \ln \left| \frac{1 + \sqrt{1-u^2}}{u} \right| + C = \frac{-\sin^{-1} ax}{x} - a \ln \left| \frac{1 + \sqrt{1-(ax)^2}}{ax} \right| + C.$$

8.7.83

a. Note that $\int_0^b f(x) \, dx = \int_0^{b/2} f(x) \, dx + \int_{b/2}^b f(x) \, dx$. For the second integral, let $u = b - x$. Then

$du = -dx$. The second integral is equal to $-\int_{b-(b/2)}^0 f(b-u) \, du = \int_0^{b/2} f(b-u) \, du$. Because this can

be written as $\int_0^{b/2} f(b-x) \, dx$, we have $\int_0^b f(x) \, dx = \int_0^{b/2} (f(x) + f(b-x)) \, dx$.

b.

$$\begin{aligned} \int_0^{\pi/4} \ln(1 + \tan x) \, dx &= \int_0^{\pi/8} (\ln(1 + \tan x) + \ln(1 + \tan(\pi/4 - x))) \, dx \\ &= \int_0^{\pi/8} \left(\ln(1 + \tan x) + \ln \left(1 + \frac{\tan \pi/4 - \tan x}{1 + \tan \pi/4 \tan x} \right) \right) \, dx \\ &= \int_0^{\pi/8} \left(\ln(1 + \tan x) + \ln \left(1 + \frac{1 - \tan x}{1 + \tan x} \right) \right) \, dx \\ &= \int_0^{\pi/8} \left(\ln(1 + \tan x) + \ln \left(\frac{2}{1 + \tan x} \right) \right) \, dx = \int_0^{\pi/8} \ln(2) \, dx = \frac{\pi \ln 2}{8}. \end{aligned}$$

8.7.84

a. Let $u = \ln x$. Then $du = \frac{dx}{x}$, so $dx = x \, du = e^u \, du$. Substituting gives $\int \cos(u) e^u \, du$. Table entry 98 gives $\frac{e^u(\cos u + \sin u)}{2} + C = \frac{x(\cos \ln x + \sin \ln x)}{2} + C.$

- b. Let $u = \cos \ln x$ and $dv = dx$. Then $du = \frac{-\sin \ln x}{x} dx$ and $v = x$. Then we have $\int \cos \ln x dx = x \cos \ln x + \int \sin \ln x dx$. Now, focusing on the last integral, let $u = \sin \ln x$ and $dv = dx$. Then $du = \frac{\cos \ln x}{x} dx$ and $v = x$. Now we have $\int \cos \ln x dx = x \cos \ln x + \int \sin \ln x dx = x \cos \ln x + x \sin \ln x - \int \cos \ln x dx$. Adding $\int \cos \ln x dx$ to both sides of this equation and dividing by 2 gives

$$\int \cos \ln x dx = \frac{x \cos \ln x + x \sin \ln x}{2} + C.$$

8.7.85

- a. Using a computer algebra system, we have

θ_0	T	Relative Error
0.1	6.27927	0.000623603
0.2	6.26762	0.0024778
0.3	6.24854	0.0051388
0.4	6.22253	0.00965413
0.5	6.19021	0.0147967
0.6	6.15236	0.0208215
0.7	6.10979	0.0275963
0.8	6.06338	0.0349831
0.9	6.01399	0.0428433
1.0	5.96247	0.0510427

- b. All of these values are within 10 percent of 2π .

8.7.86

- a. $L(c) = \int_0^c \sqrt{1+4x^2} dx$. Note that $L'(c) = \sqrt{1+4c^2} > 0$, so L is increasing.
- b. Note that $L''(c) = \frac{4c}{\sqrt{1+4c^2}} \geq 0$ on $[0, \infty)$, so L is concave up on that interval.
- c. We have $\lim_{c \rightarrow \infty} \frac{L(c)}{c^2} = \lim_{c \rightarrow \infty} \frac{L'(c)}{2c} = \lim_{c \rightarrow \infty} \frac{\sqrt{1+4c^2}}{2c} = 1$. Thus $L(c)$ increases as c^2 increases.

8.7.87

- a. The result holds.

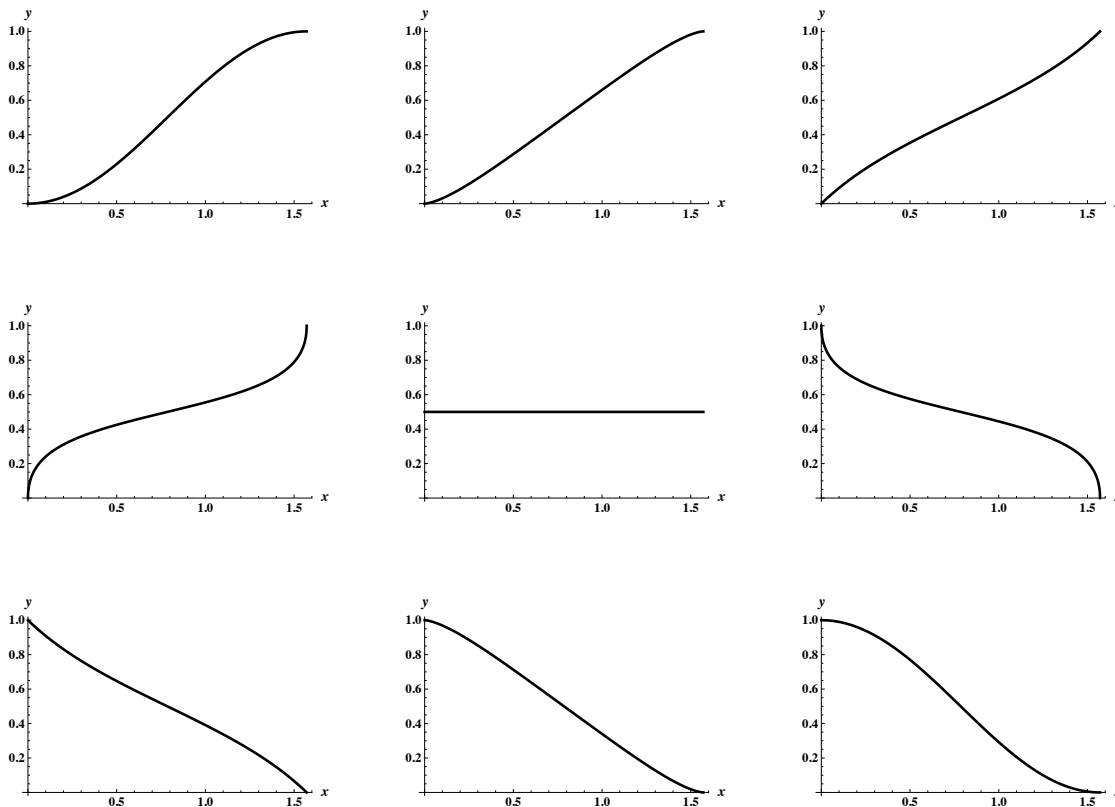
- b. First note that $\int_0^{\pi/2} \cos^n x dx = -\int_{\pi/2}^0 \cos^n(\pi/2 - \theta) d\theta = \int_0^{\pi/2} \sin^n \theta d\theta$. So we only need to show the result for $\int_0^{\pi/2} \sin^{10} x dx$. Repeatedly using the reduction formula we have

$$\begin{aligned} \int_0^{\pi/2} \sin^{10} x dx &= \frac{9}{10} \int_0^{\pi/2} \sin^8 x dx = \frac{9 \cdot 7}{10 \cdot 8} \int_0^{\pi/2} \sin^6 x dx \\ &= \frac{9 \cdot 7 \cdot 5}{10 \cdot 8 \cdot 6} \int_0^{\pi/2} \sin^4 x dx = \frac{9 \cdot 7 \cdot 5 \cdot 3}{10 \cdot 8 \cdot 6 \cdot 4} \int_0^{\pi/2} \sin^2 x dx \\ &= \frac{9 \cdot 7 \cdot 5 \cdot 3}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \int_0^{\pi/2} dx = \frac{63\pi}{2^9} \end{aligned}$$

- c. The values decrease as n increases.

8.7.88

- a. The graphs change shape, but the area under the curve remains constant.



- b. When $m = 0$, the integrand is equal to $\frac{1}{2}$ and the bounded region forms a rectangle of width $\frac{\pi}{2}$ and height $\frac{1}{2}$. For all values of m , the graph of $f(x) = \frac{1}{(1 + \tan^m x)}$ is symmetric with respect to the point $(\pi/2, 1/2)$. When $m \neq 0$, for any area located above $y = \frac{1}{2}$ and below f on one side of $x = \frac{\pi}{4}$, there is an equal amount an area below $y = \frac{1}{2}$ and above f on the other side of $x = \frac{\pi}{4}$. The result is easily verified with a computer algebra system.

8.8 Numerical Integration

8.8.1 $\Delta x = \frac{18 - 4}{28} = \frac{1}{2}.$

8.8.2 The Midpoint Rule uses the value of the function evaluated at the midpoint of each subinterval to determine the height of the approximating rectangle over each subinterval. The areas of these rectangles are added up to yield an approximation to the definite integral.

8.8.3 The Trapezoidal Rule approximates the definite integral by using a trapezoid over each subinterval rather than a rectangle.

8.8.4 It is evaluated at 1, 5, and 9, which are the midpoints of the 3 subinterval of length 4.

$$8.8.5 \quad M(4) = 2(2) + 5(2) + 6(2) + 8(2) = 42.$$

$$8.8.6 \quad T(4) = \left(\frac{1}{2}(1) + 4 + 7 + 5 + \frac{1}{2}(5) \right) 2 = 38.$$

$$8.8.7 \quad S(4) = (1 + 4(4) + 2(7) + 4(5) + 5) \frac{2}{3} = \frac{112}{3} \approx 37.33.$$

$$8.8.8 \quad S(8) = ((1 + 4(2) + 2(4) + 4(5) + 2(7) + 4(6) + 2(5) + 4(8) + 5) \frac{1}{3} = \frac{122}{3} \approx 40.67.$$

8.8.9 The endpoints of the subintervals are $-1, 1, 3, 5, 7$, and 9 . The trapezoidal rule uses the value of f at each of these endpoints.

$$8.8.10 \quad S(4) = \frac{4T(4) - 6}{3} = 4.8.$$

$$8.8.11 \quad \text{The absolute error is } |\pi - 3.14| \approx 0.0015926536. \text{ The relative error is } \frac{|\pi - 3.14|}{\pi} \approx 5 \times 10^{-4}.$$

$$8.8.12 \quad \text{The absolute error is } |\sqrt{2} - 1.414| \approx 2.14 \times 10^{-4}. \text{ The relative error is } \frac{|\sqrt{2} - 1.414|}{\sqrt{2}} \approx 1.51 \times 10^{-4}.$$

$$8.8.13 \quad \text{The absolute error is } |e - 2.72| \approx 0.0017181715. \text{ The relative error is } \frac{|e - 2.72|}{e} \approx 6.32 \times 10^{-4}.$$

$$8.8.14 \quad \text{The absolute error is } |e - 2.718| \approx 2.81 \times 10^{-4} \text{ and the relative error is } \frac{|e - 2.718|}{e} \approx 1.04 \times 10^{-4}.$$

8.8.15

For $n = 1$, we have $f(6) \cdot 8 = 72 \cdot 8 = 576$.

For $n = 2$ we have $f(4) \cdot 4 + f(8) \cdot 4 = 32 \cdot 4 + 128 \cdot 4 = 640$.

For $n = 4$, we have $f(3) \cdot 2 + f(5) \cdot 2 + f(7) \cdot 2 + f(9) \cdot 2 = 18 \cdot 2 + 50 \cdot 2 + 98 \cdot 2 + 162 \cdot 2 = 656$.

8.8.16

For $n = 1$, we have $f(5) \cdot 8 = 125 \cdot 8 = 1000$.

For $n = 2$ we have $f(3) \cdot 4 + f(7) \cdot 4 = 27 \cdot 4 + 343 \cdot 4 = 1480$.

For $n = 4$, we have $f(2) \cdot 2 + f(4) \cdot 2 + f(6) \cdot 2 + f(8) \cdot 2 = 8 \cdot 2 + 64 \cdot 2 + 216 \cdot 2 + 512 \cdot 2 = 1600$.

8.8.17

$$\frac{1}{6} (\sin(\pi/12) + \sin(\pi/4) + \sin(5\pi/12) + \sin(7\pi/12) + \sin(3\pi/4) + \sin(11\pi/12)) \approx 0.6439505509.$$

8.8.18

$$\frac{1}{8} (e^{-1/16} + e^{-3/16} + e^{-5/16} + e^{-7/16} + e^{-9/16} + e^{-11/16} + e^{-13/16} + e^{-15/16}) \approx 0.6317092095.$$

$$8.8.19 \quad \text{For } n = 2 \text{ we have } T(2) = \frac{4}{2}(f(2) + 2f(6) + f(10)) = 2(8 + 2(72) + 200) = 704.$$

$$\text{Using the results of number 15: For } n = 4, \text{ note that } T(4) = \frac{T(2) + M(2)}{2} = \frac{704 + 640}{2} = 672.$$

$$\text{Using the results of number 15: For } n = 8 \text{ we have that } T(8) = \frac{T(4) + M(4)}{2} = \frac{672 + 656}{2} = 664.$$

8.8.20

$$T(2) = \frac{T(1) + M(1)}{2}. \text{ Using problem 16 if we compute } T(1).$$

$$\text{We have } T(1) = \left(\frac{1}{2} + \frac{729}{2} \right) \cdot 8 = 2920. \text{ Thus, } T(2) = \frac{2920 + 1000}{2} = 1960.$$

$$\text{For } n = 4, \text{ we have } T(4) = \frac{T(2) + M(2)}{2} = \frac{1960 + 1480}{2} = 1720.$$

$$\text{For } n = 8 \text{ we have } T(8) = \frac{T(4) + M(4)}{2} = \frac{1720 + 1600}{2} = 1660.$$

8.8.21 We have

$$\begin{aligned} T(6) &= \frac{1}{12} \left(\sin 0 + 2 \sin \frac{\pi}{6} + 2 \sin \frac{\pi}{3} + 2 \sin \frac{\pi}{2} + 2 \sin \frac{2\pi}{3} + 2 \sin \frac{5\pi}{6} + \sin \pi \right) \\ &= \frac{1}{6} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2} \right) = \frac{1}{6} (2 + \sqrt{3}). \end{aligned}$$

8.8.22 $T(8) = \frac{1}{16} (e^{-0} + 2e^{-1/8} + 2e^{-1/4} + 2e^{-3/8} + 2e^{-1/2} + 2e^{-5/8} + 2e^{-3/4} + 2e^{-7/8} + e^{-1}) \approx 0.6329434182.$

8.8.23 $S(4) = \frac{\pi}{12} \left(\sqrt{\sin 0} + 4\sqrt{\sin \frac{\pi}{4}} + 2\sqrt{\sin \frac{\pi}{2}} + 4\sqrt{\sin \frac{3\pi}{4}} + \sqrt{\sin \pi} \right) \approx 2.28477.$

$$\begin{aligned} S(8) &\approx \frac{\pi}{24} \left(\sqrt{\sin 0} + 4\sqrt{\sin \frac{\pi}{8}} + 2\sqrt{\sin \frac{\pi}{4}} + 4\sqrt{\sin \frac{3\pi}{8}} + \right. \\ &\quad \left. 2\sqrt{\sin \frac{\pi}{2}} + 4\sqrt{\sin \frac{5\pi}{8}} + 2\sqrt{\sin \frac{3\pi}{4}} + 4\sqrt{\sin \frac{7\pi}{8}} + \sqrt{\sin \pi} \right) \approx 2.35646. \end{aligned}$$

8.8.24 $S(4) = \frac{1}{3} (\sqrt{4} + 4\sqrt{5} + 2\sqrt{6} + 4\sqrt{7} + \sqrt{8}) \approx 9.75156.$

$$S(8) \approx \frac{1}{6} (\sqrt{4} + 4\sqrt{4.5} + 2\sqrt{5} + 4\sqrt{5.5} + 2\sqrt{6} + 4\sqrt{6.5} + 2\sqrt{7} + 4\sqrt{7.5} + \sqrt{8}) \approx 9.75161.$$

8.8.25

$$\begin{aligned} S(10) &= \frac{1}{6} (e^{-4} + 4e^{-2.25} + 2e^{-1} + 4e^{-0.25} + 2e^0 + 4e^{-0.25} + 2e^{-1} \\ &\quad + 4e^{-2.25} + 2e^{-4} + 4e^{-6.25} + e^{-9}) \approx 1.7680. \end{aligned}$$

8.8.26

$$\begin{aligned} S(8) &= \frac{1}{12} (\cos \sqrt{2} + 4 \cos \sqrt{2.25} + 2 \cos \sqrt{2.5} + 4 \cos \sqrt{2.75} \\ &\quad + 2 \cos \sqrt{3} + 4 \cos \sqrt{3.25} + 2 \cos \sqrt{3.5} + 4 \cos \sqrt{3.75} + \cos \sqrt{4}) \approx -0.30082. \end{aligned}$$

8.8.27 The width of each subinterval is $1/25$, so

$$M(25) = \frac{1}{25} (\sin \pi/50 + \sin 3\pi/50 + \sin 5\pi/50 + \cdots + \sin 49\pi/50) \approx 0.63704.$$

Because $\int_0^1 \sin \pi x \, dx = \frac{2}{\pi}$, the absolute error is $|2/\pi - M(25)| \approx 4.19 \times 10^{-4}$ and the relative error is this number divided by $2/\pi$ which is approximately 6.58×10^{-4} . The Trapezoidal Rule yields approximately 0.63578, with a relative error of ≈ 0.00132 .

8.8.28 The width of each subinterval is $1/50$, so

$$M(50) = \frac{1}{50} (e^{-1/100} + (e^{-1/100})^3 + (e^{-1/100})^5 + \cdots + (e^{-1/100})^{99}) \approx 0.63211.$$

$$T(50) = \frac{1}{100} (e^0 + 2e^{-1/50} + 2e^{-2/50} + \cdots + 2e^{-49/50} + e^{-1}) \approx 0.63214.$$

The actual value of the integral is $1 - \frac{1}{e}$. The absolute error for $M(50)$ is $|1 - \frac{1}{e} - .63211| \approx 1 \times 10^{-5}$, and the relative error is that number divided by $1 - \frac{1}{e}$ which is about 1.66×10^{-5} . The absolute error for $T(50)$ is $|1 - \frac{1}{e} - .63214| \approx 2.1 \times 10^{-5}$ and the relative error is that number divided by $1 - \frac{1}{e}$ which is about 3.33×10^{-5} .

8.8.29

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	99	1	102	2
8	99.75	0.250	100.5	0.5
16	99.9375	0.0625	100.125	0.125
32	99.984375	0.0156	100.03125	0.03125

8.8.30

n	$T(n)$	Absolute Error	$M(n)$	Absolute Error
4	6	2	3	1
8	4.5	0.5	3.75	0.25
16	4.125	0.125	3.9375	0.0625
32	4.03125	0.03125	3.984375	0.015625

8.8.31

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	1.50968181	9.7×10^{-3}	1.48067370	1.9×10^{-2}
8	1.50241228	2.4×10^{-3}	1.49517776	4.8×10^{-3}
16	1.50060256	6.0×10^{-4}	1.49879502	1.2×10^{-3}
32	1.50015061	1.5×10^{-4}	1.49969879	3.0×10^{-4}

8.8.32

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	1.004785839	4.8×10^{-3}	0.99036501	9.6×10^{-3}
8	1.001210217	1.2×10^{-3}	0.99757542	2.4×10^{-3}
16	1.000303459	3.03×10^{-4}	0.99939282	6.07×10^{-4}
32	1.000075922	7.59×10^{-5}	0.99984814	1.52×10^{-4}

8.8.33 Because the given function has odd symmetry about the midpoint of the interval $[0, \pi]$, the midpoint rule calculates to be zero for all even values of n , as does the trapezoidal rule.

8.8.34

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	0.27572053	0.224	1.0373146	0.54
8	0.425459016	0.075	0.65651757	0.15
16	0.47975863	0.02	0.54098829	0.041
32	0.49482934	0.0052	0.51037346	0.01

8.8.35 $T(4) = 3(9.1 + 2(12) + 2(26) + 2(46) + 53) = 690.3$ million ft^3 .

$$S(4) = 2(9.1 + 4(12) + 2(26) + 4(46) + 53) = 692.2 \text{ million } \text{ft}^3.$$

8.8.36 $T(6) = 2(9.1 + 2(10.5) + 2(15) + 2(26) + 2(40) + 2(49) + 53) = 686.2$ million ft^3 .

$$S(6) = \frac{4}{3}(9.1 + 4(10.5) + 2(15) + 4(26) + 2(40) + 4(49) + 53) = 685.5 \text{ million } \text{ft}^3.$$

8.8.37 Answers may vary.

$$\begin{aligned} \bar{T} &= \frac{1}{12} \int_0^{12} T(t) dt \approx \frac{1}{12} \text{Trapezoid}(12) \\ &\approx \frac{1}{24}(47 + 2(50 + 46 + 45 + 48 + 52 + 54 + 61 + 62 + 63 + 63 + 59) + 55) = 54.5. \end{aligned}$$

8.8.38 Answers may vary.

$$\begin{aligned}\bar{T} &= \frac{1}{12} \int_0^{12} T(t) dt \approx \frac{1}{12} \text{Trapezoid}(12) \\ &\approx \frac{1}{24}(41 + 2(44 + 46 + 48 + 52 + 53 + 53 + 53 + 51 + 51 + 49 + 47) + 47) = 49.25.\end{aligned}$$

8.8.39 Answers may vary.

$$\begin{aligned}\bar{T} &= \frac{1}{12} \int_0^{12} T(t) dt \approx \frac{1}{12} \text{Trapezoid}(12) \\ &\approx \frac{1}{24}(35 + 2(34 + 34 + 36 + 36 + 37 + 37 + 36 + 35 + 35 + 34 + 33) + 32) \approx 35.0.\end{aligned}$$

8.8.40 Answers may vary.

$$\begin{aligned}\bar{T} &= \frac{1}{12} \int_0^{12} T(t) dt \approx \frac{1}{12} \text{Trapezoid}(12) \\ &\approx \frac{1}{24}(9 + 2(11 + 11 + 12 + 14 + 15 + 17 + 19 + 20 + 22 + 24 + 24) + 25) \approx 17.1667.\end{aligned}$$

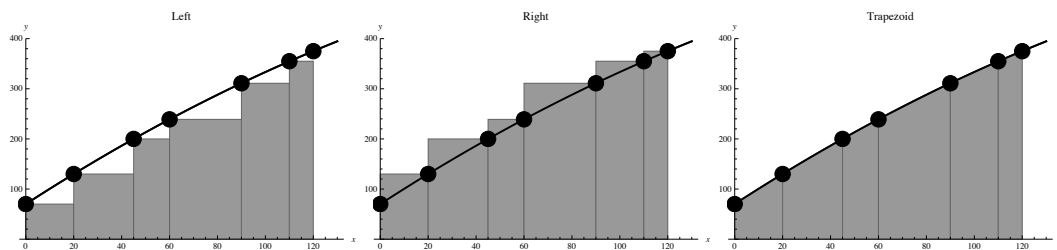
8.8.41

- a. Left Riemann sum: $\frac{1}{120}(70 \cdot 20 + 130 \cdot 25 + 200 \cdot 15 + 239 \cdot 30 + 311 \cdot 20 + 355 \cdot 10) = 204.917$.
 Right Riemann sum: $\frac{1}{120}(130 \cdot 20 + 200 \cdot 25 + 239 \cdot 15 + 311 \cdot 30 + 355 \cdot 20 + 375 \cdot 10) = 261.375$.
 Trapezoidal rule:

$$\begin{aligned}&\frac{(70 + 130) \cdot 20}{2 \cdot 120} + \frac{(130 + 200) \cdot 25}{2 \cdot 120} + \frac{(200 + 239) \cdot 15}{2 \cdot 120} + \frac{(239 + 311) \cdot 30}{2 \cdot 120} \\ &\quad + \frac{(311 + 355) \cdot 20}{2 \cdot 120} + \frac{(355 + 375) \cdot 10}{2 \cdot 120} = 233.146\end{aligned}$$

These are approximations to the average temperature of the curling iron on the interval $[0, 120]$.

- b. Because the function is increasing, the left Riemann sum is an underestimate and the right Riemann sum is an overestimate. The Trapezoidal rule appears to be a slight underestimate. The following is a plot of the points with straight line segments connecting them. The shape suggests that the actual function is concave down on the interval, so the trapezoidal rule will be an underestimate.



- c. The change in temperature over the time interval is

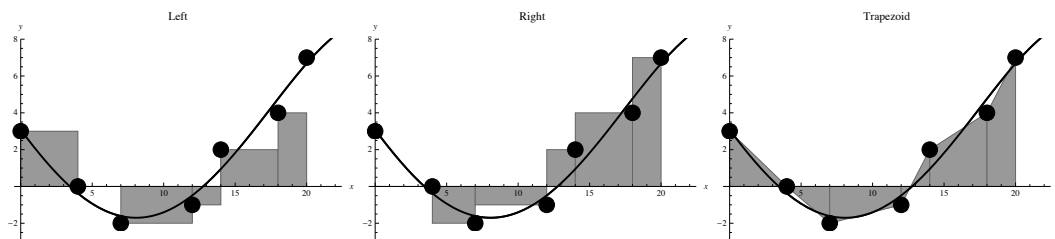
$$\int_0^{120} T'(t) dt = T(120) - T(0) = 375 - 70 = 305 \text{ degrees.}$$

8.8.42

- a. Left Riemann sum: $(3 \cdot 4 + 0 \cdot 3 + -2 \cdot 5 + -1 \cdot 2 + 2 \cdot 4 + 4 \cdot 2) = 16$.
 Right Riemann sum: $(0 \cdot 4 + -2 \cdot 3 + -1 \cdot 5 + 2 \cdot 2 + 4 \cdot 4 + 7 \cdot 2) = 23$.
 Trapezoidal rule:

$$\frac{(3+0) \cdot 4}{2} + \frac{(0+(-2)) \cdot 3}{2} + \frac{(-2+(-1)) \cdot 5}{2} + \frac{(2+(-1)) \cdot 2}{2} + \frac{(2+4) \cdot 4}{2} + \frac{(4+7) \cdot 2}{2} = 19.5$$

- b. Plots of all three approximations, together with the point themselves and a curve fitted through the points (a degree 4 curve, actually) are:



- c. We have

$$\int_4^{12} (3f'(x) - 2) dx = 3 \int_4^{12} f'(x) dx + \int_4^{12} 2 dx = 3(f(12) - f(4)) + 2 \cdot (12 - 4) = 13.$$

8.8.43

- a. The net change in elevation is $\int_0^5 v(t) dt$, which can be approximated by the trapezoidal rule to give

$$\begin{aligned} & \frac{(0+100) \cdot 1}{2} + \frac{(100+120) \cdot 0.5}{2} + \frac{(120+150) \cdot 1.5}{2} \\ & + \frac{(150+110) \cdot 0.5}{2} + \frac{(110+90) \cdot 0.5}{2} + \frac{(90+80) \cdot 1}{2} = 507.5. \end{aligned}$$

If we add the net change to the original elevation of 5400 feet, we have an elevation of approximately 5907.5 feet.

- b. The right Riemann sum is

$$100 \cdot 1 + 120 \cdot 0.5 + 150 \cdot 1.5 + 110 \cdot 0.5 + 90 \cdot 0.5 + 80 \cdot 1 = 565.$$

If we add the net change to the original elevation of 5400 feet, we have an elevation of approximately 5965 feet.

- c. The elevation can be estimated by

$$5400 + \int_0^5 g(t) dt = 5400 + \int_0^5 (3.49t^3 - 43.21t^2 + 142.43t - 1.75) dt \approx 5916.52,$$

so the elevation of the balloon is about 5917 feet.

8.8.44

- a. The trapezoidal rule gives

$$\begin{aligned} & \frac{(0+2.5) \cdot 1}{2} + \frac{(2.5+3.2) \cdot 1}{2} + \frac{(3.2+4) \cdot 1}{2} + \frac{(4+6) \cdot 1}{2} \\ & + \frac{(6+7) \cdot 2}{2} + \frac{(7+5.3) \cdot 1.5}{2} + \frac{(3+0) \cdot 0.5}{2} = 35.675. \end{aligned}$$

b. The left Riemann sum gives

$$0 \cdot 1 + 2.5 \cdot 1 + 3.2 \cdot 1 + 4 \cdot 1 + 6 \cdot 2 + 7 \cdot 1.5 + 5.3 \cdot 0.5 = 34.85.$$

c. Although the surface area of the piece appears to be less than half of $81 = 9^2$ (the area of a 9×9 piece of wood), the shape prohibits the creation of two identical pieces.

	n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
8.8.45	25	3.19623162	—	—	—
	50	3.19495398	4.3×10^{-4}	3.19452809	4.5×10^{-8}

	n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
8.8.46	30	6.411850535	—	—	—
	60	6.402962881	.003	6.400000329	3.3×10^{-7}

	n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
8.8.47	50	1.00008509	—	—	—
	100	1.00002127	2.1×10^{-5}	1.00000000	4.6×10^{-9}

	n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
8.8.48	64	0.6657662105	—	—	—
	128	0.6657718648	1.9×10^{-6}	.665774	8.3×10^{-12}

	n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
8.8.49	4	1820	284	—	—
	8	1607.75	71.8	1537	1
	16	1553.9844	18	1536.0625	6.3×10^{-2}
	32	1540.4990	4.5	1536.0039	3.9×10^{-3}

	n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
8.8.50	4	0.99036501	9.6×10^{-3}	—	—
	8	0.99757542	2.4×10^{-3}	0.99997889	2.1×10^{-5}
	16	0.99939282	6.1×10^{-4}	0.99999862	1.4×10^{-6}
	32	0.99984814	1.5×10^{-4}	0.9999999	8.7×10^{-8}

	n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
8.8.51	4	0.46911538	5.3×10^{-2}	—	—
	8	0.50826998	1.3×10^{-2}	0.52132152	2.9×10^{-4}
	16	0.51825968	3.4×10^{-3}	0.52158957	1.7×10^{-5}
	32	0.52076933	8.4×10^{-4}	0.52160588	1.1×10^{-6}

	n	$T(n)$	Absolute Error	$S(n)$	Absolute Error
8.8.52	4	2.300552032	1.30	—	—
	8	1.390088342	.39	1.086600445	8.7×10^{-2}
	16	1.103308986	.1	1.007715867	7.7×10^{-3}
	32	1.026229165	.026	1.000535892	5.4×10^{-4}

8.8.53

- a. True. Note that $\Delta x = \frac{b-a}{18}$, and $E_S(18) \leq \frac{1 \cdot (b-a)}{10 \cdot 18} \cdot (\Delta x)^4 = \frac{(\Delta x)^5}{10}$.
- b. False. Because $E_M(n) \leq \frac{k(b-a)^3}{24n^2}$, we have $E_M(3n) \leq \frac{k(b-a)^3}{24(9n^2)}$, so the error decreases by a factor of about 9.
- c. True. Because $E_T(n) \leq \frac{k(b-a)^3}{12n^2}$, we have $E_T(4n) \leq \frac{k(b-a)^3}{24(16n^2)}$, so the error decreases by a factor of about 16.

$$8.8.54 \quad \int_0^{\pi/2} \sin^6 x \, dx = \frac{5\pi}{32} \approx 0.4908738521.$$

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	0.4908738521	2.341×10^{-11}	.4908738521	2.341×10^{-11}
8	0.4908738521	≈ 0	0.49087352	≈ 0
16	0.4908738521	≈ 0	0.49087352	≈ 0
32	0.4908738521	≈ 0	0.49087352	≈ 0

$$8.8.55 \quad \int_0^{\pi/2} \cos^9 x \, dx = \frac{128}{315}.$$

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	0.40635058	1.4×10^{-6}	0.40634783	1.4×10^{-6}
8	0.40634921	7.6×10^{-10}	0.40634921	7.6×10^{-9}
16	0.40634921	6.6×10^{-13}	0.40634921	6.6×10^{-13}
32	0.40634921	8.9×10^{-16}	0.40634921	7.8×10^{-16}

$$8.8.56 \quad \int_0^1 (8x^7 - 7x^8) \, dx = \frac{2}{9}.$$

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	0.2192	2.2×10^{-3}	0.2257	3.4×10^{-3}
8	0.222	2.0×10^{-4}	0.2224	2.2×10^{-4}
16	0.2222	1.2×10^{-5}	0.2222	1.4×10^{-5}
32	0.2222	7.8×10^{-7}	0.2222	3.9×10^{-7}

$$8.8.57 \quad \int_0^{\pi} \ln(5 + 3 \cos x) x \, dx = \pi \ln(9/2).$$

n	$M(n)$	Absolute Error	$T(n)$	Absolute Error
4	4.72531820	1.2×10^{-4}	4.72507878	1.2×10^{-4}
8	4.72519851	9.1×10^{-9}	4.72519849	9.1×10^{-9}
16	4.72519850	0.	4.72519850	8.9×10^{-16}
32	4.72519850	0.	4.72519850	8.9×10^{-19}

8.8.58 $\int_0^{2\pi} \frac{1}{(5 + 3 \sin x)^2} dx = \frac{5\pi}{32} \approx 0.4908738521.$

n	$S(n)$	Absolute Error
4	0.6400995032	0.149
8	0.475006	0.0159
16	0.49050507	3.7×10^{-4}

8.8.59 $\int_0^{\pi} \frac{\cos x}{(5/4) - \cos x} dx = \frac{2\pi}{3} \approx 2.094395102.$

n	$S(n)$	Absolute Error
4	1.916439963	0.178
8	2.080919302	0.0135
16	2.094341841	5.3×10^{-5}

8.8.60 $\int_0^{\pi} \ln(2 + \cos x) dx = \pi \ln \left(\frac{2 + \sqrt{3}}{2} \right) \approx 1.959759164.$

n	$S(n)$	Absolute Error
4	1.962437352	.0027
8	1.95976612	6.96×10^{-6}

8.8.61 $\int_0^{\pi} \sin 6x \cos 3x dx = \frac{4}{9} = 0.\bar{4}.$

n	$S(n)$	Absolute Error
8	0.0305049084	0.475
16	0.4540112289	0.0096
32	0.44487	0.00087

8.8.62 $T = \frac{4}{w} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}} d\phi$, $w = \sqrt{\frac{g}{L}}$, $g = 9.8$, and $k^2 = \sin^2(\theta_0/2).$

n	$S(n)$
4	2.08732001
8	2.08732001

8.8.63 We are computing $\frac{1}{3\sqrt{2\pi}} \int_{66}^{72} e^{-(x-69)^2/18} dx$. If we use Simpson's rule we obtain

n	$S(n)$
4	0.683
8	0.683

So about 68.3%.

8.8.64 We are computing $\frac{1}{22\sqrt{2\pi}} \int_{60}^{90} e^{-(x-110)^2/968} dx$. If we compute $S(8)$, we see that $S(8) \approx 0.17012657$.

8.8.65

- For even n we have $S(n) = \frac{20}{n}(365)(f(a) + 4f(x_1) + 4f(x_2) + \dots + 4f(x_{n-1}) + f(b))$. For $n = 6$ since there are 6 decades between 1940 and 2000, we have $S(6) \approx 160,000$ millions of barrels produced.
- Following part (a) with $n = 6$ we have $S(n) \approx 68,000$ millions of barrels imported.

8.8.66

- a. $M(30) \approx 1.80967$.
- b. $f'(x) = -2x \sin x^2$, so $f''(x) = -2 \sin x^2 + (-2x)(\cos x^2)2x = -2(\sin x^2 + 2x^2 \cos x^2)$.
- c. For $-1 \leq x \leq 1$,

$$|-2(2x^2 \cos x^2 + \sin x^2)| \leq 2(2x^2 |\cos x^2| + |\sin x^2|) \leq 2(2(1)(1) + 1) = 6.$$

d. $E_M \leq \frac{k(b-a)}{24}(\Delta x)^2 = \frac{6(2)}{24} \left(\frac{1}{15}\right)^2 = \frac{1}{450} = 0.00\bar{2}.$

8.8.67

- a. $M(50) \approx 35.43456$.
- b. $f'(x) = \frac{1}{2}(x^3 + 1)^{-1/2}3x^2 = \frac{3x^2}{2(x^3 + 1)^{1/2}}$. Then $f''(x) = \frac{2(x^3 + 1)^{1/2}(6x) - 3x^2(x^3 + 1)^{-1/2}(3x^2)}{4(x^3 + 1)}$.
 Multiplying the numerator and denominator by $(x^3 + 1)^{1/2}$ yields

$$\frac{12x(x^3 + 1) - 9x^4}{4(x^3 + 1)^{3/2}} = \frac{3x^4 + 12x}{4(x^3 + 1)^{3/2}} = \frac{3x(x^3 + 4)}{4(x^3 + 1)^{3/2}}.$$

- c. Because f'' is decreasing on $[1, 6]$, $|f''(x)| \leq f''(1) = \frac{15}{8\sqrt{2}}$.
- d. Note that $\frac{15}{8\sqrt{2}} < 1.33$. We have $E_M \leq \frac{k(b-a)}{24}(\Delta x)^2 = \frac{1.33(5)}{24} \left(\frac{1}{10}\right)^2 \leq 0.0028$.

8.8.68

- a. $T(50) = \frac{1}{100}(1 + 2 \sum_{i=1}^{49} e^{i^2/50^2} + e) \approx 1.462832952$.
- b. $f'(x) = 2xe^{x^2}$, so $f''(x) = 4x^2e^{x^2} + 2e^{x^2} = 2e^{x^2}(2x + 1)$.
- c. Because both e^{x^2} and $2x + 1$ are increasing and positive on $[0, 1]$, we have that $|e^{x^2}| \leq e$ and $|2x + 1| \leq 2 \cdot 1 + 1 = 3$. Thus $|f''(x)| = 2|e^{x^2}||2x + 1| \leq 2e \cdot 3 < 2(3)^2 = 18$.
- d. Because $E_T(n) \leq k(b-a)^3/12n^2 = 18/12(50^2) = 6 \times 10^{-4}$, $T(50)$ is accurate to at least 3 decimal places.

8.8.69

- a. $T(40) = \frac{1}{80}(\sin 1 + 2 \sum_{i=1}^{39} \sin e^{i/40} + \sin e) \approx 0.8748$.
- b. $f(x) = \sin e^x$, so $f'(x) = e^x \cos e^x$. $f''(x) = -e^{2x} \sin e^x + e^x \cos e^x = e^x(\cos e^x - e^x \sin e^x)$.
- c. $|f''(x)| = |e^x| \cdot |\cos e^x - e^x \sin e^x| \leq |e^x|(|\cos e^x| + |e^x \sin e^x|) \leq e(1 + e) < 10.11$. However, the graph of the absolute value of $f''(x)$ reveals that it is actually bounded by 5.75 on the interval $[0, 1]$.
- d. Because $E_T(n) \leq k(b-a)^3/12n^2 = 6/(12(40^2)) = 0.0003125$, $T(40)$ is accurate to at least 3 decimal places.

8.8.70

- a. $S(30) \approx 0.886207336$.

b. $f'(x) = -2xe^{-x^2}$. $f''(x) = -2e^{-x^2} + (-2x)e^{-x^2}(-2x) = -2e^{-x^2}(1 - 2x^2) = 2e^{-x^2}(2x^2 - 1)$. $f'''(x) = -4xe^{-x^2}(2x^2 - 1) + 2e^{-x^2}(4x) = 4xe^{-x^2}(1 - 2x^2 + 2) = 4xe^{-x^2}(3 - 2x^2) = 4e^{-x^2}(3x - 2x^3)$. $f^{(4)}(x) = -8xe^{-x^2}(3x - 2x^3) + 4e^{-x^2}(3 - 6x^2) = 4e^{-x^2}(4x^4 - 6x^2 + 3 - 6x^2) = 4e^{-x^2}(4x^4 - 12x^2 + 3)$.

c. Using a graph of $f^{(4)}$, we see that $|f^{(4)}(x)| \leq 12$ on $[0, 3]$. So $E_S \leq \frac{k(b-a)}{180}(\Delta x)^4 = \frac{12(3)}{180} \left(\frac{1}{10}\right)^4 = \frac{1}{50,000} = 0.00002$.

8.8.71

a. $S(20) \approx 0.97775$.

b. The error is bounded by $\frac{f'''(x)}{180} \left(\frac{1}{20}\right)^4 \leq \frac{1}{180 \cdot 20^4} \leq 3.5 \times 10^{-8}$.

8.8.72 $\int_a^b (mx+k) dx = \left[\frac{mx^2}{2} + kx \right]_a^b = \frac{mb^2}{2} + kb - \frac{ma^2}{2} - ka = \frac{m}{2}(b^2 - a^2) + k(b-a) = \left(\frac{m}{2}(b+a) + k\right)(b-a) = \frac{f(a) + f(b)}{2}(b-a) = T_{[a,b]}(1)$. Now for any n , the above argument shows that the trapezoidal rule gives the exact value of $\int_{x_{i-1}}^{x_i} mx + k dx$. Thus

$$\int_a^b (mx+k) dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (mx+k) dx = \sum_{i=1}^n T_{[x_{i-1}, x_i]}(1) = T_{[a,b]}(n).$$

8.8.73 $\int_0^{2\pi} \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} dt$, $a = 4$, $b = 8$.

n	$S(n)$
4	41.88790205
8	39.05860599

8.8.74 $\text{Si}(1) = \int_0^1 \frac{\sin t}{t} dt$. We have

n	$M(n)$
4	0.9468682055
8	0.9462791963
16	0.9461320920

For $\text{Si}(10) = \int_0^{10} \frac{\sin t}{t} dt$, we have

n	$M(n)$
4	1.682149231
8	1.663648208
16	1.659636470

8.8.75

a. The exact value is $\int_0^4 x^3 dx = \frac{x^4}{4} \Big|_0^4 = 64$. Simpson's rule gives $S(2) = (0 + 4 \cdot 8 + 64) \frac{2}{3} = 64$. The values match exactly.

b. $S(4) = (0 + 4 \cdot 1 + 2 \cdot 8 + 4 \cdot 27 + 64) \frac{1}{3} = 64$. This also matches the exact value exactly.

c. The 4th derivative of x^3 is 0, so the value of K in the theorem can be taken to be 0, so the error is 0.

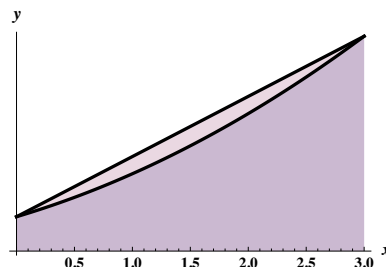
d. Any polynomial of degree 3 or less has 4th derivative equal to 0, so the value of K in the theorem can be taken to be 0, so the error is 0.

8.8.76 Let $x_j = a + j(b-a)/2n$ for $1 \leq j \leq 2n$ and let $x_i = a + i(b-a)/n$ and let $\bar{x}_i$ be the midpoint of the interval $[x_{i-1}, x_i]$. Then

$$\begin{aligned} T(2n) &= \frac{b-a}{4n} (f(a) + 2 \sum_{j=1}^{2n-1} f(x_j) + f(b)) \\ &= \frac{1}{2} \cdot \frac{b-a}{2n} (f(a) + 2 \sum_{1 \leq j \leq 2n-1, j \text{ even}} f(x_j) + f(b) + 2 \sum_{1 \leq j \leq 2n-1, j \text{ odd}} f(x_j)) \\ &= \frac{1}{2} \left(\frac{b-a}{2n} (f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b)) + \frac{b-a}{n} \sum_{i=1}^n f(\bar{x}_i) \right) = \frac{1}{2} (T(n) + M(n)). \end{aligned}$$

8.8.77

The trapezoidal rule will be an overestimate in this case. This is because of the fact that if the function is above the axis and concave up on the given interval, then each trapezoid on each subinterval lies over the area under the curve for that corresponding subinterval.



8.8.78 Let $x_i = a + i(b-a)/n$ for $1 \leq i \leq n$ and let $x_j = a + j(b-a)/2n$ for $1 \leq j \leq 2n$. We have

$$\begin{aligned} 4T(2n) - T(n) &= \frac{b-a}{2n} (2(f(a) + 2 \sum_{j=1}^{2n-1} f(x_j) + f(b)) - (f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b))) \\ &= \frac{b-a}{2n} ((f(a) + 4 \sum_{1 \leq j \leq 2n-1, j \text{ odd}} f(x_j) + 4 \sum_{2 \leq j \leq 2n-2, j \text{ even}} f(x_j) - 2 \sum_{i=1}^{n-1} f(x_i) + f(b))) \\ &= \frac{b-a}{2n} (f(a) + 4 \sum_{1 \leq j \leq 2n-1, j \text{ odd}} f(x_j) + 4 \sum_{2 \leq j \leq 2n-2} f(x_j) - 2 \sum_{2 \leq j \leq 2n-1} f(x_j) + f(b)) \\ &= \frac{b-a}{2n} (f(a) + 4 \sum_{1 \leq j \leq 2n-1, j \text{ odd}} f(x_j) + 2 \sum_{2 \leq j \leq 2n-2, j \text{ even}} f(x_j) + f(b)) = 3S(2n). \end{aligned}$$

8.8.79 Using the given formula, we have

$$S(2n) = \frac{2M(n) + T(n)}{3}.$$

$$\text{So } S(20) = \frac{1}{3} (2M(10) + T(10)) \approx 1.000001.$$

8.9 Improper Integrals

8.9.1 The interval of integration is infinite or the integrand is unbounded on the interval of integration.

8.9.2 $\lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x^3}$. We have

$$\lim_{b \rightarrow \infty} -\frac{1}{2} \left(\frac{1}{x^2} \right) \Big|_2^b = \lim_{b \rightarrow \infty} -\frac{1}{2} \left(\frac{1}{b^2} - \frac{1}{4} \right) = \frac{1}{8}.$$

8.9.3 $\lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x^{1/5}}$. We have

$$\lim_{b \rightarrow \infty} \left. \frac{5}{4} \cdot x^{4/5} \right|_2^b = \lim_{b \rightarrow \infty} \left(\frac{5}{4} \cdot b^{4/5} - \frac{5}{4} \cdot 2^{4/5} \right) = \infty.$$

The integral diverges.

8.9.4 $\int_0^1 \frac{1}{x^{1/5}} dx = \lim_{b \rightarrow 0^+} \int_b^1 \frac{dx}{x^{1/5}}$. We have

$$\lim_{b \rightarrow 0^+} \left. \frac{5}{4} \cdot x^{4/5} \right|_b^1 = \lim_{b \rightarrow 0^+} \left(\frac{5}{4} - \frac{5}{4} \cdot b^{4/5} \right) = \frac{5}{4}.$$

8.9.5 This is $\int_{-\infty}^{\infty} f(x) dx$.

8.9.6 As shown in Example 2, this integral converges if and only if $p > 1$.

8.9.7 $\int_3^{\infty} x^{-2} dx = \lim_{b \rightarrow \infty} \int_3^b x^{-2} dx = \lim_{b \rightarrow \infty} \left(-\frac{1}{x} \right) \Big|_3^b = \lim_{b \rightarrow \infty} \left(\frac{1}{3} - \frac{1}{b} \right) = \frac{1}{3}$.

8.9.8 $\lim_{b \rightarrow \infty} \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln x \Big|_2^b = \lim_{b \rightarrow \infty} (\ln b - \ln 2) = \infty$. This integral diverges.

8.9.9 $\int_2^{\infty} \frac{dx}{\sqrt{x}} = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{\sqrt{x}} = \lim_{b \rightarrow \infty} 2\sqrt{x} \Big|_2^b = \lim_{b \rightarrow \infty} 2(\sqrt{b} - \sqrt{2}) = \infty$, so the integral diverges.

8.9.10 $\int_0^{\infty} e^{-2x} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-2x} dx = \lim_{b \rightarrow \infty} \left(-\frac{1}{2} e^{-2x} \right) \Big|_0^b = \lim_{b \rightarrow \infty} \frac{1}{2} (1 - e^{-2b}) = \frac{1}{2}$.

8.9.11 $\int_0^{\infty} e^{-ax} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-ax} dx = \lim_{b \rightarrow \infty} \left. \frac{e^{-ax}}{-a} \right|_0^b = \lim_{b \rightarrow \infty} \left(-\frac{1}{ae^{ab}} + \frac{1}{a} \right) = \frac{1}{a}$.

8.9.12 $\lim_{b \rightarrow -\infty} \int_b^{-1} \frac{dx}{\sqrt[3]{x}} = \lim_{b \rightarrow -\infty} \left. \frac{3}{2} x^{2/3} \right|_b^{-1} = \lim_{b \rightarrow -\infty} \frac{3}{2} (1 - b^{2/3}) = -\infty$. This integral diverges.

8.9.13 $\int_0^{\infty} \cos x dx = \lim_{b \rightarrow \infty} \int_0^b \cos x dx = \lim_{b \rightarrow \infty} \sin x \Big|_0^b = \lim_{b \rightarrow \infty} \sin b$, which does not exist so the integral diverges.

8.9.14 $\lim_{b \rightarrow -\infty} \int_b^{-1} \frac{dx}{x^3} = \lim_{b \rightarrow -\infty} \left(-\frac{1}{2x^2} \right) \Big|_b^{-1} = \lim_{b \rightarrow -\infty} \left(-\frac{1}{2} + \frac{1}{2b^2} \right) = -\frac{1}{2}$.

8.9.15 The given integral is equal to $\lim_{b \rightarrow -\infty} \int_b^0 \frac{dx}{x^2 + 100} + \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2 + 100}$, if both limits exist.

$$\lim_{b \rightarrow -\infty} \int_b^0 \frac{dx}{x^2 + 100} = \lim_{b \rightarrow -\infty} \left. \frac{1}{10} \tan^{-1} \frac{x}{10} \right|_b^0 = \frac{1}{10} \left(0 + \frac{\pi}{2} \right) = \frac{\pi}{20}.$$

$$\lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2 + 100} = \lim_{b \rightarrow \infty} \left. \frac{1}{10} \tan^{-1} \frac{x}{10} \right|_0^b = \frac{1}{10} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{20}.$$

So the given integral has value $\frac{\pi}{20} + \frac{\pi}{20} = \frac{\pi}{10}$.

8.9.16 The given integral is equal to $\lim_{b \rightarrow -\infty} \int_b^0 \frac{dx}{x^2 + a^2} + \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2 + a^2}$, if both limits exist.

$$\lim_{b \rightarrow -\infty} \int_b^0 \frac{dx}{x^2 + a^2} = \lim_{b \rightarrow -\infty} \frac{1}{a} \tan^{-1} \frac{x}{a} \Big|_b^0 = \frac{1}{a} \left(0 + \frac{\pi}{2} \right) = \frac{\pi}{2a}.$$

$$\lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2 + a^2} = \lim_{b \rightarrow \infty} \frac{1}{a} \tan^{-1} \frac{x}{a} \Big|_0^b = \frac{1}{a} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{2a}.$$

So the given integral has value $\frac{\pi}{2a} + \frac{\pi}{2a} = \frac{\pi}{a}$.

8.9.17 $\lim_{b \rightarrow \infty} \int_7^b \frac{dx}{(x+1)^{1/3}} = \lim_{b \rightarrow \infty} \frac{3}{2} (x+1)^{2/3} \Big|_7^b = \lim_{b \rightarrow \infty} \left(\frac{3}{2} (b+1)^{2/3} - \frac{3}{2} 7^{2/3} \right) = \infty$.

The integral diverges.

8.9.18 $\int_2^\infty \frac{dx}{(x+2)^2} = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{(x+2)^2} = \lim_{b \rightarrow \infty} -\frac{1}{x+2} \Big|_2^b = \lim_{b \rightarrow \infty} -\frac{1}{b+2} + \frac{1}{4} = \frac{1}{4}$.

8.9.19 Let $u = x^3 + x$ so that $du = (3x^2 + 1) dx$ Substituting gives

$$\int_2^\infty \frac{1}{u} du = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{u} du = \lim_{b \rightarrow \infty} \ln u \Big|_2^b = \lim_{b \rightarrow \infty} (\ln b - \ln 2) = \infty.$$

The given integral diverges.

8.9.20 $\int_1^\infty 2^{-x} dx = \lim_{b \rightarrow \infty} \int_1^b 2^{-x} dx = \lim_{b \rightarrow \infty} \left(\frac{-1}{\ln 2(2^x)} \right) \Big|_1^b = \lim_{b \rightarrow \infty} \left(\frac{-1}{\ln 2(2^b)} + \frac{1}{2 \ln 2} \right) = \frac{1}{2 \ln 2}$.

8.9.21 $\int_2^\infty \frac{\cos(\pi/x)}{x^2} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{\cos(\pi/x)}{x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{1}{\pi} \sin(\pi/x) \right) \Big|_2^b = \lim_{b \rightarrow \infty} \frac{1}{\pi} (1 - \sin(\pi/b)) = \frac{1}{\pi}$.

8.9.22 Note that $\int \frac{1}{z^2} \sin(\pi/z) dz = -\frac{1}{\pi} \int \sin u du = \frac{1}{\pi} \cos u + C = \frac{1}{\pi} \cos(\pi/z) + C$. Thus,

$$\lim_{b \rightarrow -\infty} \int_b^{-2} \frac{1}{z^2} \sin(\pi/z) dz = \lim_{b \rightarrow -\infty} \frac{1}{\pi} \cos(\pi/z) \Big|_b^{-2} = \lim_{b \rightarrow -\infty} \frac{1}{\pi} (0 - \cos(\pi/b)) = -\frac{1}{\pi}.$$

8.9.23 $\int_0^\infty \frac{e^u}{e^{2u} + 1} du = \lim_{b \rightarrow \infty} \int_0^b \frac{e^u}{e^{2u} + 1} du$. Let $u = e^u$ so that $du = e^u du$. After substitution we have

$$\lim_{b \rightarrow \infty} \int_1^{e^b} \frac{1}{u^2 + 1} du = \lim_{b \rightarrow \infty} (\tan^{-1}(u)) \Big|_1^{e^b} = \lim_{b \rightarrow \infty} (\tan^{-1}(e^b) - \tan^{-1}(1)) = \pi/2 - \pi/4 = \pi/4.$$

8.9.24 $\int_{-\infty}^a \sqrt{e^x} dx = \lim_{b \rightarrow -\infty} \int_b^a e^{x/2} dx = \lim_{b \rightarrow -\infty} \left(2e^{x/2} \right) \Big|_b^a = \lim_{b \rightarrow -\infty} 2(e^{a/2} - e^{b/2}) = 2e^{a/2}$.

8.9.25 $\int_{-\infty}^\infty \frac{e^{3x}}{1 + e^{6x}} dx = \lim_{a \rightarrow -\infty} \int_a^0 \frac{e^{3x}}{1 + e^{6x}} dx + \lim_{b \rightarrow \infty} \int_0^b \frac{e^{3x}}{1 + e^{6x}} dx$. Let $u = e^{3x}$ so that $du = 3e^{3x} dx$. Note that

$$\int \frac{e^{3x}}{1 + e^{6x}} dx = \frac{1}{3} \int \frac{du}{1 + u^2} = \frac{1}{3} \tan^{-1} u + C = \frac{1}{3} \tan^{-1} e^{3x} + C.$$

So we have

$$\frac{1}{3} \lim_{a \rightarrow -\infty} (\tan^{-1} 1 - \tan^{-1} e^{3a}) + \frac{1}{3} \lim_{b \rightarrow \infty} (\tan^{-1} e^{3b} - \tan^{-1} 1) = \frac{1}{3} \left(\frac{\pi}{4} - 0 \right) + \frac{1}{3} \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi}{6}.$$

8.9.26 $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2} dx = \lim_{a \rightarrow -\infty} \int_a^0 \frac{x}{(x^2+1)^2} dx + \lim_{b \rightarrow \infty} \int_0^b \frac{x}{(x^2+1)^2} dx$. Let $u = x^2 + 1$ so that $du = 2x dx$. Note that

$$\int \frac{x}{(x^2+1)^2} dx = \frac{1}{2} \int u^{-2} du = -\frac{1}{2} \cdot \frac{1}{u} + C = -\frac{1}{2} \cdot \frac{1}{x^2+1} + C.$$

Then our integral is given by

$$\lim_{a \rightarrow -\infty} -\frac{1}{2} \left(1 - \frac{1}{a^2}\right) + \lim_{b \rightarrow \infty} -\frac{1}{2} \left(\frac{1}{b^2+1} - 1\right) = -\frac{1}{2} + \frac{1}{2} = 0.$$

8.9.27

$$\begin{aligned} \int_{-\infty}^{\infty} x e^{-x^2} dx &= \lim_{b \rightarrow -\infty} \int_b^0 x e^{-x^2} dx + \lim_{b \rightarrow \infty} \int_0^b x e^{-x^2} dx \\ &= \lim_{b \rightarrow -\infty} \left(-\frac{1}{2} e^{-x^2}\right) \Big|_b^0 + \lim_{b \rightarrow \infty} \left(-\frac{1}{2} e^{-x^2}\right) \Big|_0^b \\ &= \lim_{b \rightarrow -\infty} \left(-\frac{1}{2} + \frac{1}{2} e^{-b^2}\right) + \lim_{b \rightarrow \infty} \left(-\frac{1}{2} e^{-b^2} + \frac{1}{2}\right) = -\frac{1}{2} + \frac{1}{2} = 0. \end{aligned}$$

8.9.28 $\int_1^{\infty} \frac{\tan^{-1} s}{s^2+1} ds = \lim_{b \rightarrow \infty} \int_1^b \frac{\tan^{-1} s}{s^2+1} ds = \lim_{b \rightarrow \infty} \left(\frac{1}{2} (\tan^{-1} s)^2\right) \Big|_1^b = \lim_{b \rightarrow \infty} \frac{1}{2} \left((\tan^{-1} b)^2 - \left(\frac{\pi}{4}\right)^2\right) = \frac{1}{2} \left(\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right) = \frac{3\pi^2}{32}.$

8.9.29 By symmetry, the given integral is equal to $2 \int_0^{\infty} \frac{(\tan^{-1} t)^2}{t^2+1} dt$ if the integral exists. Let $u = \tan^{-1} t$ so that $du = \frac{dt}{t^2+1}$. We have

$$2 \lim_{b \rightarrow \infty} \int_0^{\tan^{-1} b} u^2 du = 2 \lim_{b \rightarrow \infty} \frac{u^3}{3} \Big|_0^{\tan^{-1} b} = 2 \lim_{b \rightarrow \infty} \frac{(\tan^{-1} b)^3}{3} = \frac{2(\pi/2)^3}{3} = \frac{\pi^3}{12}.$$

8.9.30 $\int_{-\infty}^0 e^x dx = \lim_{b \rightarrow -\infty} \int_b^0 e^x dx = \lim_{b \rightarrow -\infty} e^x \Big|_b^0 = \lim_{b \rightarrow -\infty} (1 - e^b) = 1.$

8.9.31 $\int_1^{\infty} \frac{1}{v(v+1)} dv = \lim_{b \rightarrow \infty} \int_1^b \left(\frac{1}{v} - \frac{1}{v+1}\right) dv = \lim_{b \rightarrow \infty} \ln \left| \frac{v}{v+1} \right| \Big|_1^b = \lim_{b \rightarrow \infty} \ln \left| \frac{b}{b+1} \right| - \ln(1/2) = 0 - \ln(1/2) = \ln 2.$

8.9.32 Write

$$\int_1^{\infty} \frac{1}{x^2(x-1)} dx = \int_1^2 \frac{1}{x^2(x-1)} dx + \int_2^{\infty} \frac{1}{x^2(x-1)} dx,$$

if both integrals converge.

However,

$$\begin{aligned} \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{x^2(x-1)} dx &= \lim_{c \rightarrow 1^+} \int_c^2 \left(-\frac{1}{x^2} + \frac{1}{x-1} - \frac{1}{x}\right) dx \\ &= \lim_{c \rightarrow 1^+} \left(\frac{1}{x} + \ln \left| \frac{x-1}{x} \right| \right) \Big|_c^2 = \lim_{c \rightarrow 1^+} \left(\frac{1}{2} + \ln \left(\frac{1}{2}\right) - \frac{1}{c} - \ln \left(\frac{c-1}{c}\right)\right) \end{aligned}$$

which diverges. Therefore, $\int_1^{\infty} \frac{1}{x^2(x-1)} dx$ diverges.

8.9.33 $\int_2^\infty \frac{dy}{y \ln y} = \lim_{b \rightarrow \infty} \int_2^b \frac{dy}{y \ln y} = \lim_{b \rightarrow \infty} (\ln(\ln y)) \Big|_2^b = \lim_{b \rightarrow \infty} (\ln(\ln b) - \ln(\ln 2)) = \infty$, so the integral diverges.

8.9.34 $\int_{-\infty}^{-4/\pi} \frac{\sec^2(1/x)}{x^2} dx = \lim_{b \rightarrow -\infty} \int_b^{-4/\pi} \frac{\sec^2(1/x)}{x^2} dx$. Let $u = 1/x$ so that $du = -1/x^2 dx$. Then we have $\lim_{b \rightarrow -\infty} \int_{1/b}^{-\pi/4} (-\sec^2 u) du = - \lim_{b \rightarrow -\infty} \tan u \Big|_{1/b}^{-\pi/4} = - \lim_{b \rightarrow -\infty} (-1 - \tan(1/b)) = 1$.

8.9.35

$$\begin{aligned} \int_{-\infty}^0 \frac{dx}{\sqrt[3]{2-x}} &= \lim_{b \rightarrow -\infty} \int_b^0 (2-x)^{-1/3} dx = \lim_{b \rightarrow -\infty} \left(-\frac{3}{2} (2-x)^{2/3} \right) \Big|_b^0 \\ &= \lim_{b \rightarrow -\infty} \left(\frac{3}{2} (-2^{2/3} + (2-b)^{2/3}) \right) = \infty, \end{aligned}$$

so the integral diverges.

8.9.36

$$\int_{e^2}^\infty \frac{dx}{x \ln^p x} = \lim_{b \rightarrow \infty} \int_{e^2}^b \frac{dx}{x \ln^p x} = \lim_{b \rightarrow \infty} \left(\frac{1}{1-p} \ln^{1-p} x \right) \Big|_{e^2}^b = \lim_{b \rightarrow \infty} \frac{1}{p-1} (2^{1-p} - \ln^{1-p} b) = \frac{1}{(p-1)2^{p-1}}.$$

$$\textbf{8.9.37} \quad \int_0^8 \frac{dx}{\sqrt[3]{x}} = \lim_{c \rightarrow 0^+} \int_c^8 x^{-1/3} dx = \lim_{c \rightarrow 0^+} \left(\frac{3}{2} x^{2/3} \right) \Big|_c^8 = \frac{3}{2} \lim_{c \rightarrow 0^+} (4 - c^{2/3}) = 6.$$

$$\textbf{8.9.38} \quad \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{\sqrt{x-1}} dx = \lim_{c \rightarrow 1^+} (2\sqrt{x-1}) \Big|_c^2 = \lim_{c \rightarrow 1^+} (2 - 2\sqrt{c-1}) = 2.$$

$$\textbf{8.9.39} \quad \int_0^{\pi/2} \tan \theta d\theta = \lim_{c \rightarrow \pi/2^-} \int_0^c \tan \theta d\theta = \lim_{c \rightarrow \pi/2^-} (\ln \sec \theta) \Big|_0^c = \lim_{c \rightarrow \pi/2^-} \ln \sec c = \infty, \text{ so the integral diverges.}$$

$$\textbf{8.9.40} \quad \lim_{c \rightarrow -3^+} \int_c^1 (2x+6)^{-2/3} dx = \lim_{c \rightarrow -3^+} \left(\frac{3}{2} \sqrt[3]{2x+6} \right) \Big|_c^1 = \lim_{c \rightarrow -3^+} \frac{3}{2} (2 - \sqrt[3]{2c+6}) = 3.$$

$$\textbf{8.9.41} \quad \lim_{b \rightarrow (\pi/2)^-} \int_0^b \sec x \tan x dx = \lim_{b \rightarrow (\pi/2)^-} (\sec x) \Big|_0^b = \lim_{b \rightarrow (\pi/2)^-} (\sec b - 1) = \infty. \text{ The given integral diverges.}$$

$$\textbf{8.9.42} \quad \lim_{c \rightarrow 3^+} \int_c^4 (z-3)^{-3/2} dz = \lim_{c \rightarrow 3^+} \left(-2(z-3)^{-1/2} \right) \Big|_c^4 = \lim_{c \rightarrow 3^+} \left(-2 - (-2(c-3)^{-1/2}) \right) = \infty. \text{ The given integral diverges.}$$

$$\textbf{8.9.43} \quad \text{Note that } \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2e^{\sqrt{x}} + C. \text{ Thus,}$$

$$\lim_{c \rightarrow 0^+} \int_c^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \lim_{c \rightarrow 0^+} \left(2e^{\sqrt{x}} \right) \Big|_c^1 = \lim_{c \rightarrow 0^+} (2e - 2e^{\sqrt{c}}) = 2e - 2.$$

8.9.44 Use the substitution $u = e^y - 1$ so that $du = e^y dy$. Then $y = 0$ corresponds to $u = 0$ and $y = \ln 3$ corresponds to $u = 2$, so we have

$$\int_0^{\ln 3} \frac{e^y}{(e^y - 1)^{2/3}} dy = \int_0^2 u^{-2/3} du = \lim_{b \rightarrow 0^+} \int_b^2 u^{-2/3} du = \lim_{b \rightarrow 0^+} 3u^{1/3} \Big|_b^2 = \lim_{b \rightarrow 0^+} (3 \cdot 2^{1/3} - 3b^{1/3}) = 3 \cdot 2^{1/3}.$$

8.9.45 $\int_0^1 \frac{x^3}{x^4-1} dx = \lim_{c \rightarrow 1^-} \int_0^c \frac{x^3}{x^4-1} dx = \lim_{c \rightarrow 1^-} \left(\frac{1}{4} \ln |x^4-1| \right) \Big|_0^c = \frac{1}{4} \lim_{c \rightarrow 1^-} \ln |c^4-1| = -\infty$, so the integral diverges.

8.9.46 This integral is improper at both limits, so we split it as $\int_1^\infty \frac{dx}{\sqrt[3]{x-1}} = \int_1^2 \frac{dx}{\sqrt[3]{x-1}} + \int_2^\infty \frac{dx}{\sqrt[3]{x-1}}$. If we let $u = x-1$ in the second integral on the right we obtain

$$\int_2^\infty \frac{dx}{\sqrt[3]{x-1}} = \int_1^\infty \frac{du}{u^{1/3}},$$

which diverges using the result in Example 2. Therefore the original integral diverges.

8.9.47

$$\begin{aligned} \int_0^{10} \frac{dx}{\sqrt[4]{10-x}} &= \lim_{c \rightarrow 10^-} \int_0^c (10-x)^{-1/4} dx = \lim_{c \rightarrow 10^-} \left(-\frac{4}{3} (10-x)^{3/4} \right) \Big|_0^c \\ &= \frac{4}{3} \lim_{c \rightarrow 10^-} \left(10^{3/4} - (10-c)^{3/4} \right) = \frac{4}{3} 10^{3/4}. \end{aligned}$$

8.9.48 This integral is improper at the point $x = 3$, so we split it as $\int_1^{11} \frac{dx}{(x-3)^{2/3}} = \int_1^3 \frac{dx}{(x-3)^{2/3}} + \int_3^{11} \frac{dx}{(x-3)^{2/3}}$ and evaluate each integral separately:

$$\int_1^3 \frac{dx}{(x-3)^{2/3}} = \lim_{c \rightarrow 3^-} \int_1^c (x-3)^{-2/3} dx = \lim_{c \rightarrow 3^-} \left(3(x-3)^{1/3} \right) \Big|_1^c = 3 \lim_{c \rightarrow 3^-} \left(2^{1/3} - (3-c)^{1/3} \right) = 3 \cdot 2^{1/3},$$

and

$$\int_3^{11} \frac{dx}{(x-3)^{2/3}} = \lim_{c \rightarrow 3^+} \int_c^{11} (x-3)^{-2/3} dx = \lim_{c \rightarrow 3^+} \left(3(x-3)^{1/3} \right) \Big|_c^{11} = 3 \lim_{c \rightarrow 3^+} \left(8^{1/3} - (c-3)^{1/3} \right) = 6,$$

$$\text{so } \int_1^{11} \frac{dx}{(x-3)^{2/3}} = 6 + 3 \cdot 2^{1/3}.$$

8.9.49 The given integrand isn't defined for $x = 1$, so we write the given integral as

$$\int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}.$$

We will show that the given integral diverges by showing that $\int_0^1 \frac{dx}{(x-1)^2}$ diverges. We have

$$\lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{(x-1)^2} = \lim_{b \rightarrow 1^-} \left(-\frac{1}{x-1} \right) \Big|_0^b = \lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} + 1 \right) = \infty.$$

8.9.50 We can write this as

$$\begin{aligned} \int_0^1 \frac{1}{(x-1)^{1/3}} dx + \int_1^9 \frac{1}{(x-1)^{1/3}} dx &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^{1/3}} dx + \lim_{c \rightarrow 1^+} \int_c^9 \frac{1}{(x-1)^{1/3}} dx \\ &= \lim_{b \rightarrow 1^-} \frac{3}{2} \left((x-1)^{2/3} \right) \Big|_0^b + \lim_{c \rightarrow 1^+} \frac{3}{2} \left((x-1)^{2/3} \right) \Big|_c^9 = -\frac{3}{2} + 6 = \frac{9}{2}. \end{aligned}$$

8.9.51 By symmetry,

$$\begin{aligned}\int_{-2}^2 \frac{dp}{\sqrt{4-p^2}} &= 2 \int_0^2 \frac{dp}{\sqrt{4-p^2}} = 2 \lim_{c \rightarrow 2^-} \int_0^c \frac{dp}{\sqrt{4-p^2}} \\ &= 2 \lim_{c \rightarrow 2^-} \left(\sin^{-1}(p/2) \right) \Big|_0^c = 2 \lim_{c \rightarrow 2^-} (\sin^{-1}(c/2) - \sin^{-1} 0) \\ &= 2(\sin^{-1} 1 - 0) = \pi.\end{aligned}$$

8.9.52 Integration by parts gives $\int x e^{-x} dx = -(x+1)e^{-x} + C$, so $\int_0^\infty x e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b x e^{-x} dx = \lim_{b \rightarrow \infty} (-(x+1)e^{-x}) \Big|_0^b = \lim_{b \rightarrow \infty} (1 - (b+1)e^{-b}) = 1$.

8.9.53 Integration by parts gives $\int \ln x dx = x \ln x - x + C$, so

$$\int_0^1 \ln x dx = \lim_{c \rightarrow 0^+} \int_c^1 \ln x dx = \lim_{c \rightarrow 0^+} (x \ln x - x) \Big|_c^1 = \lim_{c \rightarrow 0^+} (-1 - (c \ln c - c)) = -1,$$

because $\lim_{c \rightarrow 0^+} c \ln c = 0$.

8.9.54 Integration by parts gives $\int \frac{\ln x}{x^2} dx = -\frac{\ln x + 1}{x} + C$, so

$$\int_1^\infty \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{\ln x + 1}{x} \right) \Big|_1^b = \lim_{b \rightarrow \infty} \left(1 - \frac{\ln b + 1}{b} \right) = 1,$$

because $\lim_{b \rightarrow \infty} \frac{\ln b + 1}{b} = 0$.

8.9.55 Write the given integral as $\lim_{a \rightarrow 0^+} \int_a^{\ln 2} \frac{e^x}{\sqrt{e^{2x} - 1}} dx$ and then let $u = e^x$ so that $du = e^x dx$. Substituting gives $\lim_{a \rightarrow 0^+} \int_{e^a}^2 \frac{du}{\sqrt{u^2 - 1}}$. Then we let $u = \sec \theta$ with $du = \sec \theta \tan \theta d\theta$. Substituting again gives

$$\begin{aligned}\lim_{a \rightarrow 0^+} \int_{\sec^{-1} e^a}^{\pi/3} \sec \theta d\theta &= \lim_{a \rightarrow 0^+} \ln |\sec \theta + \tan \theta| \Big|_{\sec^{-1} e^a}^{\pi/3} \\ &= \lim_{a \rightarrow 0^+} \left(\ln(2 + \sqrt{3}) - \ln(e^a + \sqrt{e^{2a} - 1}) \right) = \ln(2 + \sqrt{3}).\end{aligned}$$

8.9.56 Write the given integral as $\lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{x + \sqrt{x}} = \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{(\sqrt{x} + 1)\sqrt{x}}$. Let $u = \sqrt{x} + 1$ so that $du = \frac{1}{2\sqrt{x}} dx$. Substituting gives

$$\lim_{a \rightarrow 0^+} \int_{1+\sqrt{a}}^2 \frac{2du}{u} = \lim_{a \rightarrow 0^+} 2 \ln u \Big|_{1+\sqrt{a}}^2 = \lim_{a \rightarrow 0^+} (2 \ln 2 - 2 \ln(1 + \sqrt{a})) = 2 \ln 2.$$

8.9.57 The function $e^{-|x|}$ is even, so $\int_{-\infty}^\infty e^{-|x|} dx = 2 \int_0^\infty e^{-x} dx$. We have

$$2 \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx = 2 \lim_{b \rightarrow \infty} -e^{-x} \Big|_0^b = 2 \lim_{b \rightarrow \infty} (-e^{-b} + 1) = 2.$$

8.9.58 Note that $x^2 + 2x + 5 = (x^2 + 2x + 1) + 4 = (x + 1)^2 + 4$. Also recall that $\int \frac{1}{(x + 1)^2 + 4} dx = \frac{1}{2} \tan^{-1} \left(\frac{x + 1}{2} \right) + C$.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{(x + 1)^2 + 4} &= \lim_{b \rightarrow -\infty} \int_b^0 \frac{dx}{(x + 1)^2 + 4} + \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{(x + 1)^2 + 4} \\ &= \lim_{b \rightarrow -\infty} \left. \frac{1}{2} \tan^{-1} \left(\frac{x + 1}{2} \right) \right|_b^0 + \lim_{b \rightarrow \infty} \left. \frac{1}{2} \tan^{-1} \left(\frac{x + 1}{2} \right) \right|_0^b \\ &= \lim_{b \rightarrow -\infty} \frac{1}{2} \left(\tan^{-1}(1/2) - \tan^{-1} \left(\frac{b + 1}{2} \right) \right) + \lim_{b \rightarrow \infty} \frac{1}{2} \left(\tan^{-1} \left(\frac{b + 1}{2} \right) - \tan^{-1}(1/2) \right) \\ &= \frac{1}{2} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\pi}{2}. \end{aligned}$$

8.9.59 We have the relation $B = I \int_0^{\infty} e^{-rt} dt = \frac{I}{r}$, using the result that $\int_0^{\infty} e^{-ax} dx = \frac{1}{a}$ for $a > 0$. Therefore $B = \frac{5000}{0.12} = \$41,666.67$.

8.9.60 The total amount of water drained is $W = 100 \int_0^{\infty} e^{-0.05t} dt = \frac{100}{0.05} = 2000$ gal (here we use the fact that $\int_0^{\infty} e^{-ax} dx = \frac{1}{a}$ for $a > 0$).

8.9.61 As in Example 6, we have

$$\text{AUC}_i = \int_0^{\infty} C_i(t) dt = 250 \int_0^{\infty} e^{-0.08t} dt = \frac{250}{0.08} = 3125$$

and

$$\text{AUC}_o = \int_0^{\infty} C_o(t) dt = 200 \int_0^{\infty} (e^{-0.08t} - e^{-1.8t}) dt = 200 \left(\frac{1}{0.08} - \frac{1}{1.8} \right) = \frac{21,500}{9} \approx 2389,$$

(here we use the fact that $\int_0^{\infty} e^{-ax} dx = \frac{1}{a}$ for $a > 0$). Therefore the bioavailability of the drug is

$$F = \frac{\text{AUC}_o}{\text{AUC}_i} = \frac{21,500}{9 \cdot 3125} \approx 0.764.$$

8.9.62

- We will make use of the result $\int_b^{\infty} e^{-at} dt = \frac{e^{-ab}}{a}$ for $a > 0$ (this is derived in problem 73). The probability that a chip fails after 15,000 hours of operation (or equivalently, lasts at least 15,000 hours) is $p = 0.00005 \int_{15,000}^{\infty} e^{-0.00005t} dt = e^{-0.00005 \cdot 15,000} \approx 0.472$.
- The probability that a chip fails after 30,000 hours of operation is $p = 0.00005 \int_{30,000}^{\infty} e^{-0.00005t} dt = e^{-0.00005 \cdot 30,000} \approx 0.223$. Of the chips that are still operating at 15,000 hours, the fraction that operate for at least another 15,000 hours is $\approx 0.223/0.472 = 0.472$.
- From part a, we see that $0.00005 \int_0^{\infty} e^{-0.00005t} dt = 1$, which can be interpreted as meaning that all the chips are working initially.

8.9.63 Evaluate the improper integral

$$\int_0^\infty te^{-at} dt = \lim_{b \rightarrow \infty} \int_0^b te^{-at} dt = \lim_{b \rightarrow \infty} \left(-\frac{e^{-at}(at+1)}{a^2} \right) \Big|_0^b = \frac{1}{a^2} \lim_{b \rightarrow \infty} (1 - e^{-ab}(ab+1)) = \frac{1}{a^2},$$

provided $a > 0$. Therefore $0.00005 \int_0^\infty te^{-0.00005t} dt = \frac{0.00005}{0.00005^2} = 20,000$ hrs.

8.9.64 The maximum distance is

$$D = 10 \int_0^\infty (t+1)^{-2} dt = 10 \lim_{b \rightarrow \infty} \int_0^b (t+1)^{-2} dt = 10 \lim_{b \rightarrow \infty} \left(-(t+1)^{-1} \right) \Big|_0^b = 10 \text{ mi.}$$

8.9.65 Using the result from Example 2, we see that the volume is given by $V = \pi \int_1^\infty x^{-4} dx = \frac{\pi}{4-1} = \frac{\pi}{3}$.

8.9.66 $V = \pi \int_2^\infty \frac{dx}{x^2+1} = \pi \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x^2+1} = \pi \lim_{b \rightarrow \infty} (\tan^{-1} x) \Big|_2^b = \pi \left(\frac{\pi}{2} - \tan^{-1} 2 \right) \approx 1.457.$

8.9.67 Using the result from Example 2, we see that the volume is given by

$$V = \pi \int_1^\infty \left(\frac{1}{x^2} + \frac{1}{x^3} \right) dx = \frac{\pi}{2-1} + \frac{\pi}{3-1} = \frac{3\pi}{2}.$$

8.9.68 $V = 2\pi \int_0^\infty \frac{x}{(x+1)^3} dx$; using the method of partial fractions, we see that $\frac{x}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{1}{(x+1)^3}$, so

$$V = 2\pi \left(\int_0^\infty \frac{dx}{(x+1)^2} - \int_0^\infty \frac{dx}{(x+1)^3} \right).$$

Now make the substitution $u = x+1$ and use the result from Example 2 to obtain

$$V = 2\pi \left(\int_1^\infty \frac{du}{u^2} - \int_1^\infty \frac{du}{u^3} \right) = 2\pi \left(\frac{1}{2-1} - \frac{1}{3-1} \right) = \pi.$$

8.9.69 $V = \pi \int_2^\infty \frac{dx}{x(\ln x)^2} = \pi \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x(\ln x)^2} = \pi \lim_{b \rightarrow \infty} \left(-\frac{1}{\ln x} \right) \Big|_2^b = \pi \lim_{b \rightarrow \infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln b} \right) = \frac{\pi}{\ln 2}.$

8.9.70 The volume is given by

$$\begin{aligned} V &= \pi \int_0^\infty \frac{x}{(x^2+1)^{2/3}} dx = \pi \lim_{b \rightarrow \infty} \int_0^b \frac{x}{(x^2+1)^{2/3}} dx \\ &= \pi \lim_{b \rightarrow \infty} \left(\frac{3}{2}(x^2+1)^{1/3} \right) \Big|_0^b = \pi \lim_{b \rightarrow \infty} \frac{3}{2} \left((b^2+1)^{1/3} - 1 \right) = \infty, \end{aligned}$$

so the volume is infinite.

8.9.71 Using disks, we have

$$V = \pi \int_1^2 (x-1)^{-1/2} dx = \pi \lim_{c \rightarrow 1^+} \int_c^2 (x-1)^{-1/2} dx = \pi \lim_{c \rightarrow 1^+} \left(2(x-1)^{1/2} \right) \Big|_c^2 = 2\pi \lim_{c \rightarrow 1^+} (1 - \sqrt{c-1}) = 2\pi.$$

8.9.72 Using disks, we have

$$V = \pi \int_{-1}^1 \left(1 + (x+1)^{-3/2} \right)^2 dx = \pi \int_0^2 (1 + u^{-3/2})^2 du = \pi \int_0^2 (1 + 2u^{-3/2} + u^{-3}) du$$

via the substitution $u = x+1$. Therefore

$$V = \pi \lim_{c \rightarrow 0^+} \int_c^2 (1 + 2u^{-3/2} + u^{-3}) du = \pi \lim_{c \rightarrow 0^+} \left(u - 4u^{-1/2} - \frac{1}{2}u^{-2} \right) \Big|_c^2 = \infty,$$

so the volume is infinite.

8.9.73 $V = \pi \int_0^{\pi/2} \tan^2 x \, dx = \lim_{a \rightarrow \frac{\pi}{2}^-} \pi \int_0^a \tan^2 x \, dx = \lim_{a \rightarrow \frac{\pi}{2}^-} \pi \int_0^a (\sec^2 x - 1) \, dx = \lim_{a \rightarrow \frac{\pi}{2}^-} \pi (\tan x - x) \Big|_0^a = \lim_{a \rightarrow \frac{\pi}{2}^-} \pi (\tan a - a) = \infty$, so the volume doesn't exist.

8.9.74 First note that $\int \ln^2 x \, dx = 2x + x \ln^2 x - 2x \ln x$. This is obtained by integration by parts with $u = \ln x$ and $dv = \ln x \, dx$. This gives $du = \frac{1}{x} \, dx$ and $v = x \ln x - x$. Then we have $\int \ln^2 x \, dx = x \ln^2 x - x \ln x - \int (\ln x - 1) \, dx = x \ln^2 x - x \ln x - (x \ln x - x - x) + C = x \ln^2 x - 2x \ln x + 2x + C$.

Then the volume is $V = \pi \int_0^1 (-\ln x)^2 \, dx = \lim_{a \rightarrow 0^+} \pi \int_a^1 \ln^2 x \, dx = \lim_{a \rightarrow 0^+} \pi (x \ln^2 x - 2x \ln x + 2x) \Big|_a^1 = \lim_{a \rightarrow 0^+} \pi (2 - (a \ln^2 a - 2a \ln a + 2a)) = 2\pi$. Note that

$$\lim_{a \rightarrow 0^+} a \ln a = \lim_{a \rightarrow 0^+} \frac{\ln a}{1/a} = \lim_{a \rightarrow 0^+} \frac{1/a}{-1/a^2} = \lim_{a \rightarrow 0^+} -a = 0,$$

by l'Hôpital's rule. Also

$$\lim_{a \rightarrow 0^+} a \ln^2 a = \lim_{a \rightarrow 0^+} \frac{\ln^2 a}{1/a} = \lim_{a \rightarrow 0^+} \frac{2 \ln a / a}{-1/a^2} = \lim_{a \rightarrow 0^+} -2a \ln a = 0.$$

8.9.75 Using shells, we have

$$V = 2\pi \int_0^4 x(4-x)^{-1/3} \, dx = 2\pi \int_0^4 (4-u)u^{-1/3} \, du = 2\pi \int_0^4 (4u^{-1/3} - u^{2/3}) \, du$$

via the substitution $u = 4 - x$. Therefore

$$V = 2\pi \lim_{c \rightarrow 0^+} \int_c^4 (4u^{-1/3} - u^{2/3}) \, du = 2\pi \lim_{c \rightarrow 0^+} \left(6u^{2/3} - \frac{3}{5}u^{5/3} \right) \Big|_c^4 = \frac{72 \cdot 2^{1/3} \pi}{5}.$$

8.9.76 Using shells, we have

$$\begin{aligned} V &= 2\pi \int_1^2 x(x^2-1)^{-1/4} \, dx = 2\pi \lim_{c \rightarrow 1^+} \int_c^2 x(x^2-1)^{-1/4} \, dx \\ &= \pi \lim_{c \rightarrow 1^+} \left(\frac{4}{3}(x^2-1)^{3/4} \right) \Big|_c^2 = \frac{4}{3}\pi \lim_{c \rightarrow 1^+} (3^{3/4} - (c^2-1)^{3/4}) = \frac{4\pi}{3^{1/4}}. \end{aligned}$$

8.9.77 $x^3 + 1 > x^3$ for $x \geq 1$, so $\frac{1}{x^3+1} < \frac{1}{x^3}$ for $x \geq 1$. Because $\int_1^\infty \frac{1}{x^3} \, dx$ converges, $\int_1^\infty \frac{1}{x^3+1} \, dx$ converges as well by the Comparison Test. Note that the fact that $\int_1^\infty \frac{1}{x^3} \, dx$ converges to $\frac{1}{2}$ follows from Example 2, Case 1.

8.9.78 $e^x + x + 1 > e^x$ for $x \geq 1$, so $\frac{1}{e^x+x+1} < \frac{1}{e^x}$ for $x \geq 1$. Note that

$$\int_0^\infty \frac{dx}{e^x} = \lim_{b \rightarrow \infty} \int_0^b e^{-x} \, dx = \lim_{b \rightarrow \infty} -e^{-x} \Big|_0^b = \lim_{b \rightarrow \infty} (-e^{-b} + 1) = 1.$$

Thus $\int_0^\infty \frac{dx}{e^x+x+1}$ converges by the Comparison Test.

8.9.79 For $x \geq 3$, $0 < \ln x \leq x$, so $0 < \frac{1}{x} \leq \frac{1}{\ln x}$. Note that

$$\int_3^\infty \frac{dx}{x} = \lim_{b \rightarrow \infty} \int_3^b \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln x \Big|_3^b = \lim_{b \rightarrow \infty} (\ln b - \ln 3) = \infty.$$

Using the Comparison Test, because $\int_3^\infty \frac{dx}{x}$ diverges, $\int_3^\infty \frac{dx}{\ln x}$ diverges as well.

8.9.80 For $x \geq 2$, $0 < x^4 - x - 1 \leq x^4 - 1$, so $\frac{x^3}{x^4 - x - 1} \geq \frac{x^3}{x^4 - 1} > 0$. We will show that $\int_2^\infty \frac{x^3}{x^4 - 1} dx$ is divergent, and then conclude by the Comparison Test that $\int_2^\infty \frac{x^3}{x^4 - x - 1} dx$ is divergent. In order to show that $\int_2^\infty \frac{x^3}{x^4 - 1} dx$ is divergent, we let $u = x^4 - 1$ so that $du = 4x^3 dx$. Then we have

$$\lim_{b \rightarrow \infty} \int_{15}^{b^4-1} \frac{1}{4} \left(\frac{1}{u} \right) du = \lim_{b \rightarrow \infty} \frac{1}{4} \ln u \Big|_{15}^{b^4-1} = \frac{1}{4} \lim_{b \rightarrow \infty} (\ln(b^4 - 1) - \ln 15) = \infty.$$

8.9.81 $0 \leq \sin^2 x \leq 1$, so $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ for $x \geq 1$. By Example 2, $\int_1^\infty \frac{dx}{x^2}$ converges. Therefore $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ converges as well by the Comparison Test.

8.9.82 $e^x > 1$ for $x \geq 1$, so $e^x(1 + x^2) \geq 1 + x^2$ for $x \geq 1$. It follows that

$$0 \leq \frac{1}{e^x(1 + x^2)} \leq \frac{1}{1 + x^2},$$

for $x \geq 1$. We will show that $\int_1^\infty \frac{1}{1 + x^2} dx$ converges to $\frac{\pi}{4}$ and then conclude that $\int_1^\infty \frac{1}{e^x(1 + x^2)} dx$ converges. We have

$$\lim_{b \rightarrow \infty} \int_1^b \frac{1}{1 + x^2} dx = \lim_{b \rightarrow \infty} \tan^{-1} x \Big|_1^b = \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

8.9.83 $2 + \cos x \geq 1$, so $\frac{2 + \cos x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} \geq 0$. By Example 2, $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges, so $\int_1^\infty \frac{2 + \cos x}{\sqrt{x}} dx$ diverges by the Comparison Test.

8.9.84 $0 \leq 2 + \cos x \leq 3$, which implies that $0 \leq \frac{2 + \cos x}{x^2} \leq \frac{3}{x^2}$. By Example 2, $\int_1^\infty \frac{3}{x^2} dx$ converges, so $\int_1^\infty \frac{2 + \cos x}{x^2} dx$ converges as well by the Comparison Test.

8.9.85 $\sqrt{x^{1/3} + x} \geq \sqrt{x^{1/3}} = x^{1/6}$. So for $0 \leq x \leq 1$, $0 < \frac{1}{\sqrt{x^{1/3} + x}} \leq \frac{1}{x^{1/6}}$. We will show that $\int_0^1 \frac{dx}{x^{1/6}}$ converges and conclude that $\int_0^1 \frac{1}{\sqrt{x^{1/3} + x}} dx$ converges by the Comparison Test. We have

$$\lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{x^{1/6}} = \lim_{a \rightarrow 0^+} \frac{6}{5} x^{5/6} \Big|_a^1 = \lim_{a \rightarrow 0^+} \frac{6}{5} \left(1 - a^{5/6} \right) = \frac{6}{5}.$$

8.9.86 $\sin x + 1 \geq 1$, so $\frac{\sin x + 1}{x^5} \geq \frac{1}{x^5} > 0$ for all $x > 0$. We will show that $\int_0^1 \frac{1}{x^5} dx$ diverges, and then conclude that $\int_0^1 \frac{\sin x + 1}{x^5} dx$ diverges as well by the Comparison Test. We have

$$\lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x^5} dx = \lim_{a \rightarrow 0^+} -\frac{1}{4} \cdot \frac{1}{x^4} \Big|_a^1 = \lim_{a \rightarrow 0^+} -\frac{1}{4} \left(1 - \frac{1}{a^4} \right) = \infty.$$

8.9.87

- a. True. The area under the curve $y = f(x)$ from 0 to ∞ is less than the area under $y = g(x)$ on this interval, which by assumption is finite.

- b. False. For example, take $f(x) = 1$; then $\int_0^\infty f(x) dx = \infty$.
- c. False. For example, take $p = 1/2$ and $q = 1$.
- d. True. The area under the curve $y = x^{-q}$ from 1 to ∞ is less than the area under $y = x^{-p}$ on this interval, which by assumption is finite.
- e. True. Using the result in Example 2, we see that this integral exists if and only if $3p + 2 > 1$, which is equivalent to $p > -1/3$.

8.9.88

- a. The fundamental theorem cannot be applied because the function $\frac{1}{x}$ is not continuous or bounded on $[-1, 1]$.
- b. We will show that $\int_0^1 \frac{dx}{x}$ diverges, and then we may conclude that $\int_{-1}^1 \frac{dx}{x}$ diverges as well. We have

$$\lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{x} = \lim_{a \rightarrow 0^+} \ln x \Big|_a^1 = \lim_{a \rightarrow 0^+} (\ln 1 - \ln a) = \infty.$$

8.9.89 The region R has area $A = \int_0^\infty e^{-bx} dx - \int_0^\infty e^{-ax} dx = \frac{1}{b} - \frac{1}{a}$.

8.9.90 The region R has area $A = \int_1^\infty x^{-p} dx - \int_1^\infty x^{-q} dx = \frac{1}{p-1} - \frac{1}{q-1}$, where we use the result in Example 2.

8.9.91

- a. We have

$$A(a, b) = \int_b^\infty e^{-ax} dx = \lim_{c \rightarrow \infty} \int_b^c e^{-ax} dx = \lim_{c \rightarrow \infty} \left(-\frac{1}{a} e^{-ax} \right) \Big|_b^c = \frac{1}{a} \lim_{c \rightarrow \infty} (e^{-ab} - e^{-ac}) = \frac{e^{-ab}}{a}.$$

- b. Solving $e^{-ab} = 2a$ for b gives $b = g(a) = -\frac{1}{a} \ln 2a$.

- c. The function g has $g'(x) = \frac{1}{x^2} \ln 2x - \frac{1}{x^2} = \frac{\ln 2x - 1}{x^2}$, so g has a critical point at $x = e/2$, and the first derivative test shows that g takes a minimum at this point. Hence $b^* = g(e/2) = -\frac{2}{e}$.

8.9.92 We first rewrite the integrand as $e^{-x \ln x}(\ln x + 1)$, and note that if $u = x \ln x$, then $du = (\ln x + 1) dx$. Note that as $x \rightarrow 0^+$, $u \rightarrow 0^-$, and as $x \rightarrow \infty$, $u \rightarrow \infty$, and when $x = 1$, $u = 0$. We consider the integral as

$$\int_0^1 x^{-x}(\ln x + 1) dx = \int_0^0 e^{-u} du = 0.$$

Now we consider

$$\int_1^\infty x^{-x}(\ln x + 1) dx = \int_0^\infty e^{-u} du = \lim_{b \rightarrow \infty} \int_0^b e^{-u} du = \lim_{b \rightarrow \infty} -e^{-u} \Big|_0^b = -\lim_{b \rightarrow \infty} (e^{-b} - 1) = 1.$$

8.9.93 In the following, we will make the substitution $u = \sqrt{x-1}$ so that $du = \frac{dx}{2\sqrt{x-1}}$ and $x = u^2 + 1$.

$$\begin{aligned}
 \int_1^\infty \frac{dx}{x\sqrt{x-1}} &= \int_1^2 \frac{dx}{x\sqrt{x-1}} + \int_2^\infty \frac{dx}{x\sqrt{x-1}} \\
 &= \lim_{a \rightarrow 1^+} \int_a^2 \frac{dx}{x\sqrt{x-1}} + \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x\sqrt{x-1}} \\
 &= 2 \lim_{a \rightarrow 1^+} \int_{\sqrt{a-1}}^1 \frac{du}{u^2+1} + 2 \lim_{b \rightarrow \infty} \int_1^{\sqrt{b-1}} \frac{du}{u^2+1} \\
 &= 2 \lim_{a \rightarrow 1^+} \tan^{-1} u \Big|_{\sqrt{a-1}}^1 + 2 \lim_{b \rightarrow \infty} \tan^{-1} u \Big|_1^{\sqrt{b-1}} \\
 &= 2 \lim_{a \rightarrow 1^+} (\tan^{-1} 1 - \tan^{-1} \sqrt{a-1}) + 2 \lim_{b \rightarrow \infty} (\tan^{-1} \sqrt{b-1} - \tan^{-1} 1) \\
 &= 2 \left(\frac{\pi}{4} - 0 \right) + 2 \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \pi.
 \end{aligned}$$

8.9.94 First, assume $p \neq 1$. Then

$$\begin{aligned}
 \int_0^1 x^{-p} dx &= \lim_{c \rightarrow 0^+} \int_c^1 x^{-p} dx = \lim_{c \rightarrow 0^+} \left(\frac{x^{1-p}}{1-p} \right) \Big|_c^1 \\
 &= \frac{1}{1-p} \lim_{c \rightarrow 0^+} (1 - c^{1-p}) = \frac{1}{1-p}
 \end{aligned}$$

when $p < 1$ and is infinite otherwise. In the case $p = 1$ we have

$$\int_0^1 x^{-1} dx = \lim_{c \rightarrow 0^+} \int_c^1 x^{-1} dx = \lim_{c \rightarrow 0^+} (\ln x) \Big|_c^1 = \lim_{c \rightarrow 0^+} (-\ln c) = \infty,$$

so the integral exists if and only if $p < 1$.

8.9.95 These integrals cannot be evaluated by finding an antiderivative of their integrands. However, if we make the substitution $u = \pi/2 - x$ we find that $\int_0^{\pi/2} \ln \sin x dx = -\int_{\pi/2}^0 \ln \cos u du = \int_0^{\pi/2} \ln \cos x dx$. We also have

$$\int_{\pi/2}^\pi \ln \sin x dx = \int_0^{\pi/2} \ln \sin(x + \pi/2) dx = \int_0^{\pi/2} \ln \cos x dx,$$

and therefore

$$\int_0^{\pi/2} \ln \sin x dx = \frac{1}{2} \int_0^\pi \ln \sin x dx = \int_0^{\pi/2} \ln \sin 2y dy = \int_0^{\pi/2} (\ln \sin y + \ln \cos y + \ln 2) dy.$$

This implies $\int_0^{\pi/2} \ln \cos x dx = -\int_0^{\pi/2} \ln 2 dx = -\frac{\pi \ln 2}{2}$.

8.9.96 This integral cannot be evaluated by finding an antiderivative of the integrand; however the result may be verified by numerical approximation.

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8.9.99

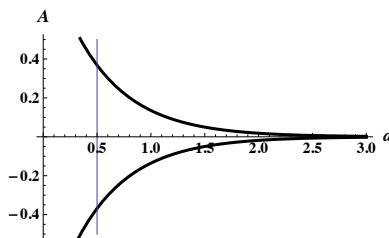
- a. We have $\int_0^\infty e^{-ax} \cos bx \, dx = \lim_{c \rightarrow \infty} \int_0^c e^{-ax} \cos bx \, dx = \lim_{c \rightarrow \infty} \left(\frac{e^{-ax}(b \sin bx - a \cos bx)}{a^2 + b^2} \right) \Big|_0^c =$
 $\lim_{c \rightarrow \infty} \frac{a + e^{-ac}(b \sin bc - a \cos bc)}{a^2 + b^2} = \frac{a}{a^2 + b^2}.$
- b. We have $\int_0^\infty e^{-ax} \sin bx \, dx = \lim_{c \rightarrow \infty} \int_0^c e^{-ax} \sin bx \, dx = \lim_{c \rightarrow \infty} \left(-\frac{e^{-ax}(a \sin bx + b \cos bx)}{a^2 + b^2} \right) \Big|_0^c =$
 $\lim_{c \rightarrow \infty} \frac{b - e^{-ac}(a \sin bc + b \cos bc)}{a^2 + b^2} = \frac{b}{a^2 + b^2}.$

8.9.100

- a. Evaluate the improper integral

$$\int_0^\infty x e^{-cx} \, dx = \lim_{b \rightarrow \infty} \int_0^b x e^{-cx} \, dx = \lim_{b \rightarrow \infty} \left(-\frac{e^{-cx}(cx + 1)}{c^2} \right) \Big|_0^b = \frac{1}{c^2} \lim_{b \rightarrow \infty} (1 - e^{-cb}(cb + 1)) = \frac{1}{c^2},$$

provided $c > 0$. We also have $\int_0^\infty e^{-cx} \, dx = \frac{1}{c}$ for $c > 0$. Hence $\bar{x} = 1/c = 1/2$ for the case $c = 2$.



- b. The curve $y = e^{-2x}$ has slope -2 at $x = 0$, so the equation of the tangent line at $(0, 1)$ is $y = 1 - 2x$. Similarly, the equation of the tangent line to $y = -e^{-2x}$ at $(0, -1)$ is $y = -1 + 2x$.
- c. Both tangent lines intersect the x -axis at $x = 1/2$.
- d. More generally, in the case $a = 0$ the center of mass is $\bar{x} = 1/c$ and the tangent lines at (0 ± 1) have equations $y = \pm(1 - cx)$, so both tangent lines meet the x -axis at the center of mass. The case for arbitrary a can be reduced to the case $a = 0$ by a horizontal shift.

8.9.101

- a. Integrate by parts with $u = \sqrt{x} \ln x$ and $v = -1/(1+x)$:

$$\int_0^\infty \frac{\sqrt{x} \ln x}{(1+x)^2} \, dx = \frac{1}{2} \int_0^\infty \frac{\ln x + 2}{\sqrt{x}(x+1)} \, dx = \frac{1}{2} \int_0^\infty \frac{\ln x}{\sqrt{x}(x+1)} \, dx + \int_0^\infty \frac{dx}{\sqrt{x}(x+1)}.$$

(The integration by parts is legitimate for this improper integral because the product uv has limit 0 as $x \rightarrow \infty$ and as $x \rightarrow 0^+$).

- b. Let $y = 1/x$. Then $dy = -\frac{1}{x^2} \, dx$.

- c. We have

$$\int_0^1 \frac{\ln x}{\sqrt{x}(x+1)} \, dx = \int_\infty^1 \frac{-\ln y}{\frac{1}{\sqrt{y}} \left(\frac{1}{y} + 1 \right)} \left(-\frac{dy}{y^2} \right) = -\int_1^\infty \frac{\ln y}{\sqrt{y}(1+y)} \, dy,$$

and hence $\int_0^\infty \frac{\ln x}{\sqrt{x}(x+1)} \, dx = 0.$

d. The change of variables $z = \sqrt{x}$ gives $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = 2 \int_0^\infty \frac{dz}{z^2+1} = \pi$.

8.9.102 The Laplace transform of $f(t) = 1$ is given by $F(s) = \int_0^\infty e^{-st} dt = \frac{1}{s}$.

8.9.103 The Laplace transform of $f(t) = e^{at}$ is given by $F(s) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}$, using the formula $\int_0^\infty e^{-cx} dx = \frac{1}{c}$ for $c > 0$.

8.9.104 The Laplace transform of $f(t) = t$ is given by $F(s) = \int_0^\infty t e^{-st} dt = \frac{1}{s^2}$ (this formula is derived in the solution to problem 100).

8.9.105 The Laplace transform of $f(t) = \sin at$ is given by $F(s) = \int_0^\infty e^{-st} \sin at dt = \frac{a}{s^2 + a^2}$ (this formula is derived in the solution to problem 99 b).

8.9.106 The Laplace transform of $f(t) = \cos at$ is given by $F(s) = \int_0^\infty e^{-st} \cos at dt = \frac{s}{s^2 + a^2}$ (this formula is derived in the solution to problem 99 a).

8.9.107

a. Make the substitution $x = y + 2$; then

$$\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}} = \int_{-1}^1 \frac{dy}{\sqrt{1-y^2}} = 2 \int_0^1 \frac{dy}{\sqrt{1-y^2}} = 2 \lim_{c \rightarrow 1^-} (\sin^{-1} x) \Big|_0^c = \pi.$$

b. The substitution $y = e^x$ gives

$$\int_1^\infty \frac{dx}{e^{x+1} + e^{3-x}} = \frac{1}{e} \int_1^\infty \frac{e^x dx}{e^{2x} + e^2} = \frac{1}{e} \int_e^\infty \frac{dy}{y^2 + e^2} = \frac{1}{e} \lim_{b \rightarrow \infty} \left(\frac{1}{e} \tan^{-1} \left(\frac{y}{e} \right) \right) \Big|_e^b = \frac{\pi}{4e^2}.$$

8.9.108 The rate at which water is draining from the tank is given by $r(t) = 100(0.95)^t = 100e^{(\ln 0.95)t}$, so the total amount of water drained from the tank is

$$W = 100 \int_0^\infty e^{(\ln 0.95)t} dt = \frac{100}{-\ln 0.95} \approx 1950 \text{ gal},$$

using the result that $\int_0^\infty e^{-ax} dx = \frac{1}{a}$ for $a > 0$. Therefore the full 3,000 gallon tank cannot be emptied at this rate.

8.9.109

a. We have $W = GMm \int_R^\infty x^{-2} dx = GMm \lim_{b \rightarrow \infty} \left(-\frac{1}{x} \right) \Big|_R^b = \frac{GMm}{R} \approx 6.279 \times 10^7 \text{ m J}$.

b. Solve $\frac{1}{2} v_e^2 = 6.279 \times 10^7$ to obtain $v_e \approx 11.207 \text{ km/s}$.

c. We need $\frac{GM}{R} \geq \frac{1}{2} c^2 \iff R \leq \frac{2GM}{c^2} \approx 9 \text{ mm}$.

8.9.110 Let r_0 be the radius of the nucleus. The work required to bring a free proton to the edge of the nucleus is given by

$$W = kQq \int_{r_0}^\infty r^{-2} dr = kQq \lim_{b \rightarrow \infty} \left(-\frac{1}{r} \right) \Big|_{r_0}^b = \frac{kQq}{r_0} = \frac{50kq^2}{r_0} = 1.92 \times 10^{-16} \text{ N m}.$$

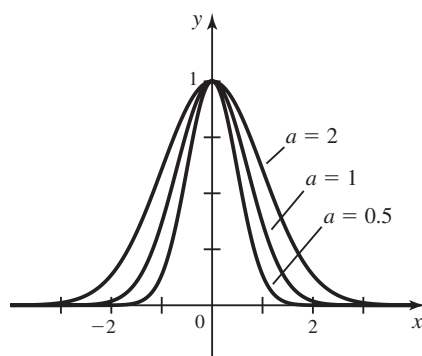
8.9.111

a. Repeatedly applying this reduction formula gives $\Gamma(p+1) = p(p-1)\cdots 2\cdot 1 \cdot \int_0^\infty e^{-x} dx = p! \cdot 1 = p!$.

b. We have $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x} dx = 2 \int_0^\infty e^{-u^2} du = \sqrt{\pi}$.

8.9.112

a.



b. The areas are $\sqrt{2\pi}$, $\sqrt{\pi}$, $\sqrt{\pi/2}$ respectively.

c. Completing the square gives

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a},$$

and therefore

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx = e^{(b^2-4ac)/(4a)} \int_{-\infty}^{\infty} e^{-a(x+(b/2a))^2} dx.$$

Make the substitution $y = x + b/(2a)$ to obtain

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx = e^{(b^2-4ac)/(4a)} \int_{-\infty}^{\infty} e^{-ay^2} dy = e^{(b^2-4ac)/(4a)} \sqrt{\frac{\pi}{a}}.$$

Chapter Eight Review

1

a. True. Two applications of integration by parts are needed to reduce to $\int x^2 e^{2x} dx$.

b. False. This integral can be done using a trigonometric substitution.

c. False. Both are correct, since $-\cos^2 x = \sin^2 x - 1$, so $\sin^2 x$ and $-\cos^2 x$ differ by a constant.

d. True. Recall that $2 \sin x \cos x = \sin 2x$.

e. False. Use long division to write the integrand as the sum of a polynomial and a proper rational function.

2 Let $u = x/2 + \pi/3$. Then $du = \frac{1}{2} dx$. Substituting gives

$$2 \int \cos u \, du = 2 \sin u + C = 2 \sin(x/2 + \pi/3) + C.$$

3 Let $u = x + 4$ so that $du = dx$. Note that $x = u - 4$. Substituting gives

$$\begin{aligned} \int \frac{3(u-4)}{\sqrt{u}} \, du &= 3 \int (u^{1/2} - 4u^{-1/2}) \, du = 3((2/3)u^{3/2} - 8u^{1/2}) + C \\ &= 2(x+4)^{3/2} - 24\sqrt{x+4} + C = 2\sqrt{x+4}(x+4-12) + C = 2\sqrt{x+4}(x-8) + C. \end{aligned}$$

4 Let $u = 3t$ and $dv = e^{-t} dt$. Then $du = 3dt$ and $v = -e^{-t}$. Thus $\int_{-1}^{\ln 2} 3te^{-t} dt$ is equal to

$$-3te^{-t} \Big|_{-1}^{\ln 2} - \int_{-1}^{\ln 2} (-3e^{-t}) \, dt = -3te^{-t} \Big|_{-1}^{\ln 2} - 3e^{-t} \Big|_{-1}^{\ln 2} = -\frac{3 \ln 2}{2} - 3e - (3/2 - 3e) = -\frac{3}{2}(\ln 2 + 1).$$

5 Let $u = x$ and $dv = (1/2)(x+2)^{-1/2} dx$. Then $du = dx$ and $v = (x+2)^{1/2}$. We have

$$\begin{aligned} \int \frac{x}{2\sqrt{x+2}} \, dx &= x\sqrt{x+2} - \int \sqrt{x+2} \, dx = x\sqrt{x+2} - \frac{2}{3}(x+2)^{3/2} + C \\ &= \frac{1}{3}\sqrt{x+2}(3x-2x-4) + C = \frac{1}{3}\sqrt{x+2}(x-4) + C. \end{aligned}$$

6 $\int (2 \sec^2 2\theta - \tan 2\theta \sec 2\theta) \, d\theta = \tan 2\theta - \frac{1}{2} \sec 2\theta + C.$

7

$$\int_{-2}^1 \frac{3}{(x+2)^2+9} \, dx = (\tan^{-1}((x+2)/3)) \Big|_{-2}^1 = \pi/4 - 0 = \pi/4.$$

8

$$\int_{\pi}^{2\pi} \frac{\cos(x/3)}{\sin(x/3)} \, dx = 3 (\ln |\sin(x/3)|) \Big|_{\pi}^{2\pi} = 3 \ln \left| \frac{\sin(2\pi/3)}{\sin(\pi/3)} \right| = 3 \ln 1 = 0.$$

9 $\int_0^{\pi/4} \cos^5 2x \sin^2 2x \, dx = \int_0^{\pi/4} \cos(2x)(1 - \sin^2 2x)^2 \sin^2 2x \, dx$. Let $u = \sin 2x$ so that $du = 2 \cos 2x \, dx$. Substituting gives

$$\frac{1}{2} \int_0^1 (1-u^2)^2 u^2 \, du = \frac{1}{2} \int_0^1 (u^2 - 2u^4 + u^6) \, du = \frac{1}{2} (u^3/3 - 2u^5/5 + u^7/7) \Big|_0^1 = \frac{1}{2} (1/3 - 2/5 + 1/7) - 0 = \frac{4}{105}.$$

10 By long division, $\frac{x^3 + 3x^2 + 1}{x^3 + 1} = 1 + \frac{3x^2}{x^3 + 1}$. Thus

$$\int \frac{x^3 + 3x^2 + 1}{x^3 + 1} \, dx = \int \left(1 + \frac{3x^2}{x^3 + 1} \right) \, dx = x + \ln |x^3 + 1| + C.$$

11 Let $u = \sqrt{t-1}$. Then $u^2 + 1 = t$, and $2u \, du = dt$. Substituting gives

$$\int \frac{u^2}{u^2 + 1} \, du = \int \left(1 - \frac{1}{u^2 + 1} \right) \, du = u - \tan^{-1} u + C = \sqrt{t-1} - \tan^{-1} \sqrt{t-1} + C.$$

12 Write $\frac{8x+5}{2x^2+3x+1} = \frac{A}{2x+1} + \frac{B}{x+1}$. Then $8x+5 = A(x+1) + B(2x+1)$. Letting $x = -1$ gives $B = 3$ and letting $x = -1/2$ gives $A = 2$. Thus $\frac{8x+5}{2x^2+3x+1} = \frac{2}{2x+1} + \frac{3}{x+1}$. Our integral is therefore equal to

$$\int \left(\frac{2}{2x+1} + \frac{3}{x+1} \right) \, dx = \ln |(2x+1)(x+1)^3| + C.$$

13 We integrate by parts. Let $u = e^{3x}$ and $dv = \sin 6x \, dx$. Then $du = 3e^{3x} \, dx$ and $v = -\frac{1}{6} \cos 6x$. Our integral is then equal to

$$I = \int_0^\pi e^{3x} \sin 6x \, dx = -\frac{1}{6} e^{3x} \cos 6x \Big|_0^\pi + \frac{1}{2} \int_0^\pi e^{3x} \cos 6x \, dx. = -\frac{1}{6} e^{3\pi} + \frac{1}{6} + \frac{1}{2} \int_0^\pi e^{3x} \cos 6x \, dx.$$

We integrate by parts again, letting $u = e^{3x}$ and $dv = \cos 6x \, dx$. Then $du = 3e^{3x} \, dx$ and $v = \frac{1}{6} \sin 6x$. We have

$$\begin{aligned} I &= -\frac{1}{6} e^{3\pi} + \frac{1}{6} + \frac{1}{2} \left(\frac{1}{6} e^{3x} \sin 6x \Big|_0^\pi - \frac{1}{2} \int_0^\pi e^{3x} \sin 6x \, dx \right) \\ &= -\frac{1}{6} e^{3\pi} + \frac{1}{6} - \frac{1}{4} I. \end{aligned}$$

So

$$\frac{5}{4} I = \frac{1}{6} (1 - e^{3\pi}),$$

and therefore

$$I = \frac{2}{15} (1 - e^{3\pi}).$$

14 We integrate by parts with $u = 6w - 10$ and $dv = e^{3w} \, dw$. Then $du = 6 \, dw$ and $v = \frac{1}{3} e^{3w}$. We have

$$(6w - 10) \frac{1}{3} e^{3w} \Big|_2^3 - \int_2^3 2e^{3w} \, dw = \frac{8}{3} e^9 - \frac{2}{3} e^6 - \left(\frac{2}{3} e^{3w} \Big|_2^3 \right) = 2e^9.$$

15 We simplify the integrand via long division and then partial fractions. We have

$$\frac{3x^5 + 48x^3 + 3x^2 + 16}{x^3 + 16x} = 3x^2 + \frac{3x^2 + 16}{x(x^2 + 16)} = 3x^2 + \frac{1}{x} + \frac{2x}{x^2 + 16}.$$

Therefore our integral is equal to

$$\int_1^2 \left(3x^2 + \frac{1}{x} + \frac{2x}{x^2 + 16} \right) dx = (x^3 + \ln |x| + \ln |x^2 + 16|) \Big|_1^2 = (8 + \ln 2 + \ln 20) - (1 + \ln 17) = 7 + \ln 40 - \ln 17.$$

16 We simplify the integrand by long division. We have

$$\frac{x^6 + 2x^4 + x^2 + 1}{(x^2 + 1)^2} = x^2 + \frac{1}{(x^2 + 1)^2}.$$

We have

$$\int \left(x^2 + \frac{1}{(x^2 + 1)^2} \right) dx = \frac{x^3}{3} + \int \frac{1}{(x^2 + 1)^2} dx.$$

For this last integral, we let $x = \tan u$ so that $dx = \sec^2 u \, du$ and $x^2 + 1 = \sec^2 u$. We have

$$\begin{aligned} \frac{x^3}{3} + \int \frac{\sec^2 u}{\sec^4 u} du &= \frac{x^3}{3} + \int \cos^2 u \, du = \frac{x^3}{3} + \frac{1}{2} \int (1 + \cos 2u) \, du \\ &= \frac{x^3}{3} + \frac{1}{2} u + \frac{1}{4} \sin 2u + C = \frac{x^3}{3} + \frac{1}{2} \tan^{-1} x + \frac{x}{2(x^2 + 1)} + C. \end{aligned}$$

In the last equation, we used the fact that $\frac{1}{4} \sin 2u = \frac{1}{2} \sin u \cos u = \frac{1}{2} \cdot \frac{x}{\sqrt{x^2 + 1}} \cdot \frac{1}{\sqrt{x^2 + 1}}.$

17 Write $\frac{2x^2 + 7x + 4}{x^3 + 2x^2 + 2x} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{C}{x}$. Then $2x^2 + 7x + 4 = Ax^2 + Bx + C(x^2 + 2x + 2)$. Letting $x = 0$ gives $C = 2$. Letting $x = 1$ gives $13 = A + B + 10$, so $A + B = 3$. Letting $x = -1$ gives $-1 = A - B + 2$, so $A - B = -3$. Solving this system of linear equations gives $A = 0$ and $B = 3$. Thus our integral is equal to

$$\int \left(\frac{3}{x^2 + 2x + 2} + \frac{2}{x} \right) dx = \int \left(\frac{3}{(x+1)^2 + 1} + \frac{2}{x} \right) dx = 3 \tan^{-1}(x+1) + 2 \ln|x| + C.$$

18

$$\begin{aligned} \int_0^{\sqrt{2}} \frac{x+1}{3x^2+6} dx &= \int_0^{\sqrt{2}} \frac{x}{3x^2+6} dx + \frac{1}{3} \int_0^{\sqrt{2}} \frac{1}{x^2+2} dx \\ &= \frac{1}{6} \ln|3x^2+6| \Big|_0^{\sqrt{2}} + \frac{1}{3\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) \Big|_0^{\sqrt{2}} \\ &= \frac{1}{6} (\ln 12 - \ln 6) + \frac{1}{3\sqrt{2}} \left(\frac{\pi}{4} \right) \\ &= \frac{1}{6} \ln 2 + \frac{\pi}{12\sqrt{2}} = \frac{1}{6} \left(\ln 2 + \frac{\pi\sqrt{2}}{4} \right). \end{aligned}$$

19 If we write

$$\frac{x^2 + 4x + 7}{(x+3)^3} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3}$$

and solve, we obtain $A = 1$, $B = -2$, and $C = 4$. We then have

$$\int \frac{1}{x+3} dx - \int \frac{2}{(x+3)^2} dx + \int \frac{4}{(x+3)^3} dx = \ln|x+3| + \frac{2}{x+3} - \frac{2}{(x+3)^2} + C.$$

20 Let $x = \sin \theta$ so that $dx = \cos \theta d\theta$ and $\sqrt{1-x^2} = \cos \theta$. Substituting gives

$$\begin{aligned} \int \frac{\cos^2 \theta}{\sin \theta} d\theta &= \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta \\ &= \int (\csc \theta - \sin \theta) d\theta = -\ln|\csc \theta + \cot \theta| + \cos \theta + C \\ &= -\ln|\csc \sin^{-1} x + \cot \sin^{-1} x| + \cos \sin^{-1} x + C \\ &= -\ln|(1/x) + \sqrt{1-x^2}/x| + \sqrt{1-x^2} + C. \end{aligned}$$

21 Let $x = \sec \theta$ so that $dx = \sec \theta \tan \theta d\theta$ and $\sqrt{x^2-1} = \tan \theta$. Substituting gives

$$\int_{\pi/4}^{\pi/3} \tan^2 \theta d\theta = \int_{\pi/4}^{\pi/3} (\sec^2 \theta - 1) d\theta = (\tan \theta - \theta) \Big|_{\pi/4}^{\pi/3} = \sqrt{3} - \pi/3 - (1 - \pi/4) = \sqrt{3} - 1 - \frac{\pi}{12}.$$

22 $\int \tan^3 \theta d\theta = \int \tan \theta (\sec^2 \theta - 1) d\theta = \int \tan \theta \sec^2 \theta d\theta - \int \tan \theta d\theta$. For the first integral, let $u = \tan \theta$ and for the second, let $w = \cos \theta$. Substituting gives

$$\int u du + \int \frac{1}{w} dw = u^2/2 + \ln|w| + C = \frac{\tan^2 \theta}{2} + \ln|\cos \theta| + C.$$

23 $\int \tan^4 t \sec^2 t dt$. Let $u = \tan t$ so that $du = \sec^2 t dt$. Substituting gives

$$\int u^4 du = u^5/5 + C = \frac{1}{5} \tan^5 t + C.$$

24 We complete the square in the denominator of the integrand to obtain

$$\int \frac{1}{\sqrt{81 - (x-9)^2}} dx = \int \frac{1}{9\sqrt{1 - \left(\frac{x-9}{9}\right)^2}} dx.$$

Then let $u = \frac{x-9}{9}$ so that $du = \frac{1}{9} dx$. Substituting gives

$$\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + C = \sin^{-1} \left(\frac{x-9}{9} \right) + C.$$

25 Complete the square in the denominator of the integrand as follows:

$$4x^2 + 12x + 10 = 4(x^2 + 3x + 9/4 - 9/4) + 10 = (4x^2 + 12x + 9) - 9 + 10 = (2x + 3)^2 + 1.$$

Then we have

$$\int_{-3/2}^{-1} \frac{dx}{(2x+3)^2 + 1}$$

. Let $u = 2x + 3$ and $du = 2 dx$. Substituting gives

$$\frac{1}{2} \int_0^1 \frac{du}{u^2 + 1} = \frac{1}{2} \tan^{-1} u \Big|_0^1 = \frac{\pi}{8}.$$

26 Let $u = \cos x$ so that $du = -\sin x dx$. We have

$$\int \tan 10x dx = \int -\frac{1}{10} \frac{1}{u} du = -\frac{1}{10} \ln |u| + C = -\frac{1}{10} \ln |\cos x| + C.$$

27 Note that

$$w^2 + 2w - 8 = w^2 + 2w + 1 - 9 = (w+1)^2 - 9.$$

Let $u = w + 1$ with $du = dw$. Substituting gives $\int \frac{1}{u^2 \sqrt{u^2 - 9}} du$. Now let $u = 3 \sec \theta$ with $du = 3 \sec \theta \tan \theta d\theta$. Now we have

$$\begin{aligned} \int \frac{1}{9 \sec^2 \theta \sqrt{9 \sec^2 \theta - 9}} 3 \sec \theta \tan \theta d\theta &= \int \frac{3 \tan \theta}{27 \sec \theta \tan \theta} d\theta \\ &= \frac{1}{9} \int \cos \theta d\theta = \frac{1}{9} \sin \theta + C = \frac{1}{9} \cdot \frac{\sqrt{u^2 - 9}}{u} + C \\ &= \frac{1}{9} \cdot \frac{\sqrt{(w+1)^2 - 9}}{w+1} + C = \frac{\sqrt{w^2 + 2w - 8}}{9(w+1)} + C. \end{aligned}$$

28 The integral can be written as $\int \tan^3 x \sec^2 x dx$. Let $u = \tan x$ so that $du = \sec^2 x dx$. Substituting gives

$$\int u^3 du = \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C.$$

29 The integral can be written as $\int \cot^4 x \csc^2 x dx$. Let $u = \cot x$ so that $du = -\csc^2 x dx$. Substituting gives

$$-\int u^4 du = -\frac{u^5}{5} + C = -\frac{\cot^5 x}{5} + C.$$

30 Let $u = \tan^{-1} 7x$ and $dv = x dx$. Then $du = \frac{7dx}{(7x)^2 + 1}$ and $v = \frac{x^2}{2}$. We have

$$\begin{aligned} \int x \tan^{-1} 7x dx &= \frac{1}{2} x^2 \tan^{-1} 7x - \frac{7}{2} \int \frac{x^2}{49x^2 + 1} dx = \frac{1}{2} x^2 \tan^{-1} 7x - \frac{1}{14} \int \left(1 - \frac{1}{(7x)^2 + 1} \right) dx \\ &= \frac{1}{2} x^2 \tan^{-1} 7x - \frac{1}{14} \left(x - \frac{\tan^{-1} 7x}{7} \right) + C = \frac{1}{2} x^2 \tan^{-1} 7x - \frac{x}{14} + \frac{\tan^{-1} 7x}{98} + C. \end{aligned}$$

31 Let $u = x$ and $dv = \sinh x \, dx$. Then $du = dx$ and $v = \cosh x$. We have

$$\int x \sinh x \, dx = x \cosh x - \int \cosh x \, dx = x \cosh x - \sinh x + C.$$

32 $\int \csc^2 6x \cot 6x \, dx$. Let $u = \cot 6x$ so that $du = -6 \csc^2 6x \, dx$. Substituting gives

$$-\frac{1}{6} \int u \, du = -\frac{u^2}{12} + C = \frac{-\cot^2 6x}{12} + C.$$

33 $\int \tan^2 3\theta \sec^2 3\theta (\sec 3\theta \tan 3\theta) \, d\theta = \int (\sec^2 3\theta - 1) \sec^2 3\theta (\sec 3\theta \tan 3\theta) \, d\theta$.

Let $u = \sec 3\theta$ so that $du = 3 \sec 3\theta \tan 3\theta \, d\theta$. Substituting gives

$$\frac{1}{3} \int (u^2 - 1)u^2 \, du = \frac{1}{3} \int (u^4 - u^2) \, du = \frac{u^5}{15} - \frac{u^3}{9} + C = \frac{1}{15} \sec^5 3\theta - \frac{1}{9} \sec^3 3\theta + C.$$

34 Let $w = 2 \sin \theta$ so that $dw = 2 \cos \theta \, d\theta$ and $\sqrt{36 - 9w^2} = 3\sqrt{4 - w^2} = 6 \cos \theta$. Substituting gives

$$\int \frac{8 \sin^3 \theta}{6 \cos \theta} \cdot 2 \cos \theta \, d\theta = \int \frac{8}{3} \sin \theta (1 - \cos^2 \theta) \, d\theta.$$

Let $u = \cos \theta$ so that $du = -\sin \theta \, d\theta$. Substituting again gives

$$\begin{aligned} \frac{8}{3} \int (u^2 - 1) \, du &= \frac{8u^3}{9} - \frac{8u}{3} + C = \frac{8}{9} \cos^3 \theta - \frac{8}{3} \cos \theta + C \\ &= \frac{8}{9} \cdot (\cos(\sin^{-1}(w/2)))^3 - \frac{8}{3} \cdot \cos(\sin^{-1}(w/2)) + C = \frac{1}{9} \cdot (\sqrt{4 - w^2})^3 - \frac{12}{9} \cdot \sqrt{4 - w^2} + C \\ &= \frac{1}{9} \cdot \sqrt{4 - w^2}(4 - w^2 - 12) = -\frac{1}{9} \sqrt{4 - w^2}(w^2 + 8) + C. \end{aligned}$$

35 Let $x = 2 \tan \theta$ so that $dx = 2 \sec^2 \theta \, d\theta$ and $\sqrt{4x^2 + 16} = 2\sqrt{x^2 + 4} = 4 \sec \theta$. Substituting gives

$$\int \frac{8 \tan^3 \theta}{4 \sec \theta} 2 \sec^2 \theta \, d\theta = 4 \int \tan \theta \sec \theta (\sec^2 \theta - 1) \, d\theta.$$

Let $u = \sec \theta$ so that $du = \sec \theta \tan \theta \, d\theta$. Substituting again gives

$$\begin{aligned} 4 \int (u^2 - 1) \, du &= \frac{4}{3} u^3 - 4u + C = \frac{4}{3} \sec^3 \theta - 4 \sec \theta + C \\ &= \frac{4}{3} (\sec(\tan^{-1}(x/2)))^3 - 4 \sec \tan^{-1}(x/2) + C = \frac{1}{6} \sqrt{x^2 + 4}(x^2 + 4) - 2\sqrt{x^2 + 4} + C \\ &= \frac{1}{6} \sqrt{x^2 + 4}(x^2 + 4 - 12) + C = \frac{1}{6} \sqrt{x^2 + 4}(x^2 - 8) + C. \end{aligned}$$

36 Note that $\frac{u^2 + 1}{u^2 - 1} = 1 + \frac{2}{u^2 - 1} = 1 + \frac{1}{u - 1} - \frac{1}{u + 1}$. Our integral is therefore equal to

$$\int_{-1/2}^{1/2} \left(-\frac{1}{u + 1} + \frac{1}{u - 1} + 1 \right) du = \left(\ln \left| \frac{u - 1}{u + 1} \right| + u \right) \Big|_{-1/2}^{1/2} = \ln(1/3) + 1/2 - (\ln 3 - 1/2) = 1 - 2 \ln 3.$$

37 Write $\frac{3x^3 + 4x^2 + 6x}{(x + 1)^2(x^2 + 4)} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x + 1} + \frac{D}{(x + 1)^2}$. Then $3x^3 + 4x^2 + 6x = (Ax + B)(x + 1)^2 + C(x + 1)(x^2 + 4) + D(x^2 + 4)$. Letting $x = -1$ gives $D = -1$. Letting $x = 0$ gives $0 = B + 4C - 4$, so $B + 4C = 4$. Letting $x = 1$ gives $13 = 4A + 4B + 10C - 5$, so $9 = 2A + 2B + 5C$. Letting $x = 2$ gives $52 = 18A + 9B + 24C - 8$, so $60 = 18A + 9B + 24C$, so $20 = 6A + 3B + 8C$. Solving the system of linear equations gives $A = 2$, $B = 0$, and $C = 1$. Our integral is thus equal to

$$\int \left(\frac{2x}{x^2 + 4} + \frac{1}{x + 1} - \frac{1}{(x + 1)^2} \right) dx = \ln |(x^2 + 4)(x + 1)| + \frac{1}{x + 1} + C.$$

38 We integrate by parts with $u = x$ and $dv = \csc^2 x \, dx$. We have $du = dx$ and $v = -\cot x$.

$$\begin{aligned}\int_{\pi/4}^{\pi/2} x \csc^2 x \, dx &= -x \cot x \Big|_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2} \cot x \, dx \\ &= -x \cot x \Big|_{\pi/4}^{\pi/2} + \ln(\sin x) \Big|_{\pi/4}^{\pi/2} \\ &= \left(-\frac{\pi}{2} \cot \frac{\pi}{2} + \ln\left(\sin \frac{\pi}{2}\right)\right) - \left(-\frac{\pi}{4} \cot \frac{\pi}{4} + \ln\left(\sin \frac{\pi}{4}\right)\right) \\ &= (0 + \ln 1) - \left(-\frac{\pi}{4} + \ln \frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} - \ln \frac{1}{\sqrt{2}} = \frac{\pi}{4} - \ln \frac{\sqrt{2}}{2}.\end{aligned}$$

39 $\int \frac{1}{2+e^t} dt = \int \frac{e^t}{(e^t+2)e^t} dt$. Let $u = e^t$, so that $du = e^t dt$. Then we have $\int \frac{1}{(u+2)u} du$. If we write

$$\frac{1}{(u+2)u} = \frac{A}{u} + \frac{B}{u+2},$$

then $A(u+2) + Bu = 1$, and we find that $A = 1/2$ and $B = -1/2$. Then we integrate

$$\int \left(\frac{1/2}{u} - \frac{1/2}{u+2} \right) du = \frac{1}{2} (\ln|u| - \ln|u+2|) + C = \frac{1}{2} (t - \ln(2+e^t)) + C.$$

40 Note that by long division $\frac{x^2-4}{x+4} = x-4 + \frac{12}{x+4}$. The original integral is therefore equal to

$$\int \left(x-4 + \frac{12}{x+4} \right) dx = x^2/2 - 4x + 12 \ln|x+4| + C.$$

41

$$\int \frac{1}{1+\cos 4\theta} \cdot \frac{1-\cos 4\theta}{1-\cos 4\theta} d\theta = \int \frac{1-\cos 4\theta}{\sin^2 4\theta} d\theta = \int (\csc^2 4\theta - \csc 4\theta \cot 4\theta) d\theta = \frac{1}{4} (-\cot 4\theta + \csc 4\theta) + C.$$

42 Two applications of integration by parts gives

$$\int x^2 \cos x \, dx = x^2 \sin x - 2 \int x \sin x \, dx = x^2 \sin x - 2 \left(x(-\cos x) + \int \cos x \, dx \right) = (x^2-2) \sin x + 2x \cos x + C.$$

43 Two applications of integration by parts gives

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx.$$

Thus,

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x,$$

so

$$\int e^x \sin x \, dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + C.$$

44 Integration by parts gives

$$\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx = \frac{x^3}{3} \left(\ln x - \frac{1}{3} \right) + C.$$

Thus,

$$\int_1^e x^2 \ln x \, dx = \frac{x^3}{3} \left(\ln x - \frac{1}{3} \right) \Big|_1^e = \frac{e^3}{3} \cdot \frac{2}{3} - \frac{1}{3} \left(-\frac{1}{3} \right) = \frac{2e^3+1}{9}.$$

45 We decompose the integrand via partial fractions. If we write $\frac{2x^3 + 5x^2 + 13x + 9}{x^2(x^2 + 4x + 9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 4x + 9}$, we can solve and obtain $A = B = C = 1$ and $D = 0$. Then we have

$$\int \frac{1}{x} dx + \int \frac{1}{x^2} dx + \int \frac{x}{x^2 + 4x + 9} dx = \ln|x| - \frac{1}{x} + \int \frac{x}{(x+2)^2 + 5} dx = \ln|x| - \frac{1}{x} + \int \frac{u-2}{u^2+5} du,$$

where $u = x + 2$ so that $du = dx$. Then we have

$$\begin{aligned} \ln|x| - \frac{1}{x} + \int \frac{u}{u^2+5} du - \int \frac{2}{u^2+5} du &= \ln|x| - \frac{1}{x} + \frac{1}{2} \ln|u^2+5| - \frac{2}{\sqrt{5}} \tan^{-1} \frac{u}{\sqrt{5}} + C \\ &= \ln|x| - \frac{1}{x} + \frac{1}{2} \ln|x^2+4x+9| - \frac{2}{\sqrt{5}} \tan^{-1} \left(\frac{x+2}{\sqrt{5}} \right) + C. \end{aligned}$$

46 We decompose the integrand via partial fractions. If we write

$$\frac{x^3 + 4x^2 + 12x + 4}{(x^2 + 4x + 10)^2} = \frac{Ax + B}{x^2 + 4x + 10} + \frac{Cx + D}{(x^2 + 4x + 10)^2},$$

and solve we obtain $A = 1$, $B = 0$, $C = 2$ and $D = 4$. We therefore have

$$\begin{aligned} \int \frac{x}{x^2 + 4x + 10} dx + \int \frac{2(x+2)}{(x^2 + 4x + 10)^2} dx &= \int \frac{x+2-2}{(x+2)^2+6} dx + \int \frac{2(x+2)}{(x^2+4x+10)^2} dx \\ &= \int \frac{x+2}{(x+2)^2+6} dx - \int \frac{2}{(x+2)^2+6} dx + \int \frac{2(x+2)}{(x^2+4x+10)^2} dx. \end{aligned}$$

For the first two integrals let $u = x + 2$ and for the last let $w = x^2 + 4x + 10$. Then we have

$$\begin{aligned} \int \frac{u}{u^2+6} du - \int \frac{2}{u^2+6} du + \int \frac{1}{w^2} dw &= \frac{1}{2} \ln|u^2+6| - \frac{2}{\sqrt{6}} \tan^{-1} \frac{u}{\sqrt{6}} - \frac{1}{w} + C \\ &= \frac{1}{2} \ln|x^2+4x+10| - \frac{2}{\sqrt{6}} \tan^{-1} \left(\frac{x+2}{\sqrt{6}} \right) - \frac{1}{x^2+4x+10} + C. \end{aligned}$$

47 Let $u = 4\theta$. Then

$$\int \cos^2 4\theta d\theta = \frac{1}{4} \int \cos^2 u du = \frac{1}{4} \int \frac{1 + \cos 2u}{2} du = \frac{u}{8} + \frac{\sin 2u}{16} + C = \frac{\theta}{2} + \frac{\sin 8\theta}{16} + C.$$

48 Let $u = \cos 3x$. Then $du = -3 \sin 3x dx$ and

$$\int \sin 3x \cos^6 3x dx = -\frac{1}{3} \int u^6 du = -\frac{u^7}{21} + C = -\frac{\cos^7 3x}{21} + C.$$

49 Let $u = \sec 2z$. Then $du = 2 \sec 2z \tan 2z dz$ and

$$\int \sec^{49} 2z \tan 2z dz = \frac{1}{2} \int u^{48} du = \frac{1}{2} \cdot \frac{u^{49}}{49} + C = \frac{\sec^{49} 2z}{98} + C.$$

50 Using the identity $\cos^2 x = \frac{1 + \cos 2x}{2}$, we obtain

$$\begin{aligned} \int_0^{\frac{\pi}{6}} \cos^4 3x dx &= \int_0^{\frac{\pi}{6}} \left(\frac{1 + \cos 6x}{2} \right)^2 dx = \frac{1}{4} \int_0^{\frac{\pi}{6}} (1 + 2 \cos 6x + \cos^2 6x) dx \\ &= \frac{1}{4} \int_0^{\frac{\pi}{6}} \left(1 + 2 \cos 6x + \frac{1 + \cos 12x}{2} \right) dx = \frac{1}{4} \int_0^{\frac{\pi}{6}} \left(\frac{3}{2} + 2 \cos 6x + \frac{1}{2} \cos 12x \right) dx \\ &= \left(\frac{3x}{8} + \frac{1}{12} \sin 6x + \frac{1}{96} \sin 12x \right) \Big|_0^{\frac{\pi}{6}} = \frac{\pi}{16}. \end{aligned}$$

51 Let $u = \cos 4\theta$ so that $du = -4 \sin \theta d\theta$. Then

$$\begin{aligned}\int_0^{\pi/4} \sin^5 4\theta d\theta &= \int_0^{\pi/4} \sin^4 4\theta \cdot \sin 4\theta d\theta = -\frac{1}{4} \int_1^{-1} (1-u^2)^2 du = \frac{1}{4} \int_{-1}^1 (1-u^2)^2 du \\ &= \frac{1}{4} \int_{-1}^1 (1-2u^2+u^4) du = \frac{1}{4} \left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right) \Big|_{-1}^1 \\ &= \frac{1}{4} \left(1 - \frac{2}{3} + \frac{1}{5} - \left(-1 + \frac{2}{3} - \frac{1}{5} \right) \right) = \frac{4}{15}.\end{aligned}$$

52 Use the identity $\tan^2 u = \sec^2 u - 1$ and the substitution $v = \tan 2u$, we have

$$\begin{aligned}\int \tan^4 2u du &= \int \tan^2 2u \sec^2 2u du - \int \tan^2 2u du = \frac{1}{2} \int v^2 dv - \int (\sec^2 2u - 1) du \\ &= \frac{v^3}{6} - \frac{1}{2} \tan 2u + u + C = \frac{1}{6} \tan^3 2u - \frac{1}{2} \tan 2u + u + C.\end{aligned}$$

53 If we let $u^6 = x$, then $6u^5 du = dx$. Substituting yields

$$\begin{aligned}\int \frac{6u^5}{u^3 + u^2} du &= 6 \int \frac{u^3}{u+1} du = 6 \int \left(u^2 - u + 1 - \frac{1}{u+1} \right) du \\ &= 2u^3 - 3u^2 + 6u - 6 \ln|u+1| + C = 2\sqrt[6]{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln(\sqrt[6]{x} + 1) + C.\end{aligned}$$

54 Let $x = (5/3) \sec \theta$ so that $dx = (5/3) \sec \theta \tan \theta d\theta$. Then

$$\int \frac{dx}{\sqrt{9x^2 - 25}} = \frac{5}{3} \int \frac{\sec \theta \tan \theta d\theta}{5 \tan \theta} = \frac{1}{3} \int \sec \theta d\theta = \frac{1}{3} \ln |\sec \theta + \tan \theta| + C.$$

Now substitute $\sec \theta = (3/5)x$, $\tan \theta = (1/5)\sqrt{9x^2 - 25}$ to obtain

$$\int \frac{dx}{\sqrt{9x^2 - 25}} = \frac{1}{3} \ln(3x + \sqrt{9x^2 - 25}) + C$$

(note that we absorb the constant $-\ln 5$ into C , and we don't need absolute values because we were given $x > 5/3$).

55 We may write the given integral as $\frac{1}{\sqrt{2}} \int \frac{dy}{y^2 \sqrt{9-y^2}}$. Let $y = 3 \sin \theta$ so that $dy = 3 \cos \theta d\theta$. Then

$$\frac{1}{\sqrt{2}} \int \frac{dy}{y^2 \sqrt{9-y^2}} = \frac{1}{\sqrt{2}} \int \frac{3 \cos \theta d\theta}{9 \sin^2 \theta \cdot 3 \cos \theta} = \frac{1}{9\sqrt{2}} \int \csc^2 \theta d\theta = -\frac{1}{9\sqrt{2}} \cot \theta + C.$$

Now use $\sin \theta = y/3$, $\cos \theta = (1/3)\sqrt{9-y^2}$ to obtain

$$\frac{1}{\sqrt{2}} \int \frac{dy}{y^2 \sqrt{9-y^2}} = -\frac{1}{9\sqrt{2}y} \sqrt{9-y^2} + C.$$

56 Let $x = \sin \theta$ so that $dx = \cos \theta d\theta$. Then

$$\begin{aligned}\int_0^{\sqrt{3}/2} \frac{x^2}{(1-x^2)^{3/2}} dx &= \int_0^{\pi/3} \frac{\sin^2 \theta}{\cos^3 \theta} \cos \theta d\theta = \int_0^{\pi/3} \tan^2 \theta d\theta = \int_0^{\pi/3} (\sec^2 \theta - 1) d\theta \\ &= (\tan \theta - \theta) \Big|_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3}.\end{aligned}$$

57 Let $x = (3/2) \tan \theta$ so that $dx = (3/2) \sec^2 \theta d\theta$, and note that $(3/2) \tan(\pi/6) = \sqrt{3}/2$. Then

$$\int_0^{\sqrt{3}/2} \frac{4}{9+4x^2} dx = \int_0^{\pi/6} \frac{4}{9 \sec^2 \theta} \cdot \frac{3}{2} \sec^2 \theta d\theta = \frac{2}{3} \int_0^{\pi/6} d\theta = \frac{\pi}{9}.$$

58 Let $u = \sin \theta$ so that $du = \cos \theta d\theta$. Then

$$\int \frac{(1-u^2)^{5/2}}{u^8} du = \int \frac{\cos^5 \theta}{\sin^8 \theta} \cos \theta d\theta = \int \cot^6 \theta \csc^2 \theta d\theta.$$

Now make the substitution $v = \cot \theta$ so that $dv = -\csc^2 \theta d\theta$ to obtain

$$\int \cot^6 \theta \csc^2 \theta d\theta = -\int v^6 dv = -\frac{v^7}{7} + C,$$

so

$$\int \frac{(1-u^2)^{5/2}}{u^8} du = -\frac{1}{7} \left(\frac{\sqrt{1-u^2}}{u} \right)^7 + C.$$

59 Let $u = \cosh x$ so that $du = \sinh x dx$. Then

$$\int \frac{\sinh x}{\cosh^2 x} dx = \int u^{-2} du = \frac{-1}{u} + C = \frac{-1}{\cosh x} + C = -\operatorname{sech} x + C.$$

60 Let $u = x^2$ and $dv = \cosh x dx$. Then $du = 2x dx$ and $v = \sinh x$. The given integral is equal to $x^2 \sinh x - \int 2x \sinh x dx$. Now let $u = x$ and $dv = \sinh x dx$, so that $du = dx$ and $v = \cosh x$. Then we have

$$x^2 \sinh x - 2 \left(x \cosh x - \int \cosh x dx \right) = x^2 \sinh x - 2x \cosh x + 2 \sinh x + C = (x^2 + 2) \sinh x - 2x \cosh x + C.$$

61 Let $u = \sinh x$ so that $du = \cosh x dx$. Substituting gives

$$\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-u^2}} du = \left(\sin^{-1}(u/2) \right) \Big|_0^{\sqrt{3}} = \frac{\pi}{3}.$$

62 Let $u = \sinh^{-1}(x)$ and $dv = dx$, so that $du = \frac{1}{\sqrt{1+x^2}} dx$ and $v = x$. Then

$$\int \sinh^{-1} x dx = x \sinh^{-1}(x) - \int \frac{x}{\sqrt{1+x^2}} dx = x \sinh^{-1}(x) - \sqrt{1+x^2} + C.$$

63 Using the method of partial fractions, we express $\frac{1}{x^2 - 2x - 15} = \frac{1}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$. Clearing denominators gives $1 = A(x+3) + B(x-5)$ and comparing coefficients gives $A+B=0$, $3A-5B=1$ which has solution $A=1/8$, $B=-1/8$. Hence

$$\int \frac{dx}{x^2 - 2x - 15} = \frac{1}{8} \int \left(\frac{1}{x-5} - \frac{1}{x+3} \right) dx = \frac{1}{8} \ln \left| \frac{x-5}{x+3} \right| + C.$$

64 Using the method of partial fractions, we express $\frac{1}{x^3 - 2x^2} = \frac{1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$. Clearing denominators gives $1 = Ax(x-2) + B(x-2) + Cx^2$ and comparing coefficients gives $A+C=0$, $-2A+B=0$ and $-2B=1$, which has solution $A=-1/4$, $B=-1/2$, $C=1/4$. Hence

$$\int \frac{dx}{x^3 - 2x^2} = \frac{1}{4} \int \left(-\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x-2} \right) dx = \frac{1}{2x} + \frac{1}{4} \ln \left| \frac{x-2}{x} \right| + C.$$

65 Using the method of partial fractions, we express $\frac{1}{(y+1)(y^2+1)} = \frac{A}{y+1} + \frac{By+C}{y^2+1}$. Clearing denominators gives $1 = A(y^2+1) + (By+C)(y+1)$ and comparing coefficients gives $A+B=0$, $B+C=0$ and $A+C=1$, which has solution $A=1/2$, $B=-1/2$, $C=1/2$. Hence

$$\int_0^1 \frac{dy}{(y+1)(y^2+1)} = \frac{1}{2} \int_0^1 \left(\frac{1}{y+1} + \frac{1-y}{y^2+1} \right) dy = \frac{1}{2} \left(\ln(y+1) + \tan^{-1} y - \frac{1}{2} \ln(y^2+1) \right) \Big|_0^1 = \frac{1}{4} \ln 2 + \frac{\pi}{8}.$$

66 First we consider $\int \frac{6x}{1+x^6} dx$. Let $u = x^2$, so that $du = 2x dx$. Then we have $3 \int \frac{1}{1+u^3} du = \int \left(\frac{1}{1+u} + \frac{2-u}{u^2-u+1} \right) du$ where this is obtained through a partial fraction decomposition. The first term evaluates to $\ln|u+1|$ while the second can be written as

$$-\frac{1}{2} \int \frac{2u-4}{u^2-u+1} du = -\frac{1}{2} \int \left(\frac{2u-1}{u^2-u+1} - \frac{3}{u^2-u+1} \right) du = -\frac{1}{2} \ln|u^2-u+1| + \frac{3}{2} \int \frac{1}{u^2-u+1} du.$$

This last integral can be written as $\frac{3}{2} \int \frac{1}{(u-1/2)^2 + 3/4} du = \frac{3}{\sqrt{3}} \tan^{-1} \left(\frac{2u-1}{\sqrt{3}} \right) + C$. Putting this all together, we have $\int \frac{6x}{1+x^6} dx = \ln(x^2+1) - \frac{1}{2} \ln(x^4-x^2+1) + \frac{3}{\sqrt{3}} \tan^{-1} \left(\frac{2x^2-1}{\sqrt{3}} \right) + C$. If we let $G(x) = \ln(x^2+1) - \frac{1}{2} \ln(x^4-x^2+1) + \frac{3}{\sqrt{3}} \tan^{-1} \left(\frac{2x^2-1}{\sqrt{3}} \right)$, then $G(1) = \frac{\sqrt{3}\pi}{6} + \ln 2$, and $\lim_{b \rightarrow \infty} G(b) = \frac{\sqrt{3}\pi}{2}$. This limit is computed as follows:

$$\lim_{b \rightarrow \infty} G(b) = \lim_{b \rightarrow \infty} \left(\ln \left(\frac{b^2+1}{\sqrt{b^4-b^2+1}} \right) + \frac{3}{\sqrt{3}} \tan^{-1} \left(\frac{2b^2-1}{\sqrt{3}} \right) \right) = \ln 1 + \frac{3}{\sqrt{3}} \cdot \frac{\pi}{2} = \frac{\sqrt{3}\pi}{2}.$$

$$\text{Thus we have } \int_1^\infty \frac{6x}{1+x^6} dx = \frac{\sqrt{3}\pi}{3} - \ln 2.$$

67 First consider $\int_0^1 \frac{dx}{\sqrt[3]{|x-1|}} = \lim_{a \rightarrow 1^-} \int \frac{dx}{\sqrt[3]{1-x}} = \lim_{a \rightarrow 1^-} -\frac{3}{2} (1-x)^{2/3} \Big|_0^a = \lim_{a \rightarrow 1^-} -\frac{3}{2} ((1-a)^{2/3} - 1) = \frac{3}{2}$.

$$\text{Now consider } \int_1^2 \frac{dx}{\sqrt[3]{|x-1|}} = \lim_{b \rightarrow 1^+} \int_b^2 \frac{dx}{\sqrt[3]{x-1}} = \lim_{b \rightarrow 1^+} \frac{3}{2} (x-1)^{2/3} \Big|_b^2 = \lim_{b \rightarrow 1^+} \frac{3}{2} (1 - (b-1)^{2/3}) = \frac{3}{2}.$$

$$\text{Thus, } \int_0^2 \frac{dx}{\sqrt[3]{|x-1|}} = \frac{3}{2} + \frac{3}{2} = 3.$$

68 Let $u = x+1$. Then we have

$$\int_{-1}^1 \frac{dx}{x^2+2x+5} = \int_0^2 \frac{du}{u^2+2^2} = \frac{1}{2} \tan^{-1} \left(\frac{u}{2} \right) \Big|_0^2 = \frac{\pi}{8}.$$

69 Factor $x^2-x-2 = (x-2)(x+1)$ and use the method of partial fractions: $\frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$. Clearing denominators gives $1 = A(x+1) + B(x-2)$ and equating coefficients gives $A+B=0$, $A-2B=1$, so $A=1/3$, $B=-1/3$. Therefore

$$\int \frac{dx}{x^2-x-2} = \frac{1}{3} \int \left(\frac{1}{x-2} - \frac{1}{x+1} \right) dx = \frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| + C.$$

70 As a preliminary step, express $\frac{3x^2+x-3}{x^2-1} = \frac{3(x^2-1+1)+x-3}{x^2-1} = 3 + \frac{x}{x^2-1}$; hence

$$\int \frac{3x^2+x-3}{x^2-1} dx = 3x + \int \frac{x}{x^2-1} dx = 3x + \frac{1}{2} \ln|x^2-1| + C,$$

where we make the substitution $u = x^2 - 1$ for the final integral above.

71 As a preliminary step, observe that

$$\frac{2x^2-4x}{x^2-4} = \frac{2x(x-2)}{(x-2)(x+2)} = \frac{2x}{x+2} = \frac{2(x+2-2)}{x+2} = 2 - \frac{4}{x+2}.$$

Hence

$$\int \frac{2x^2-4x}{x^2-4} dx = 2x - 4 \ln|x+2| + C = 2(x-2 \ln|x+2|) + C.$$

72 Make the preliminary substitution $u = 2\sqrt{x}$; then $du = dx/\sqrt{x}$ and

$$\int_{1/12}^{1/4} \frac{dx}{\sqrt{x}(1+4x)} = \int_{1/\sqrt{3}}^1 \frac{du}{1+u^2} = \tan^{-1} u \Big|_{1/\sqrt{3}}^1 = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

73 Make the preliminary substitution $x = e^{2t}$, $dx = 2e^{2t}$. Then we have

$$\int \frac{e^{2t}}{(1+e^{4t})^{3/2}} dt = \frac{1}{2} \int \frac{dx}{(1+x^2)^{3/2}}.$$

Now let $x = \tan \theta$, $dx = \sec^2 \theta d\theta$ and

$$\frac{1}{2} \int \frac{dx}{(1+x^2)^{3/2}} = \frac{1}{2} \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta = \frac{1}{2} \int \cos \theta d\theta = \frac{1}{2} \sin \theta + C.$$

Now

$$\sin \theta = \frac{\tan \theta}{\sec \theta} = \frac{x}{\sqrt{1+x^2}} = \frac{e^{2t}}{\sqrt{1+e^{4t}}}.$$

Therefore

$$\int \frac{e^{2t}}{(1+e^{4t})^{3/2}} dt = \frac{1}{2} \frac{e^{2t}}{\sqrt{1+e^{4t}}} + C.$$

74 Let $v = 1 + \sqrt{x}$. Then $dv = \frac{1}{2\sqrt{x}} dx$. We have

$$\begin{aligned} \int \frac{dx}{\sqrt{\sqrt{1+\sqrt{x}}}} &= \int \frac{1}{(1+\sqrt{x})^{1/4}} \cdot \frac{\sqrt{x}}{\sqrt{x}} dx \\ &= 2 \int \frac{v-1}{v^{1/4}} dv = 2 \int (v^{3/4} - v^{-1/4}) dv = 2 \left(\frac{4}{7} v^{7/4} - \frac{4}{3} v^{3/4} \right) + C \\ &= \frac{8}{21} v^{3/4} (3v-7) + C = \frac{8}{21} (1+\sqrt{x})^{3/4} (3\sqrt{x}-4) + C. \end{aligned}$$

75

a. Let $u = e^x$. Then $du = e^x dx$ and we have

$$\int \sec u \tan u du = \sec u + C = \sec e^x + C.$$

b. We integrate by parts with $u = e^x$ and $dv = e^x \sec e^x \tan e^x dx$. Then $du = e^x dx$ and $v = \sec e^x$ (by part a). Then we have

$$e^x \sec e^x - \int e^x \sec e^x dx = e^x \sec e^x - \ln |\sec e^x + \tan e^x| + C.$$

76 Using the table of integrals entry 84, we find that

$$\int x(ax+b)^n dx = \frac{(ax+b)^{n+1}(a(n+1)x-b)}{a^2(n+1)(n+2)} + C.$$

Therefore

$$\int x(2x+3)^5 dx = \frac{(2x+3)^6(12x-3)}{168} + C.$$

77 Using the table of integrals entry 77, we find that

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}} \tan^{-1} \left(\sqrt{\frac{ax-b}{b}} \right) + C.$$

Therefore

$$\int \frac{dx}{x\sqrt{4x-6}} = \frac{\sqrt{6}}{3} \tan^{-1} \left(\sqrt{\frac{2x-3}{3}} \right) + C.$$

78 Using the table of integrals entry 34, we find that

$$\int_0^{\pi/2} \frac{d\theta}{1+\sin 2\theta} = \left(\frac{-1}{2} \tan(\pi/4 - \theta) \right) \Big|_0^{\pi/2} = \frac{-1}{2} (\tan(-\pi/4) - \tan(\pi/4)) = \frac{1}{2} (1 - (-1)) = 1.$$

79 Using the table of integrals reduction formula (number 45) twice, we have

$$\begin{aligned} \int \sec^5 x \, dx &= \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \int \sec^3 x \, dx \\ &= \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \left(\frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x \, dx \right) \\ &= \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \left(\frac{\sec x \tan x}{2} + \frac{1}{2} (\ln |\sec x + \tan x|) \right) + C \\ &= \frac{\sec^3 x \tan x}{4} + \frac{3 \sec x \tan x}{8} + \frac{3}{8} (\ln |\sec x + \tan x|) + C. \end{aligned}$$

80 Let $u = e^x$. Then $du = e^x dx$ and we have

$$\int_1^e \frac{1}{u^2 \sqrt{16-u^2}} du = \left(-\frac{\sqrt{16-u^2}}{16u} \right) \Big|_1^e = \frac{\sqrt{15}}{16} - \frac{\sqrt{16-e^2}}{16e} = \frac{\sqrt{15}e - \sqrt{16-e^2}}{16e}.$$

81 Let $u = 1 + \sin x$ so that $du = \cos x \, dx$. We have

$$\begin{aligned} \int_1^2 \ln^3 u \, du &= (u \ln^3 u - 3u \ln^2 u + 6u \ln u - 6u) \Big|_1^2 \\ &= 2 \ln^3 2 - 6 \ln^2 2 + 12 \ln 2 - 12 - (0 - 6) = 2(\ln^3 2 - 3 \ln^2 2 + 6 \ln 2 - 3). \end{aligned}$$

82 This can be written as

$$\lim_{b \rightarrow -\infty} \int_b^{-1} \frac{1}{(x-1)^4} dx = \lim_{b \rightarrow -\infty} \left(\frac{-1}{3(x-1)^3} \right) \Big|_b^{-1} = \lim_{b \rightarrow -\infty} \left(\frac{1}{24} + \frac{1}{3(b-1)^3} \right) = \frac{1}{24}.$$

83 First evaluate

$$\int_0^b x e^{-x} dx = -e^{-x}(x+1) \Big|_0^b = 1 - (b+1)e^{-b}.$$

Then

$$\int_0^\infty x e^{-x} dx = \lim_{b \rightarrow \infty} (1 - (b+1)e^{-b}) = 1.$$

84 First consider $\int_0^{\pi/2} \sec^2 x \, dx$. We have $\int_0^{\pi/2} \sec^2 x \, dx = \lim_{a \rightarrow \pi/2^-} \int_0^a \sec^2 x \, dx = \lim_{a \rightarrow \pi/2^-} \tan x \Big|_0^a = \lim_{a \rightarrow \pi/2^-} \tan a - 0 = \infty$. This integral diverges, and thus $\int_0^\pi \sec^2 x \, dx$ diverges.

85 First take $0 < c < 3$ and evaluate

$$\int_0^c \frac{dx}{\sqrt{9-x^2}} = \sin^{-1} \left(\frac{x}{3} \right) \Big|_0^c = \sin^{-1} \left(\frac{c}{3} \right).$$

Then

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{c \rightarrow 3^-} \sin^{-1} \left(\frac{c}{3} \right) = \frac{\pi}{2}.$$

86

$$\int_{-\infty}^{\infty} \frac{x^3}{1+x^8} dx = \int_{-\infty}^0 \frac{x^3}{1+x^8} dx + \int_0^{\infty} \frac{x^3}{1+x^8} dx = \lim_{a \rightarrow -\infty} \int_a^0 \frac{x^3}{1+x^8} dx + \lim_{b \rightarrow \infty} \int_0^b \frac{x^3}{1+x^8} dx.$$

Let $u = x^4$. Then $du = 4x^3 dx$. We have

$$\int \frac{x^3}{1+x^8} dx = \frac{1}{4} \int \frac{1}{u^2} du = \frac{1}{4} \tan^{-1} u + C = \frac{1}{4} \tan^{-1} x^4 + C$$

Then the original integral is equal to

$$\lim_{a \rightarrow -\infty} \frac{1}{4} \tan^{-1} x^4 \Big|_a^0 + \lim_{b \rightarrow \infty} \frac{1}{4} \tan^{-1} x^4 \Big|_0^b = -\frac{\pi}{8} + \frac{\pi}{8} = 0.$$

87

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x(x-1)^{1/3}} &= \int_1^2 \frac{dx}{x(x-1)^{1/3}} + \int_2^{\infty} \frac{dx}{x(x-1)^{1/3}} \\ &= \lim_{a \rightarrow 1^+} \int_a^2 \frac{dx}{x(x-1)^{1/3}} + \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x(x-1)^{1/3}} \\ &= \lim_{a \rightarrow 1^+} \left(-\log(\sqrt[3]{x-1} + 1) + \frac{1}{2} \log((x-1)^{2/3} - \sqrt[3]{x-1} + 1) + \sqrt{3} \tan^{-1} \left(\frac{2\sqrt[3]{x-1}-1}{\sqrt{3}} \right) \right) \Big|_a^2 \\ &\quad + \lim_{b \rightarrow \infty} \left(-\log(\sqrt[3]{x-1} + 1) + \frac{1}{2} \log((x-1)^{2/3} - \sqrt[3]{x-1} + 1) + \sqrt{3} \tan^{-1} \left(\frac{2\sqrt[3]{x-1}-1}{\sqrt{3}} \right) \right) \Big|_2^b \\ &= \frac{\pi}{\sqrt{3}} - \ln 2 + \frac{\pi}{\sqrt{3}} + \ln 2 = \frac{2\pi}{\sqrt{3}}. \end{aligned}$$

88 Completing the square in the denominator gives $x^2 + 2x + 1 + 1 = (x+1)^2 + 1$. Let $u = x+1$ so that $du = dx$. We have

$$\lim_{a \rightarrow -\infty} \int_a^0 \frac{2}{u^2+1} du + \lim_{b \rightarrow \infty} \int_0^b \frac{2}{u^2+1} du = \lim_{a \rightarrow -\infty} (0 - 2 \tan^{-1} a) + \lim_{b \rightarrow \infty} 2 \tan^{-1} b = \pi + \pi = 2\pi.$$

89 $x^5 + x^4 + x^3 + 1 > x^5$ for $x \geq 1$, so $0 < \frac{1}{x^5 + x^4 + x^3 + 1} < \frac{1}{x^5}$. Because $\int_1^{\infty} \frac{1}{x^5} dx$ converges, so does the given integral by the Comparison Test.

90 $\sqrt{x + \sin x} > \sqrt{x}$ for $0 < x \leq 1$, so $0 < \frac{1}{\sqrt{x + \sin x}} < \frac{1}{\sqrt{x}}$ for $0 < x \leq 1$. Because $\int_0^1 \frac{1}{\sqrt{x}} dx$ converges, so does $\int_0^1 \frac{dx}{\sqrt{x + \sin x}}$ by the Comparison Test.

91 $\frac{x^3}{\sqrt{x^7-1}} \cdot \frac{1}{\sqrt{x^6}} = \frac{1}{\sqrt{x - \frac{1}{x^6}}} > \frac{1}{\sqrt{x}}$ for $x \geq 3$. Then because $\int_3^{\infty} \frac{dx}{\sqrt{x}}$ diverges, so does $\int_3^{\infty} \frac{x^3}{\sqrt{x^7-1}} dx$ by the Comparison Test.

92 Let $u = \ln x$, so that $du = \frac{1}{x} dx$. Then the related indefinite integral is

$$\int \frac{du}{u^p} = \frac{1}{p-1} u^{p-1} + C = \frac{1}{(p-1) \ln^{p-1} x} + C.$$

So

$$\begin{aligned} \int_2^\infty \frac{dx}{x \ln^p x} &= \lim_{b \rightarrow \infty} \left. \frac{1}{(p-1) \ln^{p-1} x} \right|_2^b \\ &= \lim_{b \rightarrow \infty} \left(\frac{1}{(p-1) \ln^{p-1} b} - \frac{1}{(p-1) \ln^{p-1} 2} \right) = -\frac{1}{(p-1) \ln^{p-1} 2} \text{ for } p > 1. \end{aligned}$$

For $p < 1$ this limit doesn't exist so the integral diverges. For $p = 1$, we have

$$\lim_{b \rightarrow \infty} \int_2^b \frac{1}{x \ln x} dx = \lim_{b \rightarrow \infty} \ln \ln x \Big|_2^b = \lim_{b \rightarrow \infty} (\ln \ln b - \ln \ln 2) = \infty.$$

The integral converges to $-\frac{1}{(p-1) \ln^{p-1} 2}$ for $p > 1$ and diverges for $p \leq 1$.

93 Using a Computer Algebra System, we find that

$$\int_{-1}^1 e^{-2x^2} dx \approx 1.196.$$

94 Using a Computer Algebra System, we find that

$$\int_1^{\sqrt{e}} x^3 (\ln x)^3 dx \approx 0.081.$$

95 $\Delta x = \frac{8-0}{4} = 2$. $M(4) = 2(4 + 4 + 6 + 8) = 44$. $T(4) = \frac{2}{2}(3 + 2(5) + 2(7) + 2(5) + 5) = 42$.

$$S(4) = \frac{2}{3}(3 + 4(5) + 2(7) + 4(5) + 5) = \frac{124}{3}.$$

96 $\Delta x = \frac{3-1}{4} = \frac{1}{2}$. $M(4) = \frac{1}{2}(f(1.25) + f(1.75) + f(2.25) + f(2.75)) \approx 0.235137$. $T(4) = \frac{1/2}{2}(f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)) \approx 0.248103$. $S(4) = \frac{1/2}{3}(f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)) \approx 0.239568$.

97 $\Delta x = \frac{1}{40}$. $M(40) \approx 0.398236$, $T(40) \approx 0.398771$, and $S(40) \approx 0.398416$.

98 $\Delta x = \frac{8-1}{60} = \frac{7}{60}$. $M(60) \approx 9.82388$, $T(60) \approx 9.82108$, and $S(60) \approx 9.82296$.

99

n	$T_2(n)$	$T_4(n)$	$T_8(n)$
4	0.880619	0.886319	1.036632
8	0.881704	0.886227	0.886319
16	0.881986	0.886227	0.886227
32	0.882058	0.886227	0.886227

Based on these results, we conclude that $\int_0^\infty e^{-x^2} dx \approx 0.886227$.

100 The area is given by the improper integral

$$\int_0^{\infty} ae^{-ax} dx = \lim_{b \rightarrow \infty} \int_0^b ae^{-ax} dx = \lim_{b \rightarrow \infty} \left(-e^{-ax} \Big|_0^b \right) = \lim_{b \rightarrow \infty} (1 - e^{-ab}) = 1$$

as long as $a > 0$.

101 The volume generated by revolving around the x -axis is

$$V_x = \pi \int_0^{\pi} \sin^2 x dx = \pi \left(\frac{x}{2} - \frac{\sin 2x}{4} \right) \Big|_0^{\pi} = \frac{\pi^2}{2},$$

and the volume generated by revolving around the y -axis is

$$V_y = 2\pi \int_0^{\pi} x \sin x dx = 2\pi (\sin x - x \cos x) \Big|_0^{\pi} = 2\pi^2,$$

so the greater volume is obtained by revolving around the y -axis.

102 The volume is

$$V = 2\pi \int_1^e x \ln x dx = \frac{\pi}{2} x^2 (2 \ln x - 1) \Big|_1^e = \frac{\pi}{2} (e^2 + 1).$$

103 The volume is

$$V = \pi \int_1^e (\ln x)^2 dx = \pi x ((\ln x)^2 - 2 \ln x + 2) \Big|_1^e = \pi(e - 2).$$

104 The volume is

$$V = \pi \int_1^e (1 - (1 - \ln x)^2) dx = \pi \int_1^e (2 \ln x - \ln^2 x) dx = \pi (-4x + 4x \ln x - x \ln^2 x) \Big|_1^e = \pi(4 - e).$$

105 The volume is

$$V = 2\pi \int_1^e (x - 1) \ln x dx = \frac{\pi}{2} x (2(x - 2) \ln x - x + 4) \Big|_1^e = \frac{\pi}{2} (e^2 - 3).$$

106 Note that

$$y' = \frac{1}{2} \sqrt{3 - x^2} + \frac{x}{2} \frac{-2x}{2\sqrt{3 - x^2}} + \frac{3}{2\sqrt{3}} \frac{1}{\sqrt{1 - x^2/3}} = \frac{(6 - 2x^2) - 2x^2 + 6}{4\sqrt{3 - x^2}} = \frac{3 - x^2}{\sqrt{3 - x^2}} = \sqrt{3 - x^2}.$$

$$L = \int_0^1 \sqrt{1 + 3 - x^2} dx = \int_0^1 \sqrt{4 - x^2} dx.$$

Using the table of integrals entry 51, we have

$$L = \left((x/2) \sqrt{4 - x^2} + 2 \sin^{-1}(x/2) \right) \Big|_0^1 = \frac{\sqrt{3}}{2} + \frac{\pi}{3}.$$

107

a. Observe that

$$\int_{1/2}^b \ln x dx = (x \ln x - x) \Big|_{1/2}^b \approx b \ln b - b + 0.847;$$

solve $b \ln b - b + 0.847 = 0$ numerically to obtain $b \approx 1.603$.

b. Similarly, we have

$$\int_{1/3}^b \ln x \, dx = (x \ln x - x) \Big|_{1/3}^b \approx b \ln b - b + 0.700;$$

solve $b \ln b - b + 0.700 = 0$ numerically to obtain $b \approx 1.870$.

c. In general, the pair (a, b) must satisfy the equation

$$\int_a^b \ln x \, dx = (b \ln b - b) - (a \ln a - a) = 0,$$

which gives $b \ln b - b = a \ln a - a$.

d. As a increases there is less negative area to the right of $x = 1$, so $b = g(a)$ is a decreasing function of a .

108 The arc length is given by the integral $\int_1^{e^2} \sqrt{1 + \frac{1}{x^2}} \, dx = \int_1^{e^2} \frac{\sqrt{1+x^2}}{x} \, dx$, which can be evaluated by the trigonometric substitution $x = \tan \theta$ or using a CAS. We obtain

$$\int_1^{e^2} \frac{\sqrt{1+x^2}}{x} \, dx = \left(\sqrt{x^2+1} - \ln \left(2 \left(\sqrt{x^2+1} + 1 \right) \right) + \ln x \right) \Big|_1^{e^2} \approx 6.789.$$

109 The average velocity is

$$\bar{v} = \frac{1}{\pi} \int_0^\pi 10 \sin 3t \, dt = -\frac{10}{3\pi} \cos 3t \Big|_0^\pi = \frac{20}{3\pi}.$$

110

a. The distance traveled by car A after 2 hrs is

$$\int_0^2 \frac{40}{t+1} \, dt = 40 \ln(t+1) \Big|_0^2 = 40 \ln 3 \approx 43.944 \text{ mi}$$

and the distance traveled by car B after 2 hrs is

$$\int_0^2 40e^{-t/2} \, dt = -80e^{-t/2} \Big|_0^2 = 80(1 - e^{-1}) \approx 50.570 \text{ mi},$$

so car B traveled farther.

b. The distance traveled by car A after 3 hrs is

$$\int_0^3 \frac{40}{t+1} \, dt = 40 \ln(t+1) \Big|_0^3 = 40 \ln 4 \approx 55.452 \text{ mi}$$

and the distance traveled by car B after 2 hrs is

$$\int_0^3 40e^{-t/2} \, dt = -80e^{-t/2} \Big|_0^3 = 80(1 - e^{-3/2}) \approx 62.150 \text{ mi},$$

so car B traveled farther.

c. The distance traveled by car A after t hrs is

$$\int_0^t \frac{40}{s+1} \, ds = 40 \ln(s+1) \Big|_0^t = 40 \ln(t+1)$$

and the distance traveled by car B after t hrs is

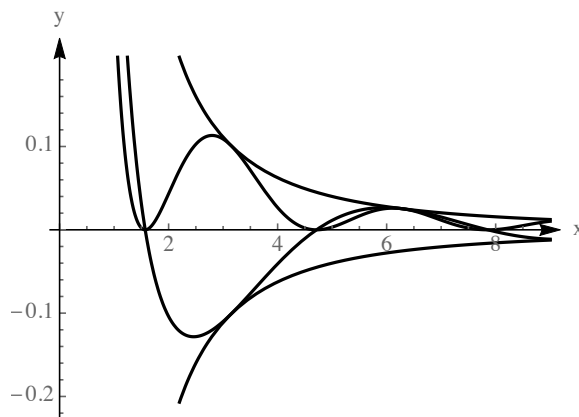
$$\int_0^t 40e^{-s/2} \, ds = -80e^{-s/2} \Big|_0^t = 80(1 - e^{-t/2}).$$

The distance traveled by car A increases without bound, whereas the distance traveled by car B approaches 80 mi as $t \rightarrow \infty$.

111 The number of cars is given by the integral

$$\int_0^4 800te^{-t/2} dt = -1600(t+2)e^{-t/2} \Big|_0^4 = 3200(1 - 3e^{-2}) \approx 1901.$$

112 Observe that both $g(x)$ and $h(x)$ lie between the functions $\pm 1/x^2$, which have finite area from 1 to ∞ . Therefore the improper integrals of both $g(x)$ and $h(x)$ from 1 to ∞ are finite.



113

a. Using integration by parts, we find that

$$I(p) = \int_1^e \frac{\ln x}{x^p} dx = -\frac{x^{1-p}}{(p-1)^2} ((p-1) \ln x + 1) \Big|_1^e = \frac{1}{(p-1)^2} (1 - pe^{1-p})$$

for $p \neq 1$, and using the substitution $u = \ln x$ gives

$$I(1) = \int_1^e \frac{\ln x}{x} dx = \frac{(\ln x)^2}{2} \Big|_1^e = \frac{1}{2}.$$

b. We have

$$\lim_{p \rightarrow \infty} I(p) = \lim_{p \rightarrow \infty} \frac{1}{(p-1)^2} (1 - pe^{1-p}) = \lim_{p \rightarrow \infty} \left(\frac{1}{(p-1)^2} - \frac{pe}{(p-1)^2} e^{-p} \right) = 0,$$

and

$$\lim_{p \rightarrow -\infty} I(p) = \lim_{p \rightarrow -\infty} \left(\frac{1}{(p-1)^2} - \frac{pe}{(p-1)^2} e^{-p} \right) = \infty,$$

since e^{-p} grows much faster than $(p-1)^2/p$ as $p \rightarrow -\infty$.

c. By inspection we see that $I(0) = 1$.

114

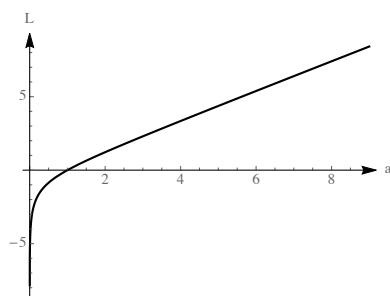
a. If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$, so $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\sqrt{x^2 + 1}}{x}$. Thus the arc length is

$$L(a) = \int_1^a \frac{\sqrt{x^2 + 1}}{x} dx.$$

If we let $u^2 = x^2 + 1$, then $2u \, du = 2x \, dx$. Substituting gives

$$\begin{aligned}
 \int_{\sqrt{2}}^{\sqrt{a^2+1}} \frac{u}{\sqrt{u^2-1}} \cdot \frac{u}{\sqrt{u^2-1}} \, du &= \int_{\sqrt{2}}^{\sqrt{a^2+1}} \frac{u^2}{u^2-1} \, du \\
 &= \int_{\sqrt{2}}^{\sqrt{a^2+1}} \left(1 + \frac{1}{u^2-1} \right) \, du \\
 &= \int_{\sqrt{2}}^{\sqrt{a^2+1}} \left(1 + \frac{1}{2} \left(\frac{1}{u-1} - \frac{1}{u+1} \right) \right) \, du \\
 &= u + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| \bigg|_{\sqrt{2}}^{\sqrt{a^2+1}} \\
 &= \sqrt{a^2+1} - \sqrt{2} + \frac{1}{2} \ln \left(\frac{\sqrt{a^2+1}-1}{\sqrt{a^2+1}+1} \right) + \frac{1}{2} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right).
 \end{aligned}$$

b.



c. Because the only term in $L(a)$ which increases as $a \rightarrow \infty$ is $\sqrt{a^2+1}$, and this term increases like a (because $\lim_{x \rightarrow \infty} \frac{\sqrt{a^2+1}}{a} = 1$), this function increases like a .

115 Use a calculator program for, say, Simpson's rule with $n = 100$, but replace the limits with 0.00000001 and 0.99999999 to avoid errors coming from trying to evaluate the function at $x = 0$ or $x = 1$; we obtain

$$\int_0^1 \frac{x^2 - x}{\ln x} \, dx \approx 0.4054651.$$

116 Numerically approximate the integral to obtain

$$\int_0^{1/2} \frac{\ln(1+2x)}{x} \, dx \approx 0.8224;$$

therefore $n = \pi^2/0.8224 = 12$. (When approximating the integral, replace the lower limit 0 with a small positive number like 0.00001 to avoid an error from evaluating the integrand at $x = 0$.)

117 Numerically approximate the integral to obtain

$$\int_0^1 \frac{\sin^{-1} x}{x} \, dx \approx 1.0889;$$

therefore $n = \pi \ln 2/1.0889 = 2$. (When approximating the integral, replace the lower limit 0 with a small positive number like 0.00001 to avoid an error from evaluating the integrand at $x = 0$.)

118

a. Following the hint, write

$$I(a) = \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} + \int_1^\infty \frac{dx}{(1+x^a)(1+x^2)}$$

and make the substitution $u = 1/x$ in the second integral to obtain

$$\begin{aligned} I(a) &= \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} - \int_1^0 \frac{du}{(1+u^{-a})(1+u^{-2})u^2} \\ &= \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} + \int_0^1 \frac{u^a du}{(1+u^a)(1+u^2)} \\ &= \int_0^1 \frac{1+x^a}{(1+x^a)(1+x^2)} dx = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x = \frac{\pi}{4}. \end{aligned}$$

b. Let

$$I = \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)} dx$$

and make the substitution $u = \pi/2 - x$ to obtain

$$I = - \int_{\pi/2}^0 \frac{f(\sin u)}{f(\sin u) + f(\cos u)} du = \int_0^{\pi/2} \frac{f(\sin u)}{f(\sin u) + f(\cos u)} du.$$

Therefore

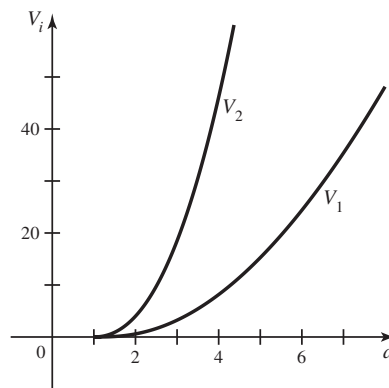
$$2I = \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)} dx + \int_0^{\pi/2} \frac{f(\sin x)}{f(\cos x) + f(\sin x)} dx = \int_0^{\pi/2} dx = \frac{\pi}{2},$$

so $I = \pi/4$.

119

- a. Using integration by parts or a CAS, we find that the volume is $V_1(a) = \pi \int_1^a (\ln x)^2 dx = \pi x((\ln x)^2 - 2 \ln x + 2) \Big|_1^a = \pi[(a \ln^2 a - 2a \ln a + 2(a-1))]$.
- b. Using integration by parts or a CAS, we find that the volume is $V_2(a) = 2\pi \int_1^a x \ln x dx = \frac{\pi}{2} x^2 (2 \ln x - 1) \Big|_1^a = \frac{\pi}{2} (2a^2 \ln a - a^2 + 1)$.

- c. As shown by the graph, $V_2(a) > V_1(a)$ for all $a > 1$.



120

- a. We have $V_1 = \pi \int_1^\infty x^{-2p} dx = \frac{\pi}{2p-1}$ for $p > \frac{1}{2}$, and $V_1 = \infty$ for $p \leq 1/2$ (see Example 2 in Section 7.4 for the evaluation of this improper integral.) Similarly $V_2 = 2\pi \int_1^\infty x^{1-p} dx = \frac{2\pi}{p-2}$ for $p > 2$, and $V_2 = \infty$ for $p \leq 2$. Observe that $V_1 = \frac{2\pi}{4p-2} < \frac{2\pi}{p-2} = V_2$ for all $p > 2$. Therefore $V_1 = V_2$ only when both are infinite.

b. We will use the fact that $\int_0^1 x^a dx = \frac{1}{a+1}$ for $a > -1$ and is infinite otherwise. We have $V_1 = \pi \int_0^1 x^{-2p} dx = \frac{\pi}{1-2p}$ for $p < \frac{1}{2}$, and $V_1 = \infty$ for $p \geq 1/2$. Similarly

$$V_2 = 2\pi \int_0^1 x^{1-p} dx = \frac{2\pi}{2-p} \text{ for } p < 2,$$

and $V_2 = \infty$ for $p \geq 2$. As above, $V_1 = V_2$ only when both are infinite.

121 We have

$$V_1 = \pi \int_0^b e^{-2ax} dx = -\frac{\pi}{2a} e^{-2ax} \Big|_0^b = \frac{\pi}{2a} (1 - e^{-2ab}),$$

and

$$V_2 = \pi \int_b^\infty e^{-2ax} dx = \lim_{c \rightarrow \infty} \left(-\frac{\pi}{2a} e^{-2ax} \Big|_b^c \right) = \frac{\pi}{2a} e^{-2ab}.$$

Equating and solving gives $e^{-2ab} = 1/2$, which is equivalent to $ab = (1/2) \ln 2$.

$$\mathbf{122} \quad R_1 = \int_0^{\pi/3} \tan x dx = -\ln |\cos x| \Big|_0^{\pi/3} = -\ln(1/2) = \ln 2.$$

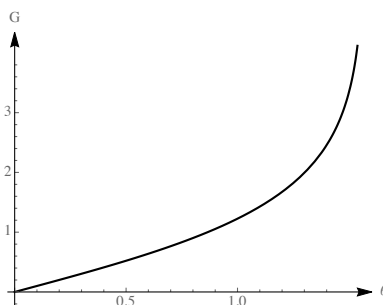
$$R_2 = \int_0^{\pi/6} \sec x dx = \ln |\sec x + \tan x| \Big|_0^{\pi/6} = \ln \sqrt{3} = \frac{\ln 3}{2}.$$

$R_1 \approx 0.693$ and $R_2 \approx 0.549$ so $R_1 > R_2$.

123

$$\begin{aligned} A &= \int_0^{\pi/4} (\sec x - \tan x) dx = (\ln |\sec x + \tan x| + \ln |\cos x|) \Big|_0^{\pi/4} \\ &= \ln(\sqrt{2} + 1) + \ln\left(\frac{\sqrt{2}}{2}\right) - (0 + 0) = \ln\left(1 + \frac{\sqrt{2}}{2}\right). \end{aligned}$$

124 Note that the function is equivalent to $\ln |\sec \theta + \tan \theta|$. Its derivative is $\sec \theta$ which is positive on the interval $[0, \pi/2)$ as its second derivative $\sec \theta \tan \theta$, so it is increasing and concave up on the given interval.



125

a. From the reduction formula in Section 8.3 we have

$$\int_0^\pi \sin^m x dx = \frac{-\sin^{m-1} x \cos x}{m} \Big|_0^\pi + \frac{m-1}{m} \int_0^\pi \sin^{m-2} x dx = \frac{m-1}{m} \int_0^\pi \sin^{m-2} x dx.$$

b. We proceed by strong induction on n . For $n = 1$, we have

$$\int_0^\pi \sin^2 x \, dx = \int_0^\pi \left(\frac{1}{2} - \frac{\sin 2x}{2} \right) dx = \left(\frac{1}{2}x + \frac{\cos 2x}{4} \right) \Big|_0^\pi = \frac{\pi}{2}.$$

Assume the result holds for $k < n$. Then by the reduction formula, we have

$$\int_0^\pi \sin^{2n} x \, dx = \frac{2n-1}{2n} \int_0^\pi \sin^{2n-2} x \, dx = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \pi.$$

c. For $n = 1$ we have $\int_0^\pi \sin 3x \, dx = \int_0^\pi \sin x(1 - \cos^2 x) \, dx = \int_0^\pi \sin x \, dx - \int_0^\pi \sin x \cos^2 x \, dx = -\cos x \Big|_0^\pi - \int_0^\pi \sin x \cos^2 x \, dx = 2 - \int_0^\pi \sin x \cos^2 x \, dx$. Let $u = \cos x$. Then $du = -\sin x \, dx$. Then we have

$$2 + \int_1^{-1} u^2 \, du = 2 + \frac{u^3}{3} \Big|_1^{-1} = 2 - \frac{2}{3} = \frac{4}{3}.$$

If the result holds for integers less than n , then by the reduction formula we have

$$\int_0^\pi \sin^{2n+1} x \, dx = \frac{-\sin^{2n}(-\cos x)}{2n+1} \Big|_0^\pi + \frac{2n}{2n+1} \int_0^\pi \sin^{2n-1} x \, dx = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot 2.$$

d. Because $0 \leq \sin x \leq 1$, if we multiply through by $\sin^m x$ we have $0 \leq \sin^{m+1} x \leq \sin^m x$ for $0 \leq x \leq \pi$. Therefore $\int_0^\pi 0 \, dx \leq \int_0^\pi \sin^{m+1} x \, dx \leq \int_0^\pi \sin^m x \, dx$. So $0 \leq \int_0^\pi \sin^{m+1} x \, dx \leq \int_0^\pi \sin^m x \, dx$.

e. By part d we have

$$\int_0^\pi \sin^{2n+1} x \, dx \leq \int_0^\pi \sin^{2n} x \, dx \leq \int_0^\pi \sin^{2n-1} x \, dx.$$

Therefore

$$\frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot 2 \leq \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \pi \leq \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \frac{2}{3} \cdot 2$$

for any integer $n \geq 2$.

f. Let $A_n = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot 2$ and let $B_n = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4}$ and let $C_n = \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \frac{2}{3} \cdot 2$. The inequality in part e can be written as

$$A_n \leq B_n \left(\frac{\pi}{2} \right) \leq C_n.$$

From this it follows that $\frac{A_n}{B_n} \leq \frac{\pi}{2}$ and that $\frac{\pi}{2} \leq \frac{C_n}{B_n}$. The inequality $\frac{A_n}{B_n} \leq \frac{\pi}{2}$ can be written as

$$\frac{(2n)^2}{2n+1} \cdot \frac{(2n-2)^2}{(2n-1)^2} \cdots \frac{2^2}{3^2} \leq \frac{\pi}{2},$$

and the inequality $\frac{\pi}{2} \leq \frac{C_n}{B_n}$ can be written as

$$\frac{\pi}{2} \leq 2n \cdot \frac{(2n-2)^2}{(2n-1)^2} \cdot \frac{(2n-4)^2}{(2n-3)^2} \cdots \frac{2^2}{3^2},$$

so

$$\frac{2n}{2n+1} \cdot \frac{\pi}{2} \leq \frac{(2n)^2}{2n+1} \cdot \frac{(2n-2)^2}{(2n-1)^2} \cdots \frac{2^2}{3^2}.$$

Putting these together gives the desired result.

g. Because $\lim_{n \rightarrow \infty} \frac{2n}{2n+1} \cdot \frac{\pi}{2} = \frac{\pi}{2}$, the result follows immediately from the Squeeze Theorem.

Chapter 9

Differential Equations

9.1 Basic Ideas

9.1.1

a. This differential equation is order 1, so there is 1 arbitrary constant.

b. It is linear.

9.1.2 We have $y(0) = C + 10 = 5$, so $C = -5$. The solution is $y(t) = -5e^{-3t} + 10$.

9.1.3 Yes. Note that $y'''(t) = 0$ and $y'(t) = 2$.

9.1.4 Yes. $y'(t) = -18e^{-3t}$, so $y'(t) - 3y(t) = -18e^{-3t} - (-18e^{-3t}) = 0$, and $y(0) = 6$.

9.1.5 The portion of the domain of $y = 2 \sec t$ that contains the point $(\pi, -2)$ is the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

9.1.6 By the definition of a solution to an initial value problem, a solution to an initial value problem is differentiable on an interval and satisfies the differential equation on that interval. If the solution's graph crossed the line $t = 1$, then the domain of the solution would be an interval that includes $t = 1$, and yet $y'(t)$ is not defined at $t = 1$.

9.1.7 Yes, it is a solution. Note that $y'(t) = -5Ce^{-5t}$, so $y'(t) + 5y(t) = 0$.

9.1.8 Yes, it is a solution. $y'(t) = 3Ct^2$, so $ty'(t) - 3y(t) = 3Ct^3 - 3Ct^3 = 0$.

9.1.9 Yes, it is a solution. $y'(t) = 4C_1 \cos 4t - 4C_2 \sin 4t$, so $y''(t) = -16C_1 \sin 4t - 16C_2 \cos 4t$, so $y''(t) + 16y(t) = 0$.

9.1.10 Yes, it is a solution. $y'(x) = -C_1e^{-x} + C_2e^x$, so $y''(x) = C_1e^{-x} + C_2e^x$, so $y''(x) - y(x) = 0$.

9.1.11 $u'(t) = Ce^{1/(4t^4)} \left(-\frac{4}{4}\right) t^{-5} = -\frac{u(t)}{t^5}$. Thus $u'(t) + \frac{u(t)}{t^5} = -\frac{u(t)}{t^5} + \frac{u(t)}{t^5} = 0$.

9.1.12

$$u'(t) = C_1e^t + C_2e^t + C_2te^t$$

, and

$$u''(t) = C_1e^t + C_2e^t + C_2e^t + C_2te^t = C_1e^t + 2C_2e^t + C_2te^t.$$

Thus,

$$u''(t) - 2u'(t) + u(t) = (C_1e^t + 2C_2e^t + C_2te^t) - 2(C_1e^t + C_2e^t + C_2te^t) + C_1e^t + C_2te^t = 0.$$

9.1.13

$$g'(x) = -2C_1e^{-2x} + C_2e^{-2x} + -2C_2xe^{-2x},$$

so

$$g''(x) = 4C_1e^{-2x} - 2C_2e^{-2x} + -2C_2e^{-2x} + 4C_2xe^{-2x} = 4C_1e^{-2x} - 4C_2e^{-2x} + 4C_2xe^{-2x}.$$

Thus,

$$\begin{aligned} g''(x) + 4g'(x) + 4g(x) &= 4C_1e^{-2x} - 4C_2e^{-2x} + 4C_2xe^{-2x} + 4(-2C_1e^{-2x} \\ &\quad + C_2e^{-2x} + -2C_2xe^{-2x}) + 4(C_1e^{-2x} + C_2xe^{-2x} + 2) = 8. \end{aligned}$$

9.1.14 $u'(t) = 2C_1t + 3C_2t^2$, so $u''(t) = 2C_1 + 6C_2t$. Thus,

$$t^2u''(t) - 4tu'(t) + 6u(t) = 2C_1t^2 + 6C_2t^3 - 4(2C_1t^2 + 3C_2t^3) + 6C_1t^2 + 6C_2t^3 = 0.$$

9.1.15 $u'(t) = 5C_1t^4 - 4C_2t^{-5} - 3t^2$, so $u''(t) = 20C_1t^3 + 20C_2t^{-6} - 6t$. Thus,

$$t^2u''(t) - 20u(t) = 20C_1t^5 + 20C_2t^{-4} - 6t^3 - 20(C_1t^5 + C_2t^{-4} - t^3) = 14t^3.$$

9.1.16 $z'(t) = -C_1e^{-t} + 2C_2e^{2t} - 3C_3e^{-3t} - e^t$. So

$$z''(t) = C_1e^{-t} + 4C_2e^{2t} + 9C_3e^{-3t} - e^t,$$

and

$$z'''(t) = -C_1e^{-t} + 8C_2e^{2t} - 27C_3e^{-3t} - e^t,$$

Thus

$$\begin{aligned} z'''(t) + 2z''(t) - 5z'(t) - 6z(t) &= -C_1e^{-t} + 8C_2e^{2t} - 27C_3e^{-3t} - e^t \\ &\quad + 2C_1e^{-t} + 8C_2e^{2t} + 18C_3e^{-3t} - 2e^t \\ &\quad + 5C_1e^{-t} - 10C_2e^{2t} + 15C_3e^{-3t} + 5e^t \\ &\quad - 6C_1e^{-t} - 6C_2e^{2t} - 6C_3e^{-3t} + 6e^t \\ &= 8e^t \end{aligned}$$

9.1.17 Yes, it is a solution. $y'(t) = 32e^{2t}$, so $y'(t) - 2y(t) = 32e^{2t} - (32e^{2t} - 20) = 20$. Also, $y(0) = 16 - 10 = 6$.

9.1.18 Yes, it is a solution. $y'(t) = 48t^5$, so $ty'(t) - 6y(t) = 48t^6 - 48t^6 + 18 = 18$. Also, $y(1) = 8 - 3 = 5$.

9.1.19 Yes, it is a solution. $y'(t) = 9 \sin 3t$, so $y''(t) = 27 \cos 3t$. Thus, $y''(t) + 9y(t) = 27 \cos 3t - 27 \cos 3t = 0$. Also, $y'(0) = 0$ and $y(0) = -3$.

9.1.20 Yes, it is a solution. $y'(x) = \frac{1}{4}(2e^{2x} + 2e^{-2x})$ and $y''(x) = \frac{1}{4}(4e^{2x} - 4e^{-2x})$. So $y''(x) - 4y(x) = 0$. Also, $y(0) = 0$ and $y'(0) = 1$.

9.1.21 $y(t) = \int (3 + e^{-2t}) dt = 3t - \frac{1}{2}e^{-2t} + C.$

9.1.22 $y(t) = \int (12t^5 - 20t^4 + 2 - 6t^{-2}) dt = 2t^6 - 4t^5 + 2t + \frac{6}{t} + C.$

9.1.23 $y(x) = \int (4 \tan 2x - 3 \cos x) dx = -2 \ln |\cos 2x| - 3 \sin x + C = 2 \ln |\sec 2x| - 3 \sin x + C.$

9.1.24 $p(x) = \int (16x^{-9} - 5 + 14x^6) dx = -2x^{-8} - 5x + 2x^7 + C.$

$$\mathbf{9.1.25} \quad y'(t) = \int (60t^4 - 4 + 12t^{-3}) dt = 12t^5 - 4t - 6t^{-2} + C. \quad y(t) = \int (12t^5 - 4t - 6t^{-2} + C) dt = 2t^6 - 2t^2 + 6t^{-1} + C_1t + C_2.$$

$$\mathbf{9.1.26} \quad y'(t) = \int (15e^{3t} + \sin 4t) dt = 5e^{3t} - \frac{1}{4} \cos 4t + C_1. \quad y(t) = \int (5e^{3t} - \frac{1}{4} \cos 4t + C_1) dt = \frac{5}{3}e^{3t} - \frac{1}{16} \sin 4t + C_1t + C_2.$$

$$\mathbf{9.1.27} \quad u'(x) = \int (55x^9 + 36x^7 - 21x^5 + 10x^{-3}) dx = 5.5x^{10} + \frac{9}{2}x^8 - \frac{7}{2}x^6 - 5x^{-2} + C_1.$$

$$u(x) = \int (5.5x^{10} + \frac{9}{2}x^8 - \frac{7}{2}x^6 - 5x^{-2} + C) dx = \frac{1}{2}x^{11} + \frac{1}{2}x^9 - \frac{1}{2}x^7 + 5x^{-1} + C_1x + C_2.$$

$$\mathbf{9.1.28} \quad v'(x) = \int \frac{5}{x^2} dx = -\frac{5}{x} + C_1, \text{ so } v(x) = \int \left(-\frac{5}{x} + C_1\right) dx = -5 \ln |x| + C_1x + C_2.$$

$$\mathbf{9.1.29} \quad u(x) = \int \frac{2x}{x^2 + 4} dx - \int \frac{2}{x^2 + 4} dx = \ln(x^2 + 4) - \tan^{-1}(x/2) + C.$$

$$\mathbf{9.1.30} \quad y(t) = \int (t \ln t + 1) dt = t + \int t \ln t dt. \text{ Let } u = \ln t \text{ and } dv = t, \text{ so that } du = \frac{dt}{t} \text{ and } v = t^2/2. \text{ Then}$$

$$y(t) = t + (t^2 \ln t)/2 - \int t/2 dt = t + (t^2 \ln t)/2 - t^2/4 + C.$$

$$\mathbf{9.1.31} \quad y'(x) = \int \frac{x}{(1-x^2)^{3/2}} dx. \text{ Let } u = 1-x^2, \text{ so that } du = -2x dx. \text{ Substituting gives}$$

$$y'(x) = -\frac{1}{2} \int u^{-3/2} du = u^{-1/2} + C_1 = \frac{1}{\sqrt{1-x^2}} + C_1 dx. \quad y(x) = \int \left(\frac{1}{\sqrt{1-x^2}} + C_1\right) dx = \sin^{-1}(x) + C_1x + C_2.$$

$$\mathbf{9.1.32} \quad \text{Note that } \frac{4}{t^2-4} = \frac{1}{t-2} - \frac{1}{t+2}. \text{ Thus, } v(t) = \int \frac{4}{t^2-4} dt = \int \left(\frac{1}{t-2} - \frac{1}{t+2}\right) dt = \ln \left|\frac{t-2}{t+2}\right| + C.$$

$$\mathbf{9.1.33} \quad y(t) = \int (1 + e^t) dt = t + e^t + C. \text{ Because } y(0) = 4 = 1 + C, \text{ we have } C = 3. \text{ Thus, } y(t) = t + e^t + 3.$$

$$\mathbf{9.1.34} \quad y(t) = \int (\sin t + \cos 2t) dt = -\cos t + \frac{1}{2} \sin 2t + C. \text{ Because } y(0) = 4 = -1 + C, \text{ we have } C = 5.$$

Thus, $y(t) = -\cos t + \frac{1}{2} \sin 2t + 5.$

$$\mathbf{9.1.35} \quad y(x) = \int (3x^2 - 3x^{-4}) dx = x^3 + x^{-3} + C. \text{ Because } y(1) = 0 = 1 + 1 + C, \text{ we have } C = -2. \text{ So}$$

$$y(x) = x^3 + x^{-3} - 2, \quad x > 0.$$

$$\mathbf{9.1.36} \quad y(x) = \int 4 \sec^2 2x dx = 2 \tan 2x + C. \text{ Because } y(0) = 8 = 0 + C, \text{ we have } C = 8. \text{ Thus, } y(x) =$$

$$2 \tan 2x + 8, \quad -\frac{\pi}{4} < x < \frac{\pi}{4}.$$

$$\mathbf{9.1.37} \quad y'(t) = \int (12t - 20t^3) dt = 6t^2 - 5t^4 + C_1. \text{ Because } y'(0) = 0 = 0 + C_1, \text{ we have } C_1 = 0. \quad y(t) =$$

$$\int (6t^2 - 5t^4) dt = 2t^3 - t^5 + C_2. \text{ Because } y(0) = 1 = 0 - 0 + C_2, \text{ we have } C_2 = 1. \text{ Thus, } y(t) = 2t^3 - t^5 + 1.$$

$$\mathbf{9.1.38} \quad u'(x) = \int (4e^{2x} - 8e^{-2x}) dx = 2e^{2x} + 4e^{-2x} + C_1. \text{ Because } u'(0) = 3 = 2 + 4 + C_1, \text{ we have } C_1 = -3.$$

$$u(x) = \int (2e^{2x} + 4e^{-2x} - 3) dx = e^{2x} - 2e^{-2x} - 3x + C_2. \text{ Because } u(0) = 1 = 1 - 2 - 0 + C_2, \text{ we have } C_2 = 2.$$

Thus, $u(x) = e^{2x} - 2e^{-2x} - 3x + 2.$

9.1.39 Using the result of number 40 below, we have $y'(t) = te^t - e^t + C_1$, and because $y'(0) = 1 = 0 - 1 + C_1$, we have $C_1 = 2$. Thus $y'(t) = te^t - e^t + 2$. $y(t) = \int y'(t) dt = \int (te^t - e^t + 2) dt = te^t - e^t - e^t + 2t + C_2 = te^t - 2e^t + 2t + C_2$. Because $y(0) = 0 = 0 - 2 + 0 + C_2$, we have $C_2 = 2$. Thus, $y(t) = te^t - 2e^t + 2t + 2$.

9.1.40 Let $u = t$ and $dv = e^t dt$. Then $du = dt$ and $v = e^t$. Thus, $y(t) = \int te^t dt = te^t - \int e^t dt = te^t - e^t + C$. Because $y(0) = -1 = 0 - 1 + C$, we have $C = 0$. Thus $y(t) = te^t - e^t$.

9.1.41 $u(x) = \int \left(\frac{1}{x^2 + 4^2} - 4 \right) dx = \frac{1}{4} \tan^{-1}(x/4) - 4x + C$. Because $u(0) = 2 = 0 - 0 + C$, we have $C = 2$. Thus, $u(x) = \frac{1}{4} \tan^{-1}(x/4) - 4x + 2$.

9.1.42 $p(x) = \int \frac{2}{x(x+1)} dx = \int \left(\frac{2}{x} - \frac{2}{x+1} \right) dx = 2 \ln \left| \frac{x}{x+1} \right| + C$. Because $p(1) = 0 = 2 \ln(1/2) + C$, we have $C = -2 \ln(1/2) = 2 \ln 2$. Thus, $p(x) = 2 \ln \left| \frac{x}{x+1} \right| + 2 \ln 2$, $x > 0$.

9.1.43

a. $v(t) = -9.8t + 29.4$. $s(t) = -4.9t^2 + 29.4t + 30$. The domain of each function is approximately $0 \leq t \leq 6.89$.

b. The object reaches its high point when $-9.8t + 29.4 = 0$, or $t = \frac{29.4}{9.8} = 3$. At that time its position is $s(3) \approx 74.1$ meters.

9.1.44

a. $v(t) = -9.8t + 49$. $s(t) = -4.9t^2 + 49t + 60$. The domain of each function is approximately $0 \leq t \leq 11.1$.

b. The object reaches its high point when $-9.8t + 49 = 0$, or $t = \frac{49}{9.8} = 5$. At that time its position is $s(5) = 182.5$ meters.

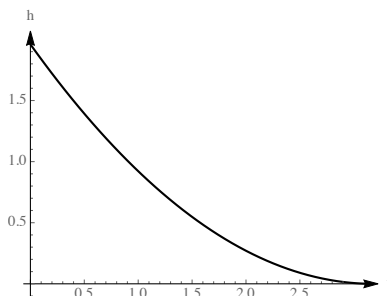
9.1.45 We have $p(t) = (1500 - 20H)e^{.05t} + 20H$. The amount of resource is increasing when $1500 - 20H > 0$, which occurs for $H < 75$. The amount of resource is constant when $1500 - 20H = 0$, which occurs for $H = 75$. If $H = 100$, the resource is zero when $(1500 - 2000)e^{.05t} + 2000 = 0$, which occurs for $t = 20 \ln 4 \approx 28$.

9.1.46 We have $p(t) = (p_0 - 10000)e^{.05t} + 10000$. The amount of resource is decreasing when $p_0 - 10000 < 0$, or $p_0 < 10,000$. The amount of resource is constant when $p_0 = 10,000$. If $p_0 = 9000$, the resource vanishes when $-1000e^{.05t} = -10000$, or $t = 20 \ln 10 \approx 46$.

9.1.47 The height function is given by

$$h(t) = \left(\sqrt{1.96} - \frac{0.3\sqrt{2 \cdot 9.8}}{1.5} \cdot \frac{t}{2} \right)^2 \approx (1.4 - 0.44t)^2 \text{ for } 0 \leq t \leq 3.16.$$

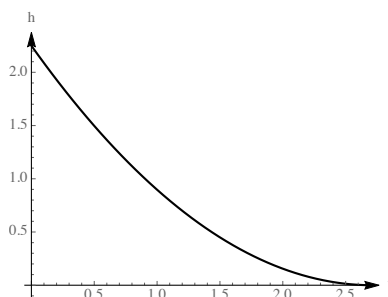
The tank is empty when $h(t) = 0$, which occurs after about 3.16 seconds.



9.1.48 The height function is given by

$$h(t) = \left(\sqrt{2.25} - \frac{0.5\sqrt{2 \cdot 9.8}}{2} \cdot \frac{t}{2} \right)^2 \approx (1.5 - 0.553t)^2 \text{ for } 0 \leq t \leq 2.71.$$

The tank is empty when $h(t) = 0$, which occurs after about 2.71 seconds.



9.1.49

- False. That is a specific solution. The general solution is $t + C$.
- False. It is second order, but is not linear.
- True. First find the general solution, and then find the specific solution which satisfies the initial condition.

9.1.50

- $y'(t) = C_1 e^t - C_2 e^{-t}$, so $y''(t) = C_1 e^t + C_2 e^{-t}$. Thus, $y''(t) - y(t) = 0$.
- $y'(t) = 2C_1 e^{2t} - 2C_2 e^{-2t}$, so $y''(t) = 4C_1 e^{2t} + 4C_2 e^{-2t}$. Thus, $y''(t) - 4y(t) = 0$.
- It appears that a general solution should be $C_1 e^{kt} + C_2 e^{-kt}$. Then $y'(t) = kC_1 e^{kt} - kC_2 e^{-kt}$, and $y''(t) = k^2 C_1 e^{kt} + k^2 C_2 e^{-kt}$. Thus, $y''(t) - k^2 y(t) = 0$.
- If $y(t) = C_1 \cosh kt + C_2 \sinh kt$, then $y'(t) = kC_1 \sinh kt + kC_2 \cosh kt$ and $y''(t) = k^2 C_1 \cosh kt + k^2 C_2 \sinh kt$. Thus $y''(t) - k^2 y(t) = 0$.

9.1.51

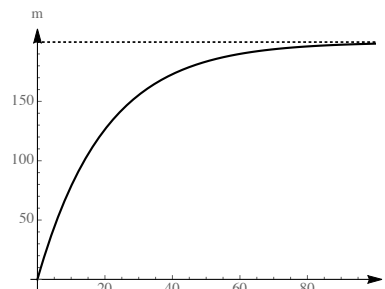
- $y'(t) = C_1 \cos t - C_2 \sin t$, so $y''(t) = -C_1 \sin t - C_2 \cos t$. Thus, $y''(t) + y(t) = 0$.
- $y'(t) = 2C_2 \cos 2t - 2C_2 \sin 2t$, so $y''(t) = -4C_2 \sin 2t - 4C_2 \cos 2t$. Thus, $y''(t) + 4y(t) = 0$.
- A general solution appears to be $y(t) = C_1 \sin kt + C_2 \cos kt$. Then $y'(t) = kC_1 \cos kt - kC_2 \sin kt$, so $y''(t) = -k^2 C_1 \sin kt - k^2 C_2 \cos kt$. And then $y''(t) + k^2 y(t) = 0$.

9.1.52

- Let $m(t) = \frac{I}{k}(1 - e^{-kt})$. Note that $m(0) = 0$. Then $m'(t) = \frac{I}{k}(ke^{-kt})$. Therefore,

$$m'(t) + km(t) = \frac{I}{k}(ke^{-kt}) + \frac{kI}{k}(1 - e^{-kt}) = Ie^{-kt} + I - Ie^{-kt} = I.$$

- b. We have $m(t) = 200(1 - e^{-0.05t})$.



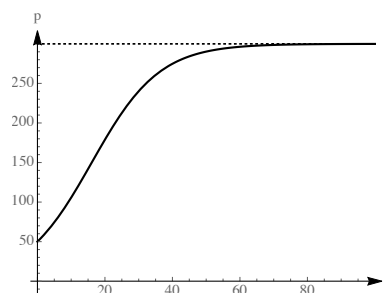
- c. It appears that $\lim_{t \rightarrow \infty} m(t) = 200$.

9.1.53

- a. Let $p(t) = \frac{K}{1 + Ce^{-rt}}$. Note that $1 - \frac{P}{K} = 1 - \frac{1}{1 + Ce^{-rt}} = \frac{Ce^{-rt}}{1 + Ce^{-rt}}$. We have

$$p'(t) = \frac{KCre^{-rt}}{(1 + Ce^{-rt})^2} = r \cdot \frac{K}{1 + Ce^{-rt}} \cdot \frac{Ce^{-rt}}{1 + Ce^{-rt}} = rp \left(1 - \frac{p}{K}\right).$$

- b. If $p(0) = 50 = \frac{K}{1 + C}$, then $50 + 50C = K$, so $C = \frac{K - 50}{50}$.



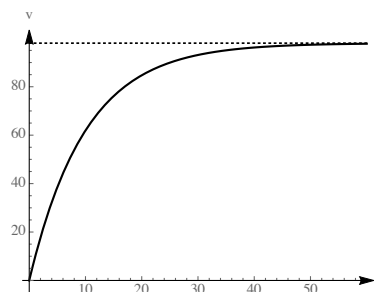
- c. We have $p(t) = \frac{300}{1 + 5e^{-0.1t}}$.

- d. $\lim_{t \rightarrow \infty} \frac{300}{1 + 5e^{-0.1t}} = \frac{300}{1 + 0} = 300$, which is consistent with the graph from part c.

9.1.54

- a. Let $v(t) = \frac{g}{b}(1 - e^{-bt})$. Then $v(0) = 0$, and

$$v'(t) = \frac{g}{b} \cdot be^{-bt} = ge^{-bt} = g - b \cdot \frac{g}{b}(1 - e^{-bt}) = g - bv.$$



- b. With $b = 0.1$, we have $v(t) = 98(1 - e^{-0.1t})$.

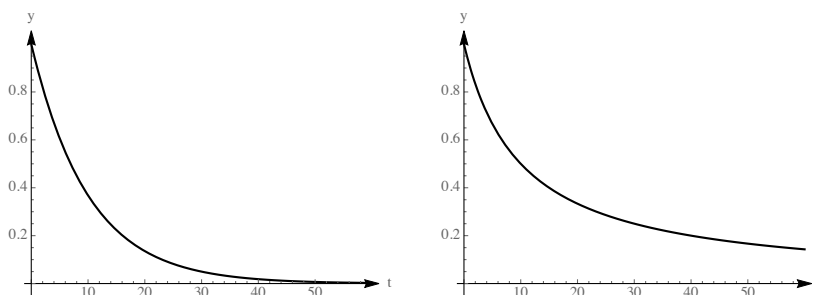
c. $\lim_{t \rightarrow \infty} v(t) = 98.$

9.1.55

a. If $y(t) = y_0 e^{-kt}$, then $y(0) = y_0$, and $y'(t) = -ky_0 e^{-kt}$, so $y'(t) = -ky(t)$.

b. Let $y(t) = \frac{y_0}{y_0 kt + 1}$. Then $y(0) = y_0$, and $y'(t) = \frac{-y_0^2 k}{(y_0 kt + 1)^2} = -k(y(t))^2$.

c. The first order reaction decays more quickly.

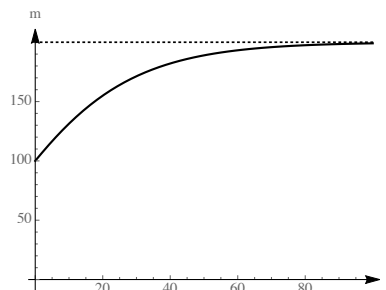


9.1.56

a. Let $M(t) = K \left(\frac{M_0}{K} \right)^{e^{-rt}}$. Note that $\ln(M(t)/K) = e^{-rt} \ln(M_0/K)$.

$$M'(t) = K \left(\frac{M_0}{K} \right)^{e^{-rt}} \ln(M_0/K) (-re^{-rt}) = -rM(t) \ln(M(t)/K). \text{ Also, } M(0) = K(M_0/K) = M_0.$$

Using $K = 200$, $M_0 = 100$, and $r = 0.05$, we have $M(t) = K \left(\frac{M_0}{K} \right)^{e^{-rt}} = 200(1/2)^{e^{-0.05t}}$.



c. $\lim_{t \rightarrow \infty} M(t) = 200 = K.$

9.2 Direction Fields and Euler's Method

9.2.1 Choose a regular grid of points in the ty -plane, and for each point P , make a small line segment with slope $f(t, y)$.

9.2.2 It will have slope $3^2 - 3(1)^2 = 6$.

9.2.3 $u_0 = y(3) = 1$. $u_1 = u_0 + f(3, 1)(.1) = 1 + .6 = 1.6$.

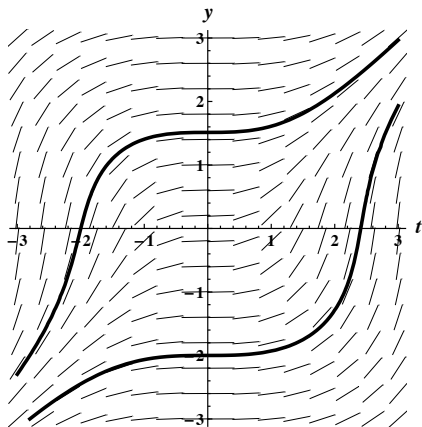
9.2.4 Because the differential equation is giving the slope at a given point, we can approximate the solution to the differential equation by starting at the initial point, and using the slope to guide where the next iteration should be. In essence, we are numerically "following the direction field" to estimate the solution to the differential equation.

9.2.5

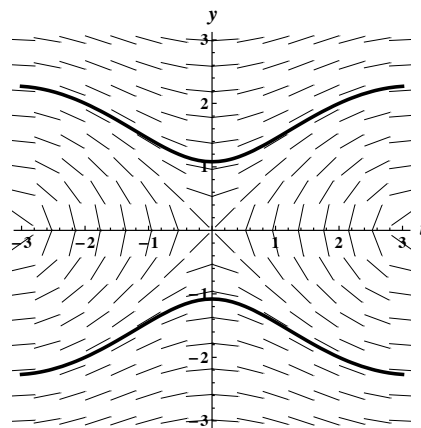
- a. This matches with D.
- b. This matches with B.
- c. This matches with A.
- d. This matches with C.

9.2.6 Note that the slopes are zero when $y = -1$ and when $t = 1$. Also, they are positive when both $y > -1$ and $t > 1$. The only differential equation with this property is choice a.

9.2.7

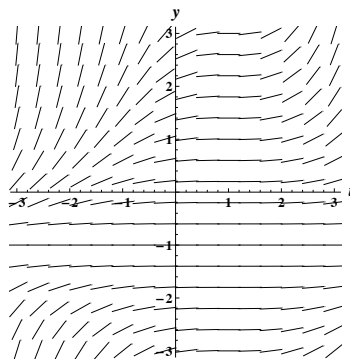


9.2.8



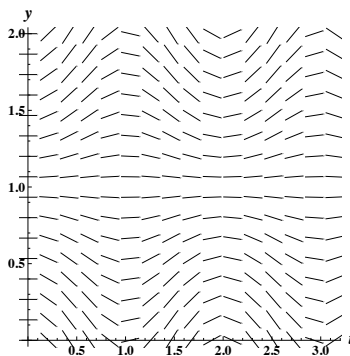
9.2.9

An initial condition of $y(0) = -1$ leads to a constant solution. For any other initial condition, the solutions are increasing over time.



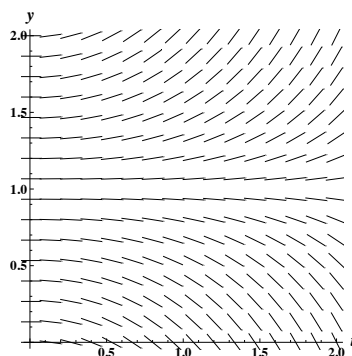
9.2.10

An initial condition of $y(0) = 1$ leads to a constant solution. For any other initial condition, the solutions oscillate between increasing and decreasing over intervals of time length one.

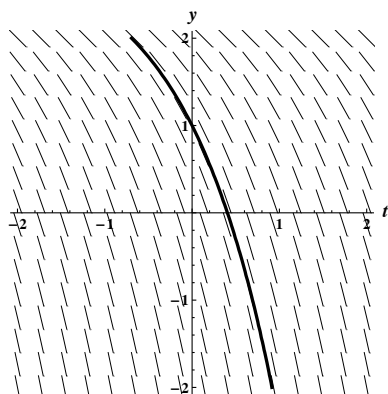


9.2.11

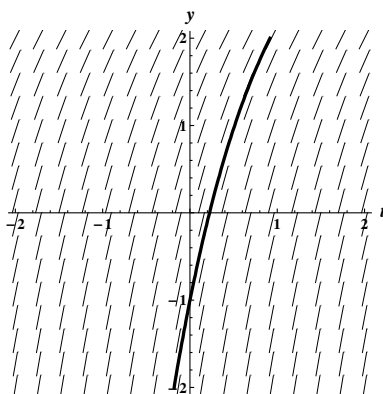
An initial condition of $y(0) = 1$ leads to a constant solution. Initial conditions $y(0) = A$ lead to solutions that are increasing over time if $A > 1$ and solutions that are decreasing over time if $A < 1$.



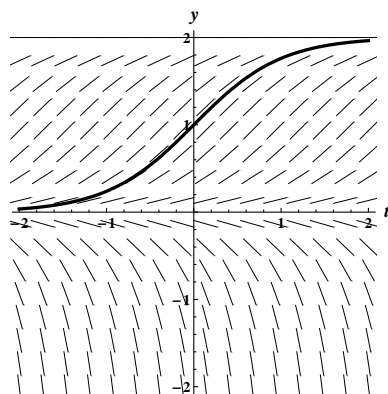
9.2.12



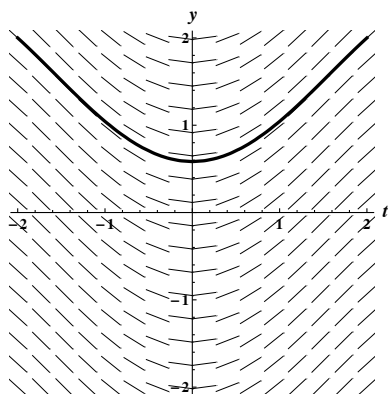
9.2.13



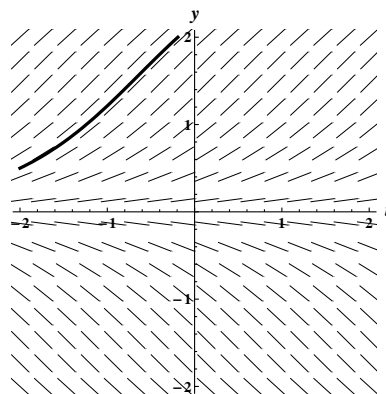
9.2.14



9.2.15



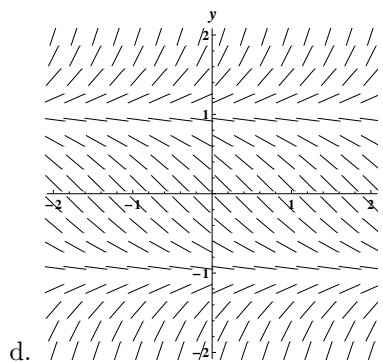
9.2.16



9.2.17

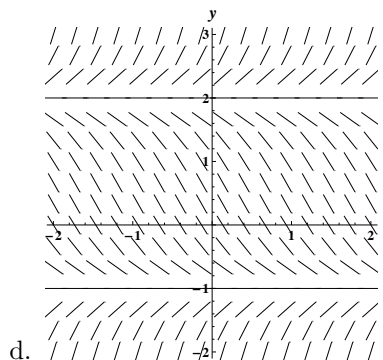
- The solutions $y = 1$ and $y = -1$ are constant.
- Solutions are increasing when both $y > 1$ and $y > -1$, (so for $y > 1$) and when both $y < 1$ and $y < -1$ (so for $y < -1$). In other words, for $|y| > 1$. Solutions are decreasing for $|y| < 1$.

- c. Initial condition $y(0) = A$ leads to solutions that are increasing over time if $|A| > 1$ and decreasing over time if $|A| < 1$.



9.2.18

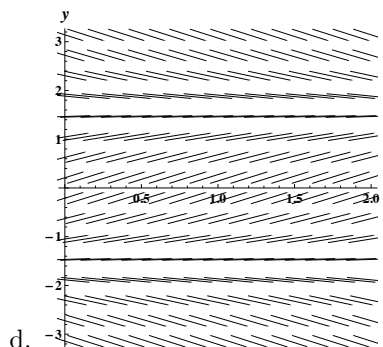
- a. The solutions $y = 2$ and $y = -1$ are constant.
- b. Solutions are increasing when both $y > 2$ and $y > -1$, (so for $y > 2$) or when both $y < 2$ and $y < -1$ (so for $y < -1$). Solutions are decreasing for $-1 < y < 2$.



- c. Initial condition $y(0) = A$ leads to solutions that are increasing over time if $A > 2$ or $A < -1$, and decreasing over time if $-1 < A < 2$.

9.2.19

- a. The solutions $y = \pi/2$ and $y = -\pi/2$ are constant.
- b. Solutions are increasing when $|y| < \pi/2$ and decreasing when $\pi/2 < |y| < \pi$.



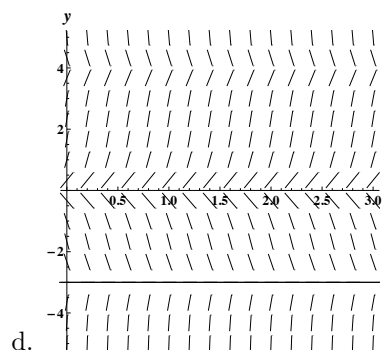
- c. Initial condition $y(0) = a$ leads to solutions that are increasing over time if $|a| < \pi/2$ and decreasing over time if $|a| > \pi/2$.

9.2.20

- a. The solutions $y = 0$, $y = -3$, and $y = 4$ are constant.

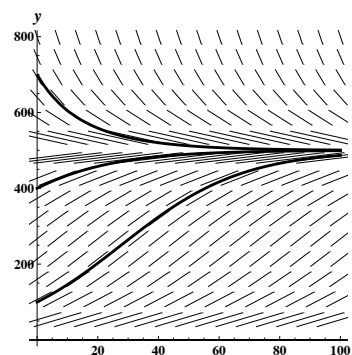
- b. Solutions are increasing when $y < -3$ and when $0 < y < 4$. Solutions are decreasing when $-3 < y < 0$ and when $y > 4$.

- c. Initial condition $y(0) = A$ leads to solutions that are increasing over time if $A < -3$ or $0 < A < 4$, and decreasing over time if $-3 < A < 0$ or $A > 4$.



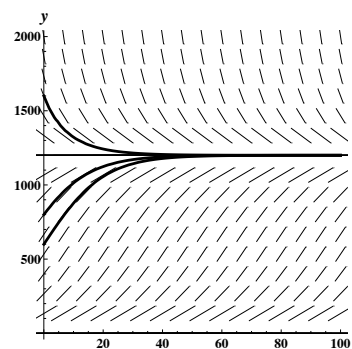
9.2.21

Equilibrium solutions are $P(t) = 0$ and $P(t) = 500$.



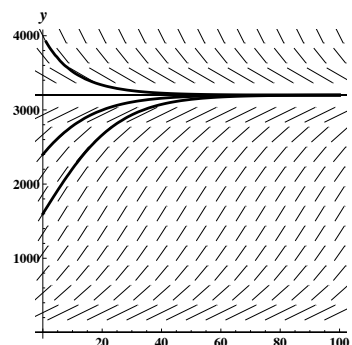
9.2.22

Equilibrium solutions are $P(t) = 0$ and $P(t) = 1200$.



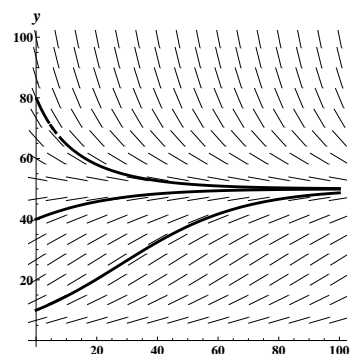
9.2.23

Equilibrium solutions are $P(t) = 0$ and $P(t) = 3200$.



9.2.24

Equilibrium solutions are $P(t) = 0$ and $P(t) = 50$.



9.2.25 $u_0 = 2$. $u_1 = 2 + f(0, 2)(0.5) = 2 + (4)(0.5) = 4$. $u_2 = 4 + f(0.5, 4)(0.5) = 4 + (8)(0.5) = 8$. So $y(0.5) \approx 4$ and $y(1) \approx 8$.

9.2.26 $u_0 = -1$. $u_1 = -1 + f(0, -1)(0.2) = -1 + 1(0.2) = -0.8$. $u_2 = -0.8 + f(0.2, -0.8)(0.2) = -0.8 + (0.8)(0.2) = -0.64$. So $y(0.2) \approx -0.8$ and $y(0.4) \approx -0.64$.

9.2.27 $u_0 = 1$. $u_1 = 1 + f(0, 1)(0.1) = 1 + 0.1 = 1.1$. $u_2 = 1.1 + f(0.1, 1.1)(0.1) = 1.1 + (0.9)(0.1) = 1.19$. So $y(0.1) \approx 1.1$ and $y(0.2) \approx 1.19$.

9.2.28 $u_0 = 4$. $u_1 = 4 + f(0, 4)(0.5) = 4 + (4)(0.5) = 6$. $u_2 = 6 + f(0.5, 6)(0.5) = 6 + (6.5)(0.5) = 9.25$. So $y(0.5) \approx 6$ and $y(1) \approx 9.25$.

9.2.29

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
0.2	0.8	0.64
a. 0.1	0.81	0.65610
0.05	0.81451	0.66342
0.025	0.81665	0.66692

b. $e^{-0.2} \approx 0.818731$ and $e^{-0.4} \approx 0.67032$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
0.2	0.018371	0.03032
0.1	0.00873	0.01422
0.05	0.00422	0.00690
0.025	0.00208	0.00340

- c. The time step $\Delta t = 0.025$ has the smallest errors. A smaller time step generally produces more accurate results.
- d. Halving the time steps results in approximately halving the error.

9.2.30

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
0.2	2.2	2.42
a. 0.1	2.205	2.43101
0.05	2.20763	2.43681
0.025	2.20897	2.43987

- b. $2e^{0.1} \approx 2.21034$ and $2e^{0.2} \approx 2.44281$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
0.2	0.0103418	0.0228055
0.1	0.00534184	0.011793
0.05	0.00271605	0.00599972
0.025	0.00136963	0.00302642

- c. The time step $\Delta t = 0.025$ has the smallest errors. A smaller time step generally produces more accurate results.
- d. Halving the time steps results in approximately halving the error.

9.2.31

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
0.2	3.2	3.36
a. 0.1	3.19	3.3439
0.05	3.18549	3.33658
0.025	3.18335	3.33308

- b. $4 - e^{-0.2} \approx 3.18127$ and $4 - e^{-0.4} \approx 3.32968$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
0.2	0.0187308	0.03032
0.1	0.00873075	0.01422
0.05	0.0042245	0.00689961
0.025	0.0020789	0.00339988

- c. The time step $\Delta t = 0.025$ has the smallest errors. A smaller time step generally produces more accurate results.
- d. Halving the time steps results in approximately halving the error.

9.2.32

Δt	approximation of $y(0.2)$	approximation of $y(0.4)$
0.2	0.2	0.48
a. 0.1	0.22	0.52
0.05	0.23	0.54
0.025	0.235	0.55

- b. $0.2^2 + 0.2 = 0.24$ and $0.4^2 + 0.4 = 0.56$.

Δt	error in approximation of $y(0.2)$	error in approximation of $y(0.4)$
0.2	0.04	0.08
0.1	0.02	0.04
0.05	0.01	0.02
0.025	0.005	0.01

- c. The time step $\Delta t = 0.025$ has the smallest errors. A smaller time step generally produces more accurate results.
- d. Halving the time steps results in approximately halving the error.

9.2.33

- a. The computations yield:

t_k	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
u_k	1	0.6	0.36	0.216	0.1296	0.07776	0.046656	0.0279936	0.0167962	0.0100777	0.00604662

So $y(2) \approx 0.00604662$.

- b. $y(2) = e^{-4} \approx 0.0183156$, so the error is about $0.0183156 - 0.00604662 = 0.012269$.

- c. The computations yield:

t_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
u_k	1	0.8	0.64	0.512	0.4096	0.32768	0.262144	0.209715	0.167772	0.134218	0.107374

t_k	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
u_k	0.0859	0.0687	0.0550	0.0440	0.0352	0.02815	0.02252	0.01801	0.01441	0.01153

So $y(2) \approx 0.01153$ The error is about $0.0183156 - 0.01153 = 0.0067856$.

- d. The error with twice as many steps is about half the other error.

9.2.34

- a. The computations yield:

t_k	0	0.2	0.4	0.6	0.8	1	1.2	1.4
u_k	-1	0.6	1.56	2.136	2.4816	2.68896	2.81338	2.88803

t_k	1.6	1.8	2	2.2	2.4	2.6	2.8	3
u_k	2.93282	2.95969	2.97581	2.98549	2.99129	2.99478	2.99687	2.99812

So $y(3) \approx 2.99812$.

- b. $y(3) = (3 - 4e^{-6}) \approx 2.99008$, so the error is about $2.99812 - 2.99008 = 0.00804$.

- c. The computations yield:

t_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
u_k	-1	-0.2	0.44	0.952	1.3616	1.68928	1.95142	2.16114	2.32891	2.46313	2.5705

t_k	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
u_k	2.6564	2.72512	2.7801	2.82408	2.85926	2.88741	2.90993	2.92794	2.94235	2.95388

t_k	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3
u_k	2.96311	2.97049	2.97639	2.98111	2.98489	2.98791	2.99033	2.99226	2.99381	2.99505

So $y(3) \approx 2.99505$ The error is about $2.99812 - 2.99505 = 0.00307$.

- d. The error with twice as many steps is less than half the other error.

9.2.35

- a. After many calculations, we arrive at $y(4) \approx 3.05765$.
- b. $y(4) = 3 + 5e^{-4} \approx 3.09158$, so the error is about $3.09158 - 3.05765 = .03393$.
- c. After many calculations, we arrive at $y(4) = 3.0739$. The error is about $3.09158 - 3.0739 = 0.01768$.
- d. The error with twice as many steps is about half the other error.

9.2.36

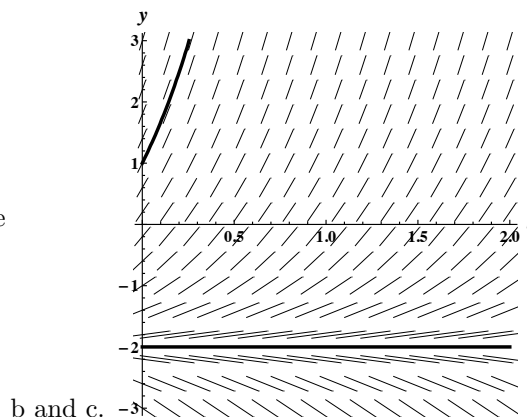
- a. After many calculations, we arrive at $y(2) \approx 4.45125$.
- b. $y(4) = \sqrt{20} \approx 4.47214$, so the error is about $4.47214 - 4.45125 = .02089$.
- c. After many calculations, we arrive at $y(2) = 4.46173$. The error is about $4.47214 - 4.46173 = 0.01041$.
- d. The error with twice as many steps is about half the other error.

9.2.37

- a. True.
- b. False. It allows you to compute approximations.

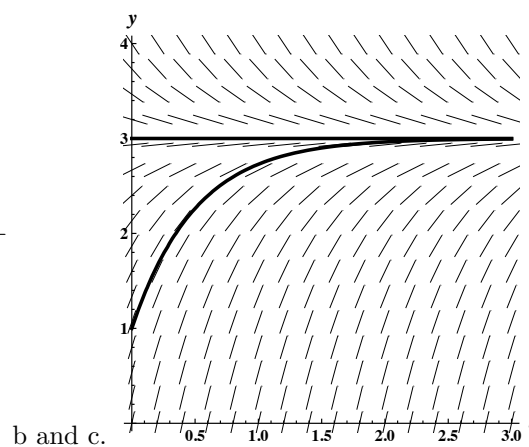
9.2.38

- a. $y = -2$ is an equilibrium solution, because $2(-2) + 4 = 0$.



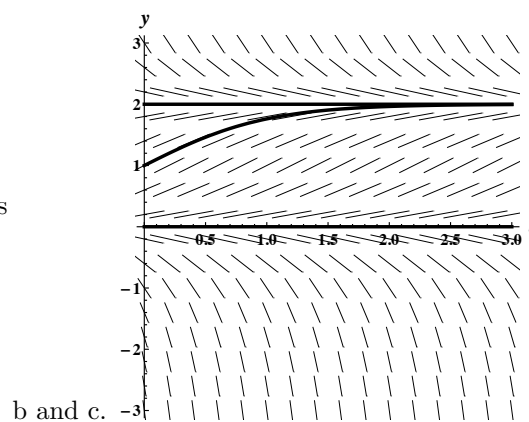
9.2.39

- a. $y = 3$ is an equilibrium solution, because $6 - 2(3) = 0$.



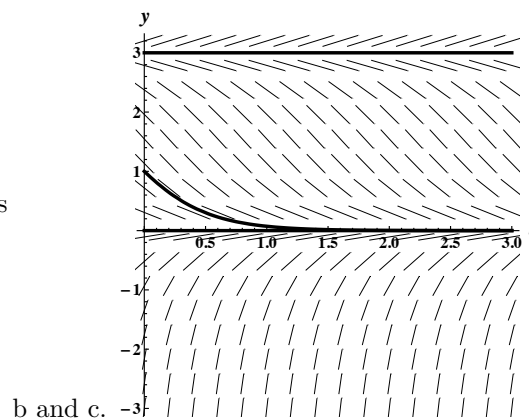
9.2.40

- a. Solve $y(2-y) = 0$ to get equilibrium solutions $y = 0$ and $y = 2$.



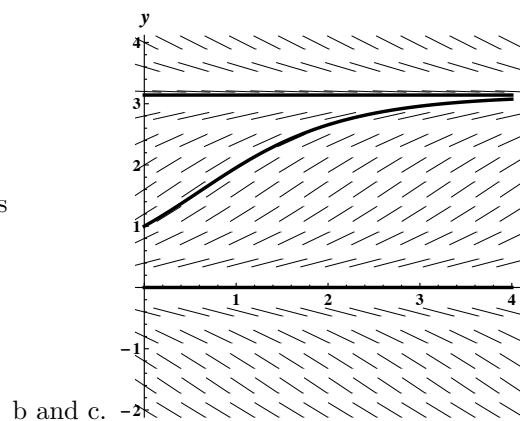
9.2.41

- a. Solve $y(y-3) = 0$ to get equilibrium solutions $y = 0$ and $y = 3$.



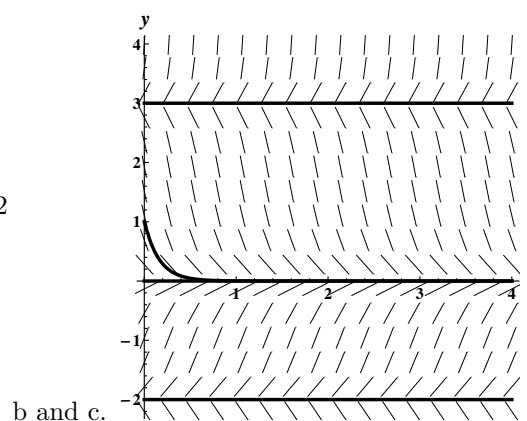
9.2.42

- a. Solve $\sin y = 0$ to get equilibrium solutions $y = k\pi$, where k is any integer.



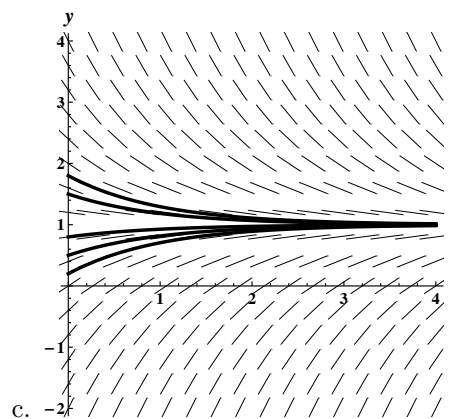
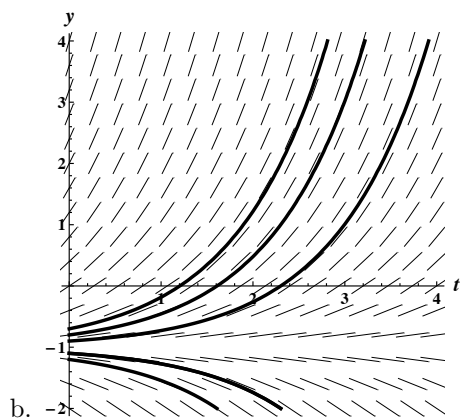
9.2.43

- a. The equilibrium solutions are $y = 0$, $y = -2$ and $y = 3$.



9.2.44

- a. Solving $y' = 0$ gives the equilibrium solution $y = -b/a$, which is a horizontal line.



Note that the general solutions $y = \left(A + \frac{b}{a}\right)e^{at} - \frac{b}{a}$ increases without bound if $a > 0$ and $A > -\frac{b}{a}$, and decreases without bound if $a > 0$ but $A < -\frac{b}{a}$. But if $a < 0$, the general solutions have limit $-\frac{b}{a}$.

but must increase to it if $-A < \frac{b}{a}$ and decrease to it if $A > -\frac{b}{a}$.

9.2.45

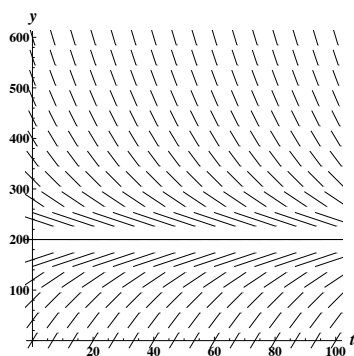
a. $\Delta t = \frac{b-a}{N}$.

b. Recall that $u_0 = A$ and $t_0 = a$. So $u_1 = A + f(a, A) \left(\frac{b-a}{N} \right)$.

c. $u_{k+1} = u_k + f(t_k, u_k) \left(\frac{b-a}{N} \right)$, where $t_k = a + k \left(\frac{b-a}{N} \right)$ for $k = 0, 1, 2, \dots, N-1$.

9.2.46

a.

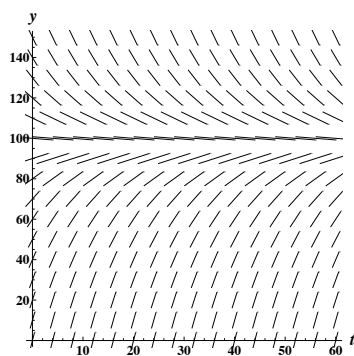


b. The equilibrium solution is $m(t) = 200$.

c. The solutions are increasing for $A < 200$ and decreasing for $A > 200$.

9.2.47

a.

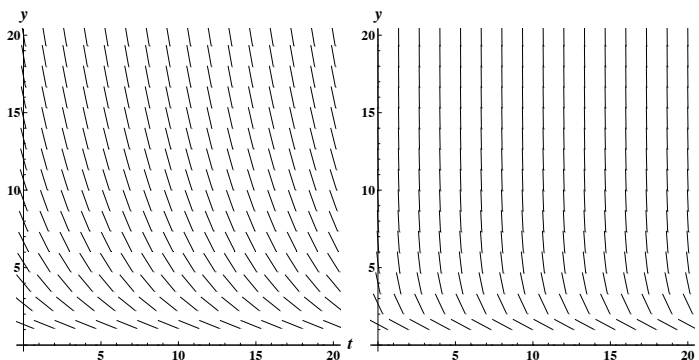


b. The solutions are increasing for $A < 98$ and decreasing for $A > 98$.

c. The equilibrium solution is $v(t) = 98$.

9.2.48

a. In both cases, the equilibrium is 0.



- b. The second order approaches more quickly.

9.2.49

- We have $u_0 = y(0) = 1$, and $u_{k+1} = u_k + f(t_k, u_k)h = u_k + au_k h = u_k(1 + ah)$ for $k = 0, 1, 2, \dots$
- Suppose $u_k = (1 + ah)^k$. Then $u_{k+1} = u_k(1 + ah) = (1 + ah)^k(1 + ah) = (1 + ah)^{k+1}$, $k = 0, 1, 2, \dots$
- $\lim_{h \rightarrow 0} u_k = \lim_{h \rightarrow 0} (1 + ah)^k = \lim_{h \rightarrow 0} (1 + ah)^{t_k/h} = \lim_{h \rightarrow 0} \left((1 + ah)^{1/h} \right)^{t_k} = (e^a)^{t_k} = e^{at_k} = y(t_k)$.

9.2.50

- We have $u_0 = y(0) = 1$, and $u_{k+1} = u_k + f(t_k, u_k)h = u_k - au_k h = u_k(1 - ah)$ for $k = 0, 1, 2, \dots$
- Suppose $u_k = (1 - ah)^k$. Then $u_{k+1} = u_k(1 - ah) = (1 - ah)^k(1 - ah) = (1 - ah)^{k+1}$, $k = 0, 1, 2, \dots$
- The function r^z increases as z increases exactly when $|r| > 1$ and decreases exactly when $|r| < 1$. So $u_k = (1 - ah)^k$ will increase when k increases exactly when $|1 - ah| > 1$ and will decrease as k increases exactly when $|1 - ah| < 1$.
- Suppose $|1 - ah| < 1$. Then $-1 < 1 - ah < 1$. Subtracting one from everything gives $-2 < -ah < 0$. Dividing through by $-a$ (which is a negative number) gives $0 < h < 2/a$, as desired.

9.3 Separable Differential Equations

9.3.1 A separable first-order differential equation is one that can be written in the form $g(y)y'(t) = h(t)$, where the factor $g(y)$ is a function of y and $h(t)$ is a function of t .

9.3.2 Yes, this equation is separable because it can be written in the form $y^2 y'(t) = t^{-2}(t + 4)$.

9.3.3 No, this equation cannot be written in the required form.

9.3.4 Integrate both sides with respect to t and convert the integral on the left side to an integral with respect to y .

9.3.5 We have $y'(t) = t^3$, so $\int \frac{dy}{dt} dt = \int t^3 dt$, so $y = \frac{t^4}{4} + C$.

9.3.6 $y'(t) = 5e^{-4t}$, so $\int y'(t) dt = \int 5e^{-4t} dt$, so $y = -\frac{5}{4}e^{-4t} + C$.

9.3.7 $y \frac{dy}{dt} = 3t^2$, so $\int y dy = \int 3t^2 dt$. Thus, $\frac{y^2}{2} = t^3 + C$, and thus $y = \pm \sqrt{2t^3 + C}$.

9.3.8 We have $\frac{1}{y} \frac{dy}{dx} = x^2 + 1$, so $\int \frac{1}{y} \frac{dy}{dx} dx = \int (x^2 + 1) dx$. Thus, $\ln y = \frac{x^3}{3} + x + C$, and $y = Ae^{(x^3/3)+x}$.

9.3.9 We have $\int e^{-y/2} dy = \int \sin t dt$, and so $-2e^{-y/2} = -\cos t + C$. Thus, $y = -2 \ln \left(\frac{1}{2} \cos t + C \right)$.

9.3.10 We have $\int w^{-1/2} dw = \int \frac{3x+1}{x^2} dx = \int \left(\frac{3}{x} + \frac{1}{x^2} \right) dx$, so $2w^{1/2} = 3 \ln |x| - \frac{1}{x} + C$, and thus $w = \left(\frac{3}{2} \ln |x| - \frac{1}{2x} + C \right)^2$.

9.3.11 $\frac{1}{y^2} \frac{dy}{dx} = \frac{1}{x^2}$, so $\int \frac{1}{y^2} \frac{dy}{dx} dx = \int \frac{1}{x^2} dx$, so $-\frac{1}{y} = -\frac{1}{x} + C = \frac{Cx-1}{x}$. Thus, $y = \frac{x}{1-Cx}$. If we replace the arbitrary constant C by its opposite, this can be written as $y = \frac{x}{1+Cx}$.

9.3.12 $\frac{y}{y^2+4} y'(t) = \frac{t}{(t^2+1)^3}$. Therefore, $\int \frac{y}{y^2+4} dy = \int \frac{t}{(t^2+1)^3} dt$. We have

$$\frac{1}{2} \ln(y^2+4) = -\frac{1}{4} \frac{1}{(t^2+1)^2} + C,$$

so

$$\ln(y^2+4) = C_1 - \frac{1}{2(t^2+1)^2}.$$

This can be written as $y^2+4 = Ae^{-(t^2+1)^{-2}/2}$, so $|y| = \sqrt{Ae^{-(t^2+1)^{-2}/2} - 4}$.

9.3.13 $-\frac{2}{y^3} \frac{dy}{dt} = \sin t$, so $\int \frac{-2}{y^3} \frac{dy}{dt} dt = \int \sin t dt$. Thus, $\frac{1}{y^2} = -\cos t + C$. Solving for y gives $y = \pm \frac{1}{\sqrt{C - \cos t}}$.

9.3.14 $\frac{1}{y^2+4} y'(t) = e^{-t/2}$, so $\int \frac{1}{y^2+4} dy = \int e^{-t/2} dt$, so

$$\frac{1}{2} \tan^{-1}(y/2) = -2e^{-t/2} + C.$$

Thus, $\tan^{-1}(y/2) = C_1 - 4e^{-t/2}$, and $y = 2 \tan(C_1 - 4e^{-t/2})$.

9.3.15 $e^u u'(x) = e^{2x}$, so $\int e^u du = \int e^{2x} dx$, and $e^u = \frac{1}{2} e^{2x} + C$. Thus, $u = \ln \left(\frac{e^{2x}}{2} + C \right)$.

9.3.16 $\frac{1}{u^2-4} u'(x) = \frac{1}{x}$, so $\int \frac{1}{u^2-4} du = \int \frac{1}{x} dx$, and thus

$$\frac{1}{4} \ln \left| \frac{u-2}{u+2} \right| = \ln |x| + C,$$

so

$$\ln \left| \frac{u-2}{u+2} \right| = 4 \ln |x| + C_1.$$

Therefore, $\left| \frac{u-2}{u+2} \right| = Ax^4$, so $1 - \frac{4}{u+2} = \pm Ax^4$, and $\frac{4}{u-2} = 1 \pm Ax^4$, so $u-2 = \frac{4}{1 \pm Ax^4}$, and $u = 2 + \frac{4}{1 \pm Ax^4}$.

9.3.17 This is separable, and is already written in the desired form. We have $\int 2y dy = \int 3t^2 dt$, so $y^2 = t^3 + C$. Because $y(0) = 9$, we have $81 = C$, so $y = \sqrt{t^3 + 81}$.

9.3.18 This equation is not separable.

9.3.19 This equation is not separable.

9.3.20 This equation is separable. We have $\int \frac{dy}{y} = \int (4t^3 + 1) dt$, and thus $\ln |y| = t^4 + t + C$. Therefore, $y = \pm e^{(t^4+t+C)} = Ae^{t^4+t}$. Substituting $y(0) = 4$ gives $A = 4$, so the solution to this initial value problem is $y = 4e^{t^4+t}$.

9.3.21 The equation is separable. We have $\int \frac{1}{y} dy = \int e^t dt$, so $\ln |y| = e^t + C$, and $|y| = e^{e^t} e^C = Ae^{e^t}$. Then $|y(0)| = 1 = Ae$, so $A = \frac{1}{e}$ and $y = -\frac{1}{e}e^{e^t} = -e^{-1}e^{e^t} = -e^{e^t-1}$.

9.3.22 The equation is separable. We have $\int \frac{1}{y} dy = \int \cos x dx$, so $\ln |y| = \sin x + C$, and $|y| = e^{\sin x} e^C$. Because $y(0) = 3$ we have $3 = e^C$. So $y(x) = 3e^{\sin x}$.

9.3.23 This equation is separable. We have $\int e^y dy = \int e^x dx$, and thus $e^y = e^x + C$. Therefore, $y = \ln(e^x + C)$. Substituting $y(0) = \ln 3$ gives $\ln 3 = \ln(1 + C)$, so $C = 2$ and the solution to this initial value problem is $y = \ln(e^x + 2)$.

9.3.24 This equation is separable. We have $\int \sec^2 y dy = \int dt$, so $\tan y = t + C$. Because $y(1) = \pi/4$ we have $1 = 1 + C$, so $C = 0$. Thus, $y = \tan^{-1}(t)$.

9.3.25 This equation is separable. We have $\int e^y dy = \int \frac{\ln^3 t}{t} dt$. For the integral on the right-hand side, let $u = \ln t$ so that $du = \frac{1}{t} dt$. Then $\int \frac{\ln^3 t}{t} dt = \int u^3 du = \frac{u^4}{4} + C = \frac{\ln^4 t}{4} + C$. Thus we have

$$e^y = \frac{\ln^4 t}{4} + C.$$

Because $y(1) = 0$, we have $1 = 0 + C$, so $y = \ln\left(\frac{\ln^4 t}{4} + 1\right)$.

9.3.26 The equation is separable. We have $\int \frac{dy}{y(y+1)} = \int \frac{1}{t} dt$. For the integral on the left-hand side, we have $\int \frac{dy}{y(y+1)} = \int \left(\frac{1}{y} - \frac{1}{1+y}\right) dy = \ln |y| - \ln |y+1| + C$. Thus we have $\ln \left|\frac{y}{y+1}\right| = \ln |t| + C$, so $\frac{y}{y+1} = e^C t$, and because $y(3) = 1$ we have $e^C = \frac{1}{6}$, so

$$\frac{y}{y+1} = \frac{t}{6}.$$

Cross multiplying gives $6y = ty + t$, so $6y - ty = y(6 - t) = t$, and $y = \frac{t}{6-t}$, for $t < 6$.

9.3.27 The equation is separable. We have $\int 2y dy = \int \sec^2 t dt$, so $y^2 = \tan t + C$. Because $y(\pi/4) = 1$ we have $C = 0$, and $y = \sqrt{\tan t}$. Note that the original differential equation is undefined for $y = 0$, which corresponds to $t = 0$ and $t = \pi$, so the solution is $y = \sqrt{\tan t}$ for $0 < t < \pi/2$.

9.3.28 The equation is separable. Assume $y > -3$ and $t > -6/5$. $\int \left(\frac{1}{y+3} \right) y'(t) dt = \int \frac{1}{5t+6} dt$, so $\ln(y+3) = \frac{\ln(5t+6)}{5} + C$. We can write this as $5\ln(y+3) = \ln(5t+6) + 5C$, so $(y+3)^5 = A(5t+6)$. Because $y(2) = 0$, we have $3^5 = 16A$, so $A = 3^5/16$. We have $(y+3) = \frac{3\sqrt[5]{2}}{2}(5t+6)^{1/5}$. Thus, $y = -3 + \frac{3\sqrt[5]{2}}{2}(5t+6)^{1/5} = \frac{3}{2}(-2 + 2^{1/5}(6+5t)^{1/5})$ for $t > -\frac{6}{5}$.

9.3.29 The equation is separable.

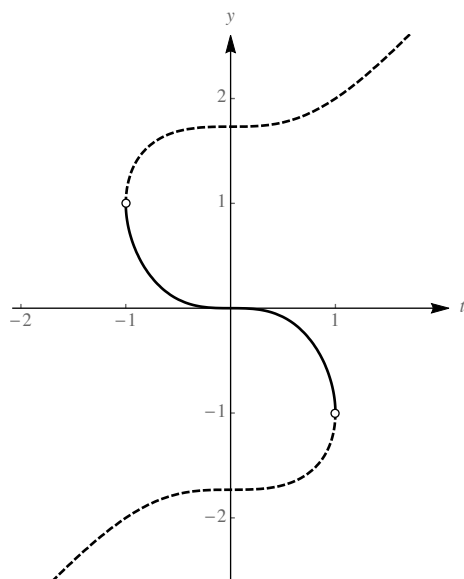
$yy'(t) = t$, so $\int y dy = \int t dt$, so $\frac{y^2}{2} = \frac{t^2}{2} + C$. Because $y(1) = 2$, we have $2 = 1/2 + C$, so $C = 3/2$. So $y^2 = t^2 + 3$, and $y = \sqrt{t^2 + 3}$.

9.3.30 The equation is separable. We have $\int \frac{1}{y^3} dy = \int \sin t dt$, so $-\frac{1}{2y^2} = -\cos t + C$. Because $y(0) = 1$ we have $C = \frac{1}{2}$ and $\frac{1}{2y^2} = \cos t - \frac{1}{2}$. Then $\frac{1}{y^2} = 2\cos t - 1$, and $y^2 = \frac{1}{2\cos t - 1}$, so $y = \frac{1}{\sqrt{2\cos t - 1}}$ for $-\frac{\pi}{3} < t < \frac{\pi}{3}$.

9.3.31 This is separable, and can be written as $y'(t) = \frac{1}{t}$. Thus, $\int y'(t) dt = \int \frac{dt}{t} = \ln t + C$, so $y(t) = \ln t + C$. Because $y(1) = 2 = 0 + C$, we have $C = 2$. Thus, $y(t) = \ln t + 2$.

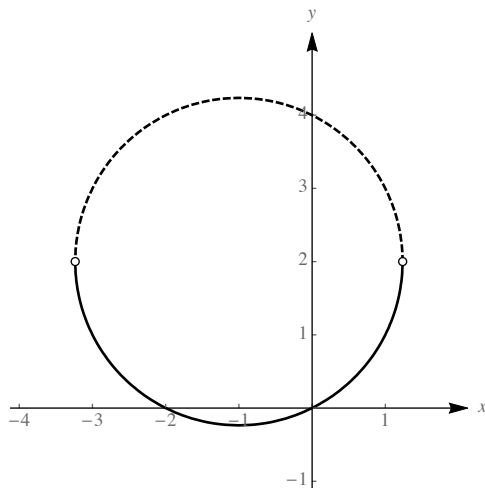
9.3.32 This is separable, and can be written as $y'(t) = \cos t$. Thus $y(t) = \sin t + C$, and because $y(0) = 1 = 0 + C$, we have $C = 1$. Thus, $y(t) = \sin t + 1$. Note that the original differential equation only makes sense for $t \neq \frac{k\pi}{2}$ for odd integers k . So our solution is $y(t) = \sin t + 1$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

9.3.33 We may separate variables and write $\int (y^2 - 1) dy = \int 2t^2 dt$. Then $\frac{y^3}{3} - y = \frac{2t^3}{3} + C$. Because $y(0) = 0$, it follows that $C = 0$. So we have $y^3 - 3y = 2t^3$. From the original differential equation, we see that $y'(t)$ is undefined for $y = \pm 1$. Using the solution, $y = 1$ for $2t^3 = -2$, or $t = -1$, and likewise $y = -1$ implies that $t = 1$. So the solution is $y^3 - 3y = 2t^3$ for $-1 < t < 1$, and the graph is the portion of the curve passing through the origin.



9.3.34 $(2 - y)y'(x) = 1 + x$, so $\int (2 - y) dy = \int (1 + x) dx$, and $2y - \frac{y^2}{2} = x + \frac{x^2}{2} + C$. Because $y(1) = 1$, we have $2 - 1/2 = 1 + 1/2 + C$, so $C = 0$. Thus, $2y - \frac{y^2}{2} = x + \frac{x^2}{2}$ describes the solution. This can be written as $x + \frac{x^2}{2} + \frac{y^2}{2} - 2y = 0$. The original differential equation is undefined for $y = 2$ which corresponds to $x + \frac{x^2}{2} - 2 = 0$ or $x^2 + 2x - 4 = 0$. Using the quadratic formula, this gives $x = -1 \pm \sqrt{5}$. So the solution is

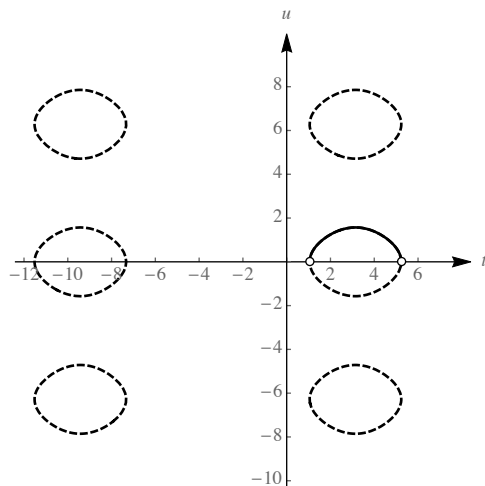
$$x + \frac{x^2}{2} + \frac{y^2}{2} - 2y = 0, \text{ for } -1 - \sqrt{5} < x < -1 + \sqrt{5}.$$



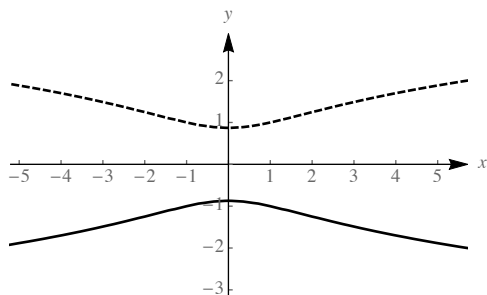
9.3.35 $(\sin u)u'(x) = \cos(x/2)$, so $\int \sin u du = \int \cos(x/2) dx$. We have $-\cos u = 2\sin(x/2) + C$. Because $u(\pi) = \pi/2$, we have $-\cos \pi/2 = 2\sin(\pi/2) + C$, so $0 = 2(1) + C$, so $C = -2$. Thus, $-\cos u = 2\sin(x/2) - 2$ describes the solution. Note that the original differential equation is undefined for $u = 0$ which corresponds to $x = \pi/3$ and $x = 5\pi/3$. So the solution is

$$-\cos u = 2\sin\left(\frac{x}{2}\right) - 2, \text{ for } \frac{\pi}{3} < x < \frac{5\pi}{3}.$$

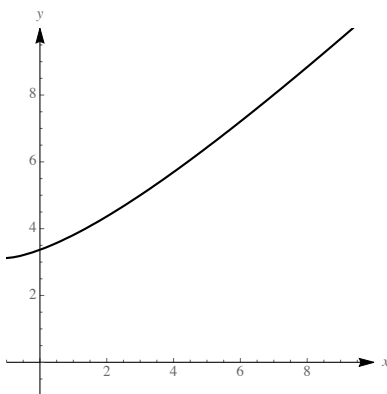
Not that the non-dotted curve contains the point $(\pi, \pi/2)$.



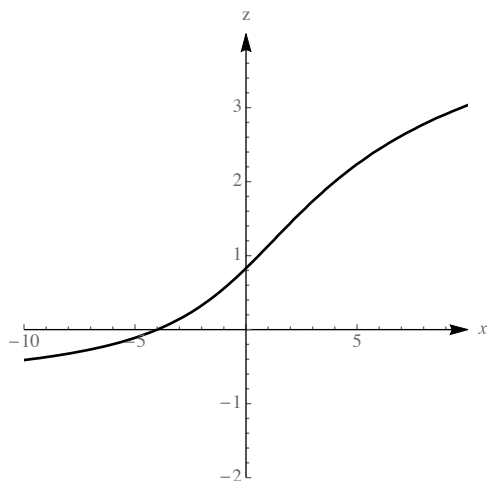
9.3.36 $(2 + y^2)^2 yy'(x) = 2x$, so $\int (2 + y^2)^2 yy'(x) dx = \int 2x dx$, and thus $\frac{(2 + y^2)^3}{6} = x^2 + C$. Because $y(1) = -1$, we have $C = \frac{21}{6}$. Thus, $(2 + y^2)^3 = 6x^2 + 21$, and thus $y^2 = \sqrt[3]{6x^2 + 21} - 2$.



9.3.37 $\sqrt{y+4}y'(x) = \sqrt{x+1}$, so $\int \sqrt{y+4} dy = \int \sqrt{x+1} dx$, so $\frac{2}{3}(y+4)^{3/2} = \frac{2}{3}(x+1)^{3/2} + C$. Because $y(3) = 5$, we have $\frac{2}{3}(27) = \frac{16}{3} + C$, so $C = \frac{38}{3}$. Thus, $\frac{2}{3}(y+4)^{3/2} = \frac{2}{3}(x+1)^{3/2} + \frac{38}{3}$, so $(y+4)^{3/2} = (x+1)^{3/2} + 19$

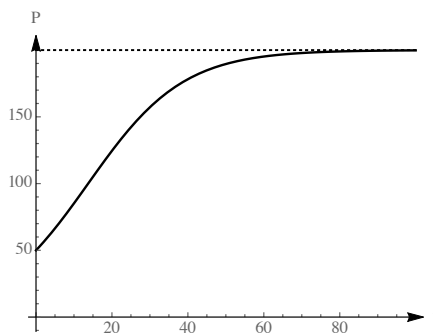


9.3.38 $\frac{1}{z^2 + 4}z'(x) = \frac{1}{x^2 + 16}$, so $\int \frac{dz}{z^2 + 4} = \int \frac{dx}{x^2 + 16}$. Thus, $\frac{1}{2}\tan^{-1}(z/2) = \frac{1}{4}\tan^{-1}(x/4) + C$. Because $z(4) = 2$, we have $\pi/8 = \pi/16 + C$, so $C = \pi/16$. Thus, $2\tan^{-1}(z/2) = \tan^{-1}(x/4) + \pi/4$ describes the solution.



9.3.39

- a. This equation is separable, so we have $\int \frac{200}{P(200-P)} dP = \int 0.08 dt$, so $\int \left(\frac{1}{P} + \frac{1}{200-P} \right) dP = 0.08t + C$. Therefore, $\ln \left| \frac{P}{200-P} \right| = 0.08t + C$. Substituting $P(0) = 50$ gives $-\ln 3 = C$, and solving for P gives $P(t) = \frac{200}{3e^{-0.08t} + 1}$.

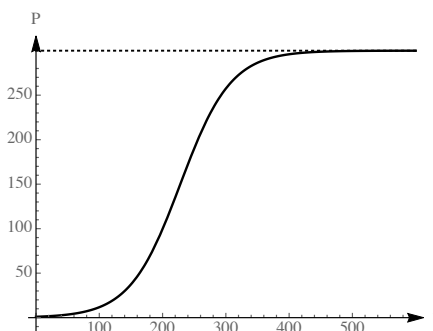


- b. The steady-state population is $\lim_{t \rightarrow \infty} P(t) = 200$.

9.3.40

- a. This equation is separable, so we have $\int \frac{A}{P(A-P)} dP = \int k dt$, so $\int \left(\frac{1}{P} + \frac{1}{A-P} \right) dP = kt + D$. Therefore, $\ln \left| \frac{P}{A-P} \right| = kt + D$, which is equivalent to $\frac{P}{A-P} = Ce^{kt}$. Substituting $P(0) = P_0$ gives $C = P_0/(A - P_0)$, and solving for P gives $P(t) = \frac{AP_0}{P_0 + (A - P_0)e^{-kt}}$.

- b.



- c. The denominator in $P(t)$ above is positive for all $t \geq 0$ when $0 < P_0 < A$, so $P(t)$ is defined for all $t \geq 0$; we have $\lim_{t \rightarrow \infty} P(t) = A$, which is the steady-state solution.

9.3.41

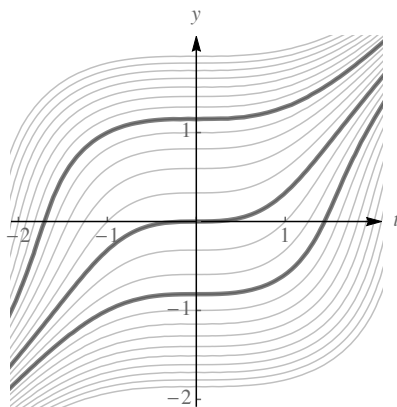
- a. True. It can be written as $u^7 u'(x) = x^{-2}$.
- b. False.
- c. True. When separated, we have $ye^y y'(x) = x$, and the left-hand side of the equation can be integrated by parts.

9.3.42

a. $\int (y^2 + 1) dy = \int t^2 dt$, so $\frac{y^3}{3} + y = \frac{t^3}{3} + C_1$, and $y^3 + 3y = t^3 + C$, where $C = 3C_1$.

b. For $y(-1) = 1$ we have $C = 5$. For $y(0) = 0$ we have $C = 0$. For $y(-1) = -1$ we have $C = -3$.

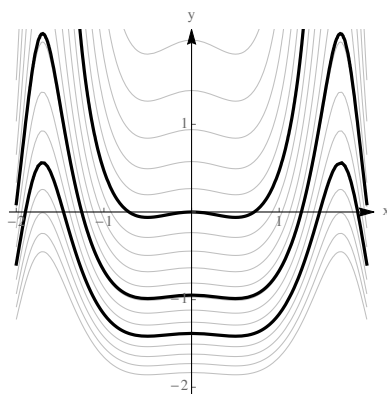
c.

**9.3.43**

a. $\int e^{-y/2} dy = \int (4x \sin x^2 - x) dx$. Thus, $-2e^{-y/2} = -2 \cos x^2 - \frac{x^2}{2} + K$. Then $e^{-y/2} = \cos x^2 + \frac{x^2}{4} + C$, so $-\frac{y}{2} = \ln \left(\cos x^2 + \frac{x^2}{4} + C \right)$, and $y = -2 \ln \left(\cos x^2 + \frac{x^2}{4} + C \right)$.

b. When $y(0) = 0$ we have $0 = -2 \ln(1 + C)$, so $C = 0$. When $y(0) = \ln(1/4)$, we have $\ln(1/4) = -2 \ln(1 + C)$, so $\ln 2 = \ln(1 + C)$, so $C = 1$. When $y(\sqrt{\pi/2}) = 0$, we have $0 = -2 \ln(0 + \pi/8 + C)$, so $C = 1 - \frac{\pi}{8}$.

c.

**9.3.44**

a. $4x + 2y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{4x}{2y} = -\frac{2x}{y}$.

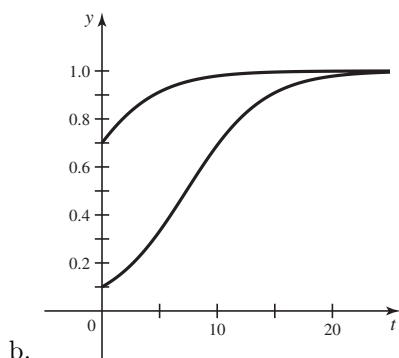
b. Curves are orthogonal when their slopes are negative reciprocals of each other, and the negative reciprocal of $-\frac{2x}{y}$ is $\frac{y}{2x}$.

- c. We have $\frac{2dy}{y} = \frac{dx}{x}$, so $2 \ln |y| = \ln y^2 = \ln |x| + C$. Thus, $y^2 = e^C |x|$. We can write $\pm e^C = k$, so we have $y^2 = kx$.

9.3.45 Differentiating implicitly gives $2x + 2yy' = 0$, so $y' = -\frac{x}{y}$. We are thus seeking curves so that $\frac{dy}{dx} = -\frac{y}{x}$. We have $\frac{dy}{y} = -\frac{dx}{x}$, so $\ln |y| = -\ln |x| + C$ so $y = e^C |x|^{-1} = k/x$. So the family of curves we are seeking is the collection of curves $y = k/x$.

9.3.46

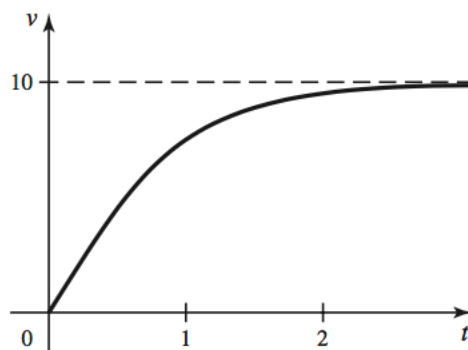
- a. This equation is separable, so we have $\int \frac{1}{y(1-y)} dy = \int k dt$, so $\int \left(\frac{1}{y} + \frac{1}{1-y} \right) dy = kt + D$. Therefore, $\ln \left| \frac{y}{1-y} \right| = kt + D$, which is equivalent to $\frac{y}{1-y} = Ce^{kt}$. Substituting $y(0) = y_0$ gives $C = y_0/(1-y_0)$, and thus $y = \frac{y_0}{(1-y_0)e^{-kt} + y_0}$.



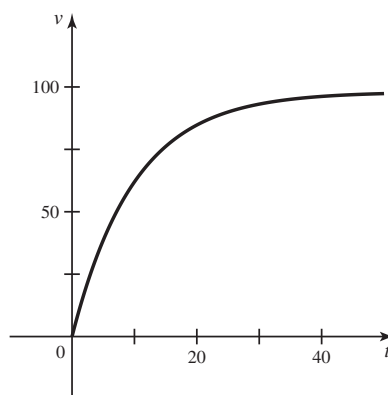
- c. The denominator in $y(t)$ above is positive for all $t \geq 0$ when $0 < y_0 < 1$, so $y(t)$ is defined for all $t \geq 0$; we have $\lim_{t \rightarrow \infty} y(t) = 1$, which is the steady-state solution.

9.3.47

- a. We have $mv'(t) = mg - kv^2$, so $v'(t) = g - av^2$ with $a = k/m$.
- b. We solve $av^2 = g$ to obtain the terminal velocity $\tilde{v} = \sqrt{g/a} = \sqrt{gm/k}$.
- c. This equation is separable, so we have $\int \frac{1}{g - av^2} dv = \int dt$, so $-\frac{1}{a} \int \frac{1}{v^2 - \tilde{v}^2} dv = t + D$. Thus, $-\frac{1}{2a\tilde{v}} \ln \left(\frac{v - \tilde{v}}{v + \tilde{v}} \right) = t + D$, and $-\frac{1}{2a\tilde{v}} \ln \left| \frac{v - \tilde{v}}{v + \tilde{v}} \right| = t + D$, hence $\frac{v - \tilde{v}}{v + \tilde{v}} = Ce^{-2a\tilde{v}t}$. The initial condition $v(0) = 0$ gives $C = -1$, and solving for v gives $v = \frac{1 - e^{-2a\tilde{v}t}}{1 + e^{-2a\tilde{v}t}} \tilde{v}$. This can be written as $v = (\sqrt{g/a}) \frac{e^{2\sqrt{ag}t} - 1}{e^{2\sqrt{ag}t} + 1}$.
- d. We have $a = 0.1$, $\tilde{v} \approx 9.90$ m/s

**9.3.48**

- a. We have $mv'(t) = mg - Rv$, so $v'(t) = g - bv$ with $b = R/m$.
- b. Solve $bv = g$ to obtain terminal velocity $\tilde{v} = g/b = mg/R$.
- c. The equation $v' = g - bv$ is first-order linear, with general solution $v = Ce^{-bv} + \tilde{v}$. The initial condition $v(0) = 0$ gives $C = -\tilde{v}$, which gives $v = \tilde{v}(1 - e^{-bt})$.

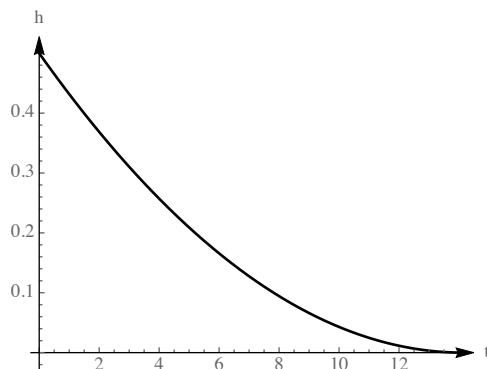


- d. We have $b = 0.1$, $\tilde{v} = 98$ m/s.

9.3.49

- a. The equation $h' = -k\sqrt{h}$ is separable, so we have $\int \frac{dh}{\sqrt{h}} = \int -k dt$, so $2\sqrt{h} = -kt + C$. The initial condition $h(0) = H$ gives $C = 2\sqrt{H}$, so the solution is $h = \left(\sqrt{H} - \frac{kt}{2}\right)^2$.
- b. The solution for $k = 0.1$ and $H = 0.5$ is $h = (\sqrt{0.5} - 0.05t)^2$, $0 \leq t \leq 14.1$.
- c. The tank is drained when $h(t) = 0$, which gives $t = \frac{\sqrt{0.5}}{0.05} \approx 14.1$ s.

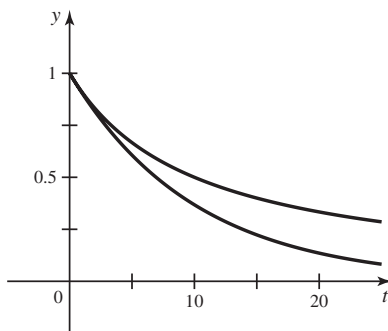
d.

**9.3.50**

a. The general solution to $y' = -ky$ is $y = Ce^{-kt}$, $t \geq 0$.

b. The equation $y' = -ky^2$ is separable, so we have $-\int \frac{dy}{y^2} = \int k dt$, so $\frac{1}{y} = kt + C$. The initial condition $y(0) = y_0$ gives $C = 1/y_0$, and solving for y gives $y = \frac{1}{kt + 1/y_0} = \frac{y_0}{1 + ky_0 t}$, for $t \geq 0$.

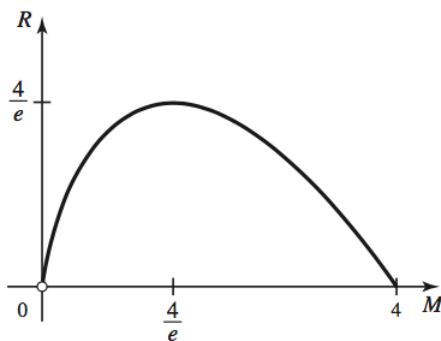
c.

**9.3.51**

a. The growth rate is positive when $0 < M < 4$. The function $R(M)$ has derivative

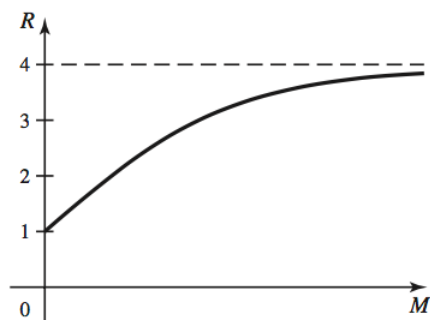
$$R'(M) = -r \left(\ln \left(\frac{M}{K} \right) + M \cdot \frac{K}{M} \cdot \frac{1}{K} \right) = -r \left(\ln \left(\frac{M}{K} \right) + 1 \right),$$

which is 0 when $M/4 = 1/e$ or $M = 4/e$. We also observe that $\lim_{M \rightarrow 0^+} R(M) = 0$ and $R(4) = 0$, so $R(M)$ takes its maximum at the critical point $M = 4/e$.



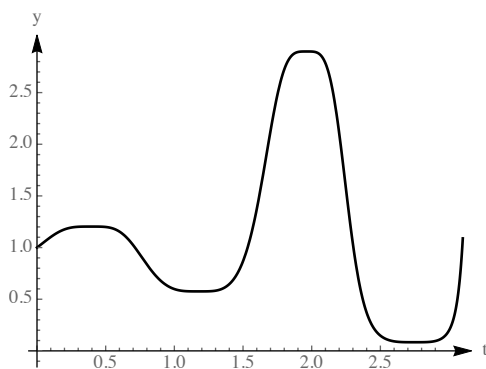
- b. The equation is separable, so we have $\int \frac{dM}{M(\ln M - \ln K)} = -\int r dt$, so $\ln |\ln M - \ln K| = -rt + D$, and thus $\ln \left(\frac{M}{K} \right) = Ce^{-rt}$. Therefore $M = Ke^{Ce^{-rt}}$.

The conditions $r = 1$, $K = 4$ and $M_0 = 1$ give $C = -\ln 4$ and $M = 4e^{(-\ln 4)e^{-t}} = 4^{1-e^{-t}}$. Observe that $\lim_{t \rightarrow \infty} M(t) = 4e^0 = 4$, so the limiting size of the tumor is 4. In general, the limiting size of the tumor is $\lim_{t \rightarrow \infty} Ke^{Ce^{-rt}} = K$, because $r > 0$.



9.3.52 Solving with a CAS gives

$$y = e^{\left(\frac{e^t(1740 \sin(4t) + 204 \sin(12t) + 435 \cos(4t) + 17 \cos(12t))}{9860} - \frac{113}{2465} \right)}.$$



9.3.53

a. $\frac{1}{y^2}y'(t) = 1$, so $\int \frac{dy}{y^2} = \int 1 dt$. Thus $-\frac{1}{y} = t + C$. Because $y_0 = 1$ we have $-1 = C$. Thus, $y = \frac{1}{1-t}$ for $t < 1$.

b. $\frac{1}{y^3}y'(t) = 1$, so $\int \frac{dy}{y^3} = \int 1 dt$. Thus $-\frac{1}{2y^2} = t + C$. Because $y_0 = \frac{1}{\sqrt{2}}$ we have $-1 = C$. Thus, $\frac{1}{2y^2} = 1 - t$ and $y = \frac{1}{\sqrt{2}\sqrt{1-t}}$ for $t < 1$.

c. $\frac{1}{y^{n+1}}y'(t) = 1$, so $\int \frac{dy}{y^{n+1}} = \int 1 dt$. Thus $-\frac{1}{ny^n} = t + C$. Because $y_0 = n^{-1/n}$ we have $-1 = C$. Thus, $\frac{1}{ny^n} = 1 - t$, and $ny^n = \frac{1}{1-t}$. Thus, $y = \frac{1}{(n(1-t))^{1/n}}$ for $t < 1$.

We have $\lim_{t \rightarrow 1^-} \frac{1}{(n(1-t))^{1/n}} = \infty$.

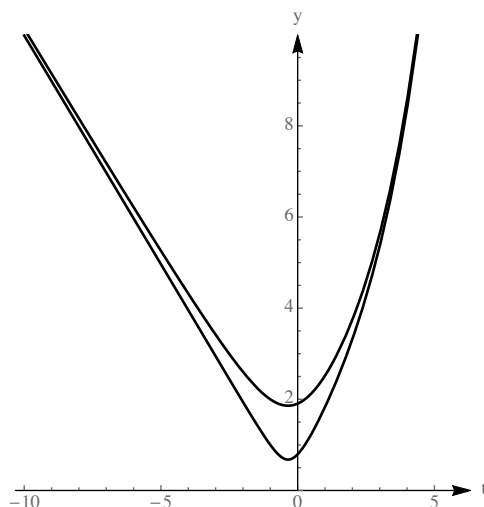
9.3.54

a. $\int yy'(t) dt = \int (e^t/2 + t) dt$, so $y^2/2 = e^t/2 + t^2/2 + C$, so $y^2 = e^t + t^2 + C_1$. So $y = \pm\sqrt{e^t + t^2 + C_1}$.

b. If $y(-1) = 1$, then $1 = \sqrt{1/e + 1 + C_1}$ so $C_1 = -1/e$, so $y = \sqrt{e^t + t^2 - 1/e}$.

If $y(-1) = 2$, then $2 = \sqrt{1/e + 1 + C_1}$ so $C_1 = 3 - 1/e$, so $y = \sqrt{e^t + t^2 + 3 - 1/e}$.

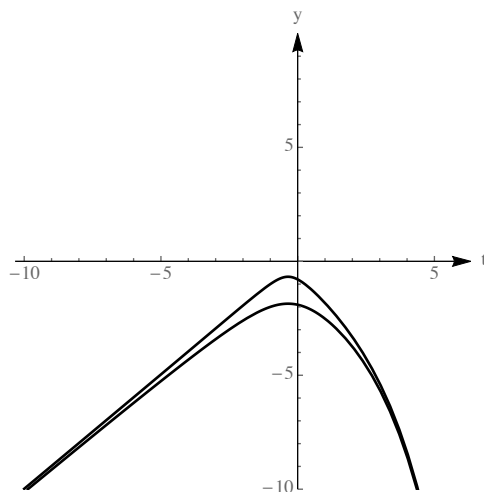
c. For $t > 0$, the solutions increase without bound as t increases.



d. If $y(-1) = -1$, then $-1 = -\sqrt{1/e + 1 + C_1}$, so $C_1 = -1/e$ and $y = -\sqrt{e^t + t^2 - 1/e}$.

If $y(-1) = -2$, then $-2 = -\sqrt{1/e + 1 + C_1}$ so $C_1 = 3 - 1/e$, so $y = -\sqrt{e^t + t^2 + 3 - 1/e}$.

e. For $t > 0$, the solutions decrease without bound as t increases.



9.4 Special First-Order Linear Differential Equations

9.4.1 Because $y(0) = 4$, we have $4 = C - 13$, so $C = 17$. Thus, the solution is $y = 17e^{-10t} - 13$.

9.4.2 The general solution is $y = Ce^{3t} - \left(-\frac{12}{3}\right) = Ce^{3t} + 4$.

9.4.3 The general solution is $y = Ce^{-4t} - \left(\frac{6}{-4}\right) = Ce^{-4t} + \frac{3}{2}$.

9.4.4 The equilibrium solution is $y = 3$. It is unstable.

9.4.5 Because $k = 3$ and $b = -4$, we have $y = Ce^{3t} + \frac{4}{3}$.

9.4.6 Because $k = -1$ and $b = 2$, we have $y = Ce^{-x} + 2$.

9.4.7 Because $k = -2$ and $b = -4$, we have $y = Ce^{-2x} - 2$.

9.4.8 Because $k = 2$ and $b = 6$, we have $y = Ce^{2x} - 3$.

9.4.9 Because $k = -12$ and $b = 15$, we have $u = Ce^{-12t} + \frac{5}{4}$.

9.4.10 Because $k = \frac{1}{2}$ and $b = 14$, we have $v = Ce^{y/2} - 28$.

9.4.11 Because $k = 3$ and $b = -6$, we have $y = Ce^{3t} + 2$. Because $y(0) = 9$, we have $9 = C + 2$, so $C = 7$. Thus, the solution is $y = 7e^{3t} + 2$.

9.4.12 Because $k = -1$ and $b = 2$, we have $y = Ce^{-x} + 2$. Because $y(0) = -2$, we have $-2 = C + 2$, so $C = -4$. Thus, the solution is $y = -4e^{-x} + 2$.

9.4.13 Because $k = 2$ and $b = 8$, we have $y = Ce^{2t} - 4$. Because $y(0) = 0$, we have $0 = C - 4$, so $C = 4$. Thus, the solution is $y = 4e^{2t} - 4$.

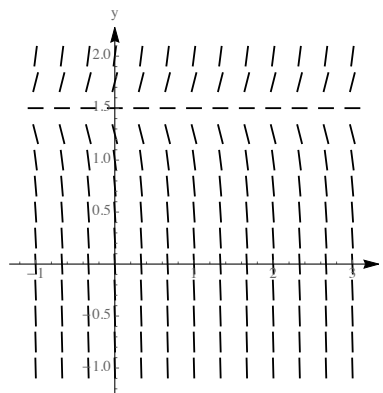
9.4.14 Because $k = 2$ and $b = 6$, we have $u = Ce^{2x} - 3$. Because $u(1) = 6$, we have $6 = Ce^2 - 3$, so $C = 9e^{-2}$. Thus, the solution is $u = 9e^{2x-2} - 3$.

9.4.15 Because $k = 3$ and $b = 12$, we have $y = Ce^{3t} - 4$. Because $y(1) = 4$, we have $4 = Ce^3 - 4$, so $C = 8e^{-3}$. Thus, the solution is $y = 8e^{3t-3} - 4$.

9.4.16 Because $k = -1/2$ and $b = 6$, we have $z = Ce^{-t/2} + 12$. Because $z(-1) = 0$, we have $0 = Ce^{1/2} + 12$, so $C = -12e^{-1/2}$. Thus, the solution is $z = -12e^{-t/2-1/2} + 12$.

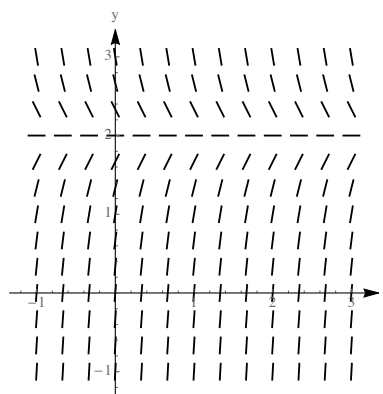
9.4.17

The equilibrium solution is $y = \frac{18}{12} = \frac{3}{2}$. The solution is unstable.



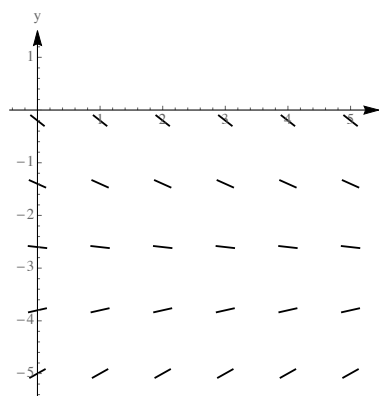
9.4.18

The equilibrium solution is $y = 2$. The solution is stable.



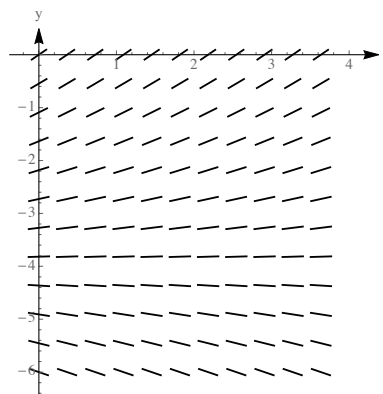
9.4.19

The equilibrium solution is $y = -3$. The solution is stable.



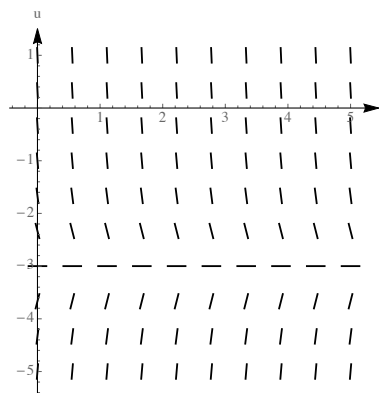
9.4.20

The equilibrium solution is $y = -4$. The solution is unstable.



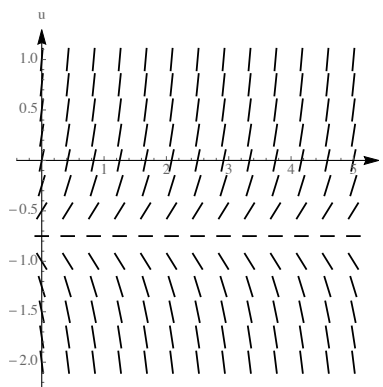
9.4.21

The equilibrium solution is $u = -3$. The solution is stable.



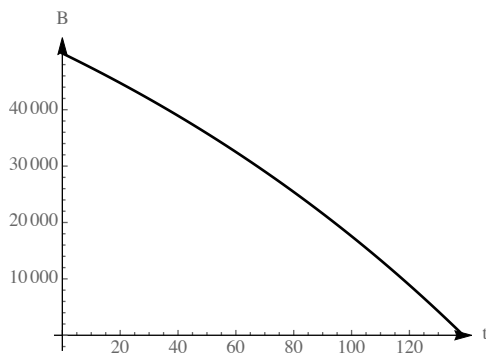
9.4.22

The equilibrium solution is $u = -3/4$. The solution is unstable.



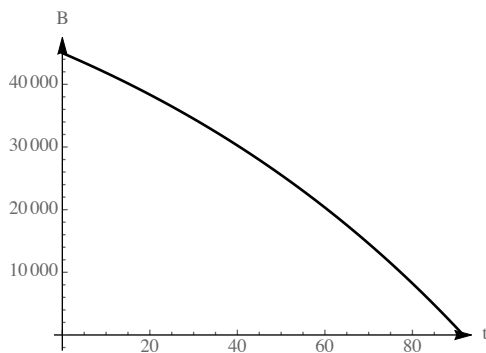
9.4.23 Because $k = 0.005$ and $b = -500$, we have $B = Ce^{0.005t} + \frac{500}{0.005} = Ce^{0.005t} + 100000$. Because $B(0) = 50,000$, we have $50000 = C + 100000$, so $C = -50000$. Thus, $B = -50000e^{0.005t} + 100000$.

The balance is zero when $t = \ln(2)/0.005 \approx 139$ months.



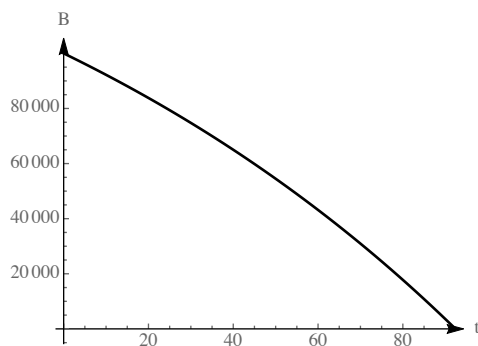
9.4.24 Because $k = 0.01$ and $b = -750$, we have $B = Ce^{0.01t} + \frac{750}{0.01} = Ce^{0.01t} + 75000$. Because $B(0) = 45,000$, we have $45000 = C + 75000$, so $C = -30000$. Thus, $B = -30000e^{0.01t} + 75000$.

The balance is zero when $t = \ln(5/2)/0.01 \approx 92$ months.



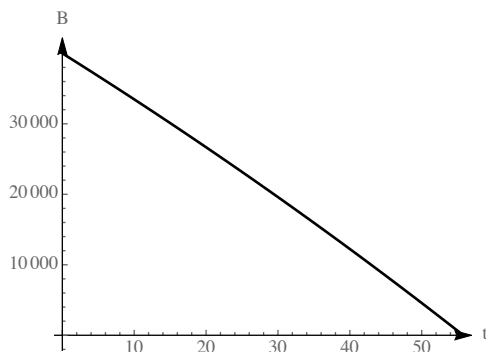
9.4.25 Because $k = 0.0075$ and $b = -1500$, we have $B = Ce^{0.0075t} + \frac{1500}{0.0075} = Ce^{0.0075t} + 200000$. Because $B(0) = 100,000$, we have $100000 = C + 200000$, so $C = -100000$. Thus, $B = -100000e^{0.0075t} + 200000$.

The balance is zero when $t = \ln(2)/0.0075 \approx 93$ months.



9.4.26 Because $k = 0.004$ and $b = -800$, we have $B = Ce^{0.004t} + \frac{800}{0.004} = Ce^{0.004t} + 200000$. Because $B(0) = 40,000$, we have $40000 = C + 200000$, so $C = -160000$. Thus, $B = -160000e^{0.004t} + 200000$.

The balance is zero when $t = \ln(5/4)/0.004 \approx 56$ months.



9.4.27 We know that the solution has the form $T(t) = (90 - 25)e^{-kt} + 25 = 65e^{-kt} + 25$. Because $T(1) = 85$, we have $65e^{-k} + 25 = 85$. Thus, $k = -\ln(60/65) \approx 0.08$. Thus, $T(t) = 65e^{-0.08t} + 25$. We have $T(t) = 30$ when $65e^{-0.08t} + 25 = 30$, or when $e^{-0.08t} = \frac{1}{13}$. This occurs when $t = \ln(13)/0.08 \approx 32$ minutes after the coffee is first poured.

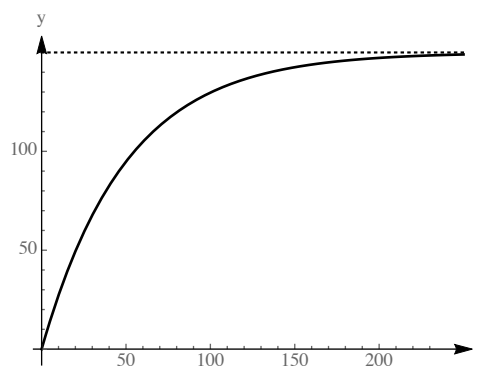
9.4.28 We have $T(t) = (900 - 30)e^{-0.02t} + 30$. We have $T(t) = 100$ when $870e^{-0.02t} + 30 = 100$, which occurs when $e^{-0.02t} = \frac{7}{87}$. This occurs when $t = \frac{\ln(7/87)}{-0.02} \approx 126$.

9.4.29 $T(t) = (5 - 20)e^{-kt} + 20 = -15e^{-kt} + 20$. Because $T(1) = 7$, we have $-15e^{-k} + 20 = 7$, so $k = \ln(15/13) \approx 0.143$. Thus $T(t) = -15e^{-0.143t} + 20$. The milk will reach 18 degrees when $-15e^{-0.143t} + 20 = 18$, which occurs when $e^{-0.143t} = 2/15$. Thus, when $t \approx \frac{\ln(2/15)}{-0.143} \approx 14$ minutes.

9.4.30 $T(t) = (100 - 10)e^{-kt} + 10 = 90e^{-kt} + 10$. We are given that $T(30) = 80$, so $80 = 90e^{-30k} + 10$, so $e^{-30k} = 7/9$. Thus $k = \frac{\ln(7/9)}{-30} \approx 0.0084$. The soup will reach 30 degrees when $30 = 90e^{-0.0084t} + 10$, which occurs when $e^{-0.0084t} = 2/9$, or $t = \frac{\ln(2/9)}{-0.0084} \approx 179$ minutes.

9.4.31

- a. The general solution is $y(t) = Ce^{-0.02t} + 150$; substitute $y(0) = 0$ to obtain $C + 150 = 0$, so $C = -150$; hence $y(t) = 150(1 - e^{-0.02t})$.



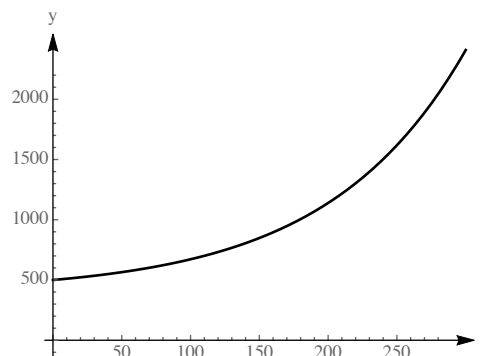
- b. The steady-state level is $\lim_{t \rightarrow \infty} 150(1 - e^{-0.02t}) = 150$ mg.

- c. We have $150(1 - e^{-0.02t}) = 0.9 \cdot 150$, so $e^{-0.02t} = 0.1$, and thus $t = \frac{\ln 10}{0.02} \approx 115.1$ hours.

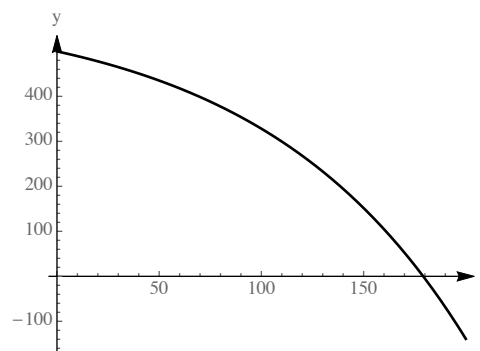
9.4.32

- a. The general solution is $y(t) = Ce^{0.01t} + 10b$; substitute $y(0) = 500$ to obtain $C + 10b = 500$, so $C = 500 - 10b$; hence $y(t) = (500 - 10b)e^{0.01t} + 10b$.

- b. In this case the solution is $y(t) = 100e^{0.01t} + 400$, which approaches ∞ as $t \rightarrow \infty$.



- c. In this case the solution is $y(t) = -100e^{0.01t} + 600$, which reaches 0 when $e^{0.01t} = 6$ or $t = 100 \ln 6 \approx 179$ yrs.

**9.4.33**

- a. The equation $y'(t) = 0.008y - h$ has steady-state solution $y = h/0.008 = 125h$, so we solve $y_0 = 2000 = 125h$ to obtain $h = 16$.
- b. If $h = 200$, then the steady-state solution is $y = 125 \cdot 200 = 25,000$.

9.4.34

- a. The equation $B' = rB - m$ is first-order linear, with general solution $B = Ce^{rt} + m/r$; in this case $r = 0.05$, $m/r = 20,000$, and the initial condition $B_0 = 15,000$ gives $C = -5000$, so $B = 20,000 - 5000e^{0.05t}$. The balance decreases.
- b. The steady-state (constant balance) solution is $B = m/r = \$50,000$, which gives $m = 0.05 \cdot 50,000 = \2500 .

9.4.35

- a. False. It is $y(t) = Ce^{2t} + 9$ where C is an arbitrary constant.
- b. True. Note that if $y(t) = 0$, then $y'(t) = 0$, not $k(0) - b = -b$ as dictated by the equation.
- c. False. It is not separable.
- d. False. It approaches it asymptotically.

9.4.36 First note that if $ky + b < 0$, then $|ky + b| = -ky - b$. We will use this fact in what follows. We have $\frac{1}{ky + b}y'(t) = 1$, so $\int \frac{dy}{ky + b} = \int dt$, so $\frac{1}{k} \ln(-ky - b) = t + C_1$. Thus $\ln(-ky - b) = kt + C_2$, so $-ky - b = C_3 e^{kt}$, and $y = Ce^{-kt} - \frac{b}{k}$.

9.4.37

- Note that $k = 0.03$ and $b = -600$. Thus, $B(t) = Ce^{0.03t} + 20,000$. If $B_0 = B(0)$ is the amount borrowed, then $B_0 = C + 20,000$, so $C = B_0 - 20,000 = 20,000$.
Thus, $B(t) = 20,000e^{0.03t} + 20,000$. This is an increasing function because its derivative is positive. It is occurring because the amount paid monthly is less than the monthly accruing interest.
- Because $20000 \cdot 0.03 = 600$, the amount borrowed should be less than 20000 if the balance is to be decreasing.
- The maximum amount that can be borrowed and not have the unpaid balance increase is B_0 when $rB_0 - m = 0$, or $B_0 = \frac{m}{r}$.

9.4.38

- $T(t) = (T_0 - A)e^{-kt} + A$. We are seeking $t_{1/2}$ so that $T(t_{1/2}) = \frac{T_0}{2}$. Thus we have $T_0/2 - A = (T_0 - A)e^{-k(t_{1/2})}$, so $\frac{T_0 - 2A}{2(T_0 - A)} = e^{-k(t_{1/2})}$. Thus, $t_{1/2} = -\frac{1}{k} \ln \left| \frac{T_0 - 2A}{2(T_0 - A)} \right|$.
- As k increases, $t_{1/2}$ decreases.
- If $A > T_0/2$, then the equation $T_0/2 - A = (T_0 - A)e^{-k(t_{1/2})}$ would have no solution, because the left-hand side is negative and the right-hand side is positive.

9.4.39 $\int (ty'(t) + y(t)) dt = \int (1 + t) dt$, so $ty(t) = t + \frac{t^2}{2} + C$ for $t > 0$. Thus, $y(t) = 1 + \frac{t}{2} + \frac{C}{t}$. Because $y(1) = 4$, we have $4 = 1 + 1/2 + C$, so $C = 5/2$. Thus, $y(t) = 1 + \frac{t}{2} + \frac{5}{2t}$ for $t > 0$.

9.4.40 $\int (t^3 y'(t) + 3t^2 y(t)) dt = \int (1/t + 1) dt$. Thus, $t^3 y(t) = \ln|t| + t + C$ for $t > 0$. Therefore, $y(t) = \frac{\ln|t|}{t^3} + \frac{1}{t^2} + \frac{C}{t^3}$. Because $y(1) = 6$, we have $6 = 0 + 1 + C$, so $C = 5$. The solution is thus $y(t) = \frac{\ln|t|}{t^3} + \frac{1}{t^2} + \frac{5}{t^3}$ for $t > 0$.

9.4.41 $\int (e^{-t} y'(t) - e^{-t} y(t)) dt = \int e^{2t} dt$, so $e^{-t} y(t) = e^{2t}/2 + C$. Thus, $y(t) = e^{3t}/2 + Ce^t$. Because $y(0) = 4$, we have $4 = 1/2 + C$, so $C = 7/2$. The solution is therefore $y(t) = e^{3t}/2 + 7e^t/2$.

9.4.42 $\int ((t^2 + 1)y'(t) + 2ty(t)) dt = \int 3t^2 dt$, so $(t^2 + 1)y(t) = t^3 + C$. Thus, $y(t) = \frac{t^3}{t^2 + 1} + \frac{C}{t^2 + 1}$. Because $y(2) = 8$, we have $8 = 8/5 + C/5$, so $C = 32$. The solution is therefore $y(t) = \frac{t^3 + 32}{t^2 + 1}$.

9.4.43

- Let $v = y^{1-p}$. Then $v'(t) = (1-p)y^{-p}y'(t)$, so $y'(t) = \frac{y^p}{1-p}v'(t)$.
- Given $y'(t) + ay = by^p$, we have $\frac{y^p}{1-p}v'(t) + ay = by^p$, so $\frac{1}{1-p}v'(t) + ay^{1-p} = b$, so

$$v'(t) = (p-1)av(t) + b(1-p).$$

Then $v(t) = Ce^{(p-1)at} - \frac{b(1-p)}{(p-1)a} = Ce^{(p-1)at} + \frac{b}{a}$. Therefore,

$$y(t) = \left(Ce^{(p-1)at} + \frac{b}{a} \right)^{1/(1-p)}.$$

9.4.44

a. Using the results of the previous problem, we have $a = 1$, $b = 2$, and $p = 2$. Thus

$$y(t) = (Ce^t + 2)^{-1}.$$

b. We have $a = -2$, $p = -1$, and $b = 3$, so

$$y(t) = \left(Ce^{4t} + \frac{-3}{2}\right)^{1/2}.$$

c. We have $a = 1$, $b = 1$, and $p = 1/2$, so

$$y(t) = \left(Ce^{-t/2} + 1\right)^2.$$

9.4.45 Let $p(t) = \exp\left(\int 1/t \, dt\right) = e^{\ln t} = t$. The original differential equation can be written as $t(y'(t) + (1/t)y(t)) = 0$, or $ty'(t) + y(t) = 0$. Integrating both sides with respect to t gives $ty(t) = C$, so $y(t) = \frac{C}{t}$. Because $y(1) = 6$, we have $C = 6$, and $y = \frac{6}{t}$ for $t > 0$.

9.4.46 Let $p(t) = \exp\left(\int 3/t \, dt\right) = e^{3 \ln t} = t^3$. The original differential equation can be written as $t^3(y'(t) + (3/t)y(t)) = t^3 - 2t^4$, or $t^3y'(t) + 3t^2y(t) = t^3 - 2t^4$. Integrating both sides with respect to t gives $t^3y(t) = t^4/4 - 2t^5/5 + C$, so $y(t) = t/4 - 2t^2/5 + C/t^3$. Because $y(2) = 0$, we have $0 = \frac{1}{2} - \frac{8}{5} + \frac{C}{8}$, and solving for C gives $C = \frac{44}{5}$. Thus,

$$y(t) = \frac{t}{4} - \frac{2t^2}{5} + \frac{44}{5t^3}$$

for $t > 0$.

9.4.47 Let $p(t) = \exp\left(\int \frac{2t}{t^2+1} \, dt\right) = \exp(\ln(t^2+1)) = t^2+1$. The original differential equation can be written as $(t^2+1)y'(t) + (2t)y(t) = (t^2+1)(1+3t^2)$. Integrating both sides with respect to t gives

$$(t^2+1)y(t) = \int (3t^4 + 4t^2 + 1) \, dt = 3t^5/5 + 4t^3/3 + t + C.$$

Because $y(1) = 4$, we have $8 = 3/5 + 4/3 + 1 + C$, so $C = \frac{76}{15}$. Thus,

$$y(t) = \frac{3t^5/5 + 4t^3/3 + t + \frac{76}{15}}{t^2+1} = \frac{9t^5 + 20t^3 + 15t + 76}{15(t^2+1)}.$$

9.4.48 Let $p(t) = \exp\left(\int 2t \, dt\right) = e^{t^2}$. The original differential equation can be written as $e^{t^2}y'(t) + e^{t^2}(2t)y(t) = 3te^{t^2}$. Integrating both sides with respect to t gives $e^{t^2}y(t) = \frac{3}{2}e^{t^2} + C$. Because $y(0) = 1$, we have $1 = \frac{3}{2} + C$, so $C = -\frac{1}{2}$. Thus, $y = \frac{\frac{3}{2}e^{t^2} - \frac{1}{2}}{e^{t^2}} = \frac{3e^{t^2} - 1}{2e^{t^2}}$.

9.5 Modeling with Differential Equations

9.5.1 The growth rate function specifies the rate of growth of the population. If the growth rate function is positive, then the population is increasing, while the population is decreasing when the growth rate function is negative.

9.5.2 The carrying capacity is the upper limit of the size of a population, due to limitations in resources. Mathematically, it appears as a horizontal asymptote as $t \rightarrow \infty$.

9.5.3 If the growth rate function is positive and decreasing, then the population is increasing. Whether or not the population is increasing is completely determined by whether the growth rate function is positive or negative.

9.5.4 A stirred tank reaction takes place in a tank that is filled with a soluble substance like salt or sugar. The tank has an inflow and an outflow pipe, so the volume of the solution is constant. The tank is assumed to be completely stirred at all times, so the solution is uniform. The problem is to find the mass of the substance in the tank at all times.

9.5.5 Is is a linear, first-order differential equation.

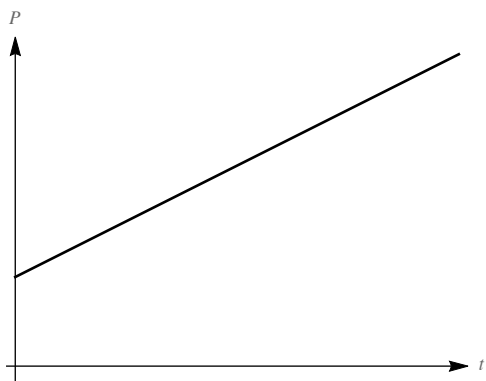
9.5.6

- In the absence of prey, the predator population decreases exponentially, while encounters between prey and predators increase the predator population (the prey are the food supply).
- In the absence of predators, the prey population increases exponentially, while encounters between the prey and predators deplete the prey population (the predators eat the prey).

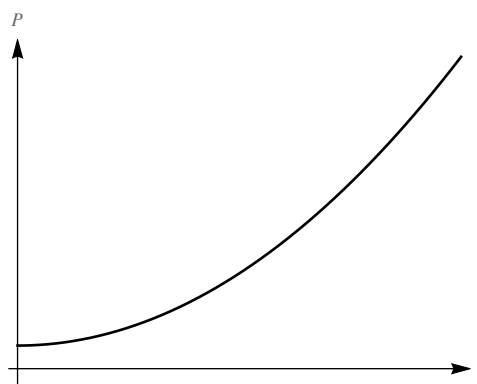
9.5.7 The solution curves in the FH -plane are closed curves that circulate around the equilibrium point.

9.5.8 They both oscillate cyclically, with the prey population peaking slightly before the predator population.

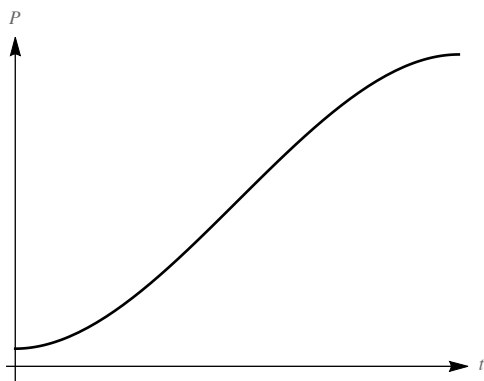
9.5.9



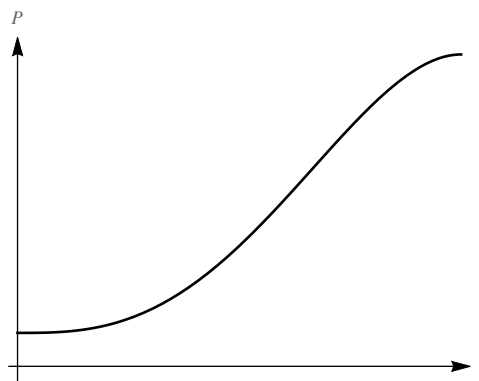
9.5.10



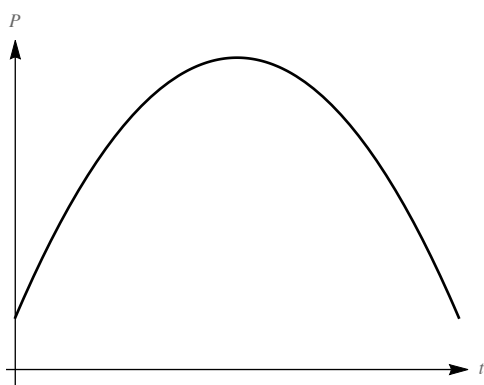
9.5.11



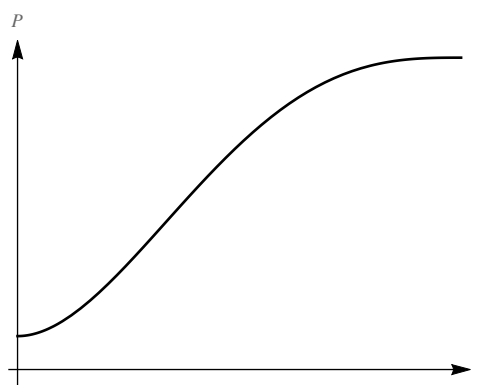
9.5.12



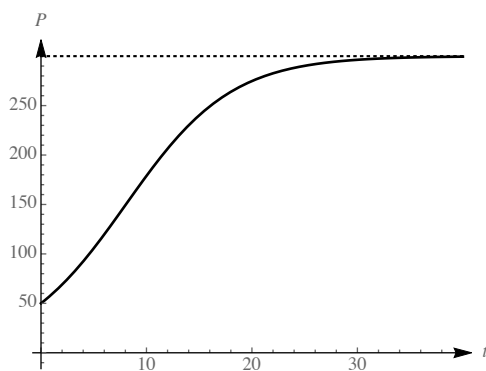
9.5.13



9.5.14

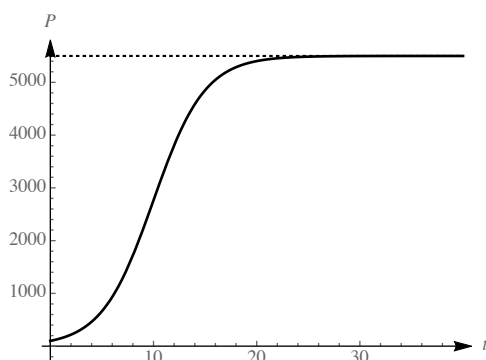


9.5.15 We must solve $P'(t) = 0.2P \left(1 - \frac{P}{300}\right)$. We have $\int \frac{300P'(t)}{P(300-P)} dt = \int 0.2 dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{300-P(t)} \right) dt = \int 0.2 dt$. Thus, $\ln \left(\frac{P(t)}{300-P(t)} \right) = 0.2t + C$. Taking the exponential of both sides and reciprocating gives $\frac{300-P(t)}{P(t)} = Ae^{-0.2t}$. Because $P(0) = 50$, we have $A = 5$. Thus $\frac{300}{P(t)} = 5e^{-0.2t} + 1$, so $P(t) = \frac{300}{5e^{-0.2t} + 1}$ for $t \geq 0$.

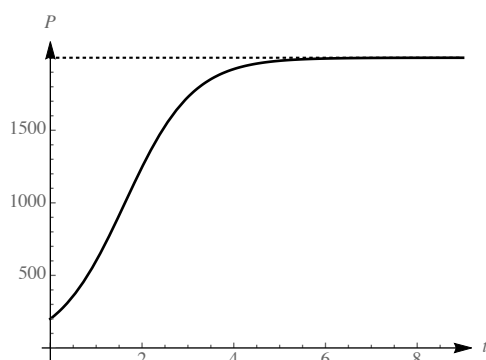


9.5.16 We must solve $P'(t) = 0.4P \left(1 - \frac{P}{5500}\right)$. We have $\int \frac{5500P'(t)}{P(5500-P)} dt = \int 0.4 dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{5500-P(t)} \right) dt = \int 0.4 dt$. Thus, $\ln \left(\frac{P(t)}{5500-P(t)} \right) = 0.4t + C$. Taking the

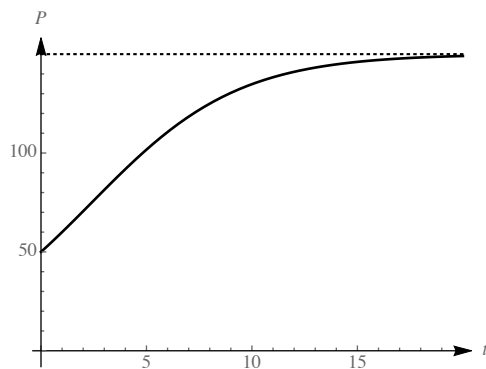
exponential of both sides and reciprocating gives $\frac{5500 - P(t)}{P(t)} = Ae^{-0.4t}$. Because $P(0) = 100$, we have $A = 54$. Thus $\frac{5500}{P(t)} = 54e^{-0.4t} + 1$, so $P(t) = \frac{5500}{54e^{-0.4t} + 1}$ for $t \geq 0$.



9.5.17 We must solve $P'(t) = rP \left(1 - \frac{P}{2000}\right)$. We have $\int \frac{2000P'(t)}{P(2000 - P)} dt = \int r dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{2000 - P(t)} \right) dt = \int r dt$. Thus, $\ln \left(\frac{P(t)}{2000 - P(t)} \right) = rt + C$. Taking the exponential of both sides and reciprocating gives $\frac{2000 - P(t)}{P(t)} = Ae^{-rt}$. Because $P(0) = 200$, we have $A = 9$. Thus $\frac{2000}{P(t)} = 9e^{-rt} + 1$, so $P(t) = \frac{2000}{9e^{-rt} + 1}$. Now because $P(1) = 600$, we have $r = \ln(27/7)$. So $P(t) = \frac{2000}{9e^{\ln(7/27)t} + 1}$ for $t \geq 0$.



9.5.18 We must solve $P'(t) = rP \left(1 - \frac{P}{150}\right)$. We have $\int \frac{150P'(t)}{P(150 - P)} dt = \int r dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{150 - P(t)} \right) dt = \int r dt$. Thus, $\ln \left(\frac{P(t)}{150 - P(t)} \right) = rt + C$. Taking the exponential of both sides and reciprocating gives $\frac{150 - P(t)}{P(t)} = Ae^{-rt}$. Because $P(0) = 50$, we have $A = 2$. Thus $\frac{150}{P(t)} = 2e^{-rt} + 1$, so $P(t) = \frac{150}{2e^{-rt} + 1}$. Now because $P(1) = 60$, we have $r = \ln(4/3)$. So $P(t) = \frac{150}{2e^{\ln(3/4)t} + 1}$ for $t \geq 0$.



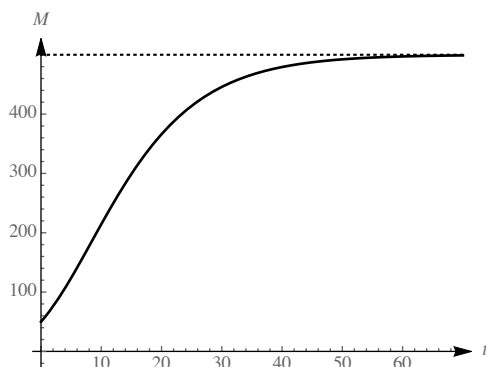
9.5.19 We have $\frac{M'(t)}{M \ln(M/K)} = -r$, so $\int \frac{M'(t)}{M \ln(M/K)} dt = \int -r dt$.

Note that $\frac{d}{dt} \ln(M/K) = (K/M)(1/K)M'(t) = M'(t)/M$. Thus integrating gives $\ln |\ln(M/K)| = -rt + C$, and thus $\ln(M/K) = Ae^{-rt}$. Because $M(0) = M_0$, we have $A = \ln(M_0/K)$.

Thus $M(t) = K(\exp(\ln(M_0/K)e^{-rt})) = K \left(\frac{M_0}{K} \right)^{e^{-rt}}$ for $t \geq 0$.

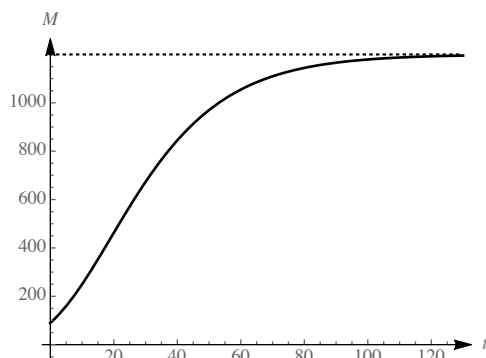
9.5.20 With $r = 0.1$, $K = 500$, and $M_0 = 50$, we have

$$M(t) = 500 \left(\frac{1}{10} \right)^{e^{-0.1t}} \quad \text{for } t \geq 0.$$



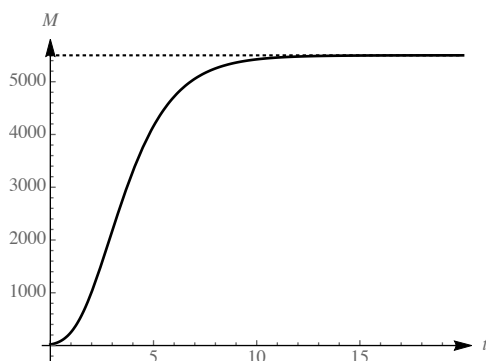
9.5.21 With $r = 0.05$, $K = 1200$, and $M_0 = 90$, we have

$$M(t) = 1200 \left(\frac{3}{40} \right)^{e^{-0.05t}} \quad \text{for } t \geq 0.$$



9.5.22 With $r = 0.6$, $K = 5500$, and $M_0 = 20$, we have

$$M(t) = 5500 \left(\frac{1}{275} \right) e^{-0.6t} \quad \text{for } t \geq 0.$$



9.5.23

- The initial mass of copper sulfate is $m_0 = 0$. We have $m'(t) = -\frac{4}{500}m(t) + 20 \cdot 4 = -\frac{1}{125}m(t) + 80$.
- This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-0.008t} + 10,000$. Because $m(0) = 0$, we have $C = -10,000$. Thus, $m(t) = -10,000e^{-0.008t} + 10,000$ for $t \geq 0$.

9.5.24

- The initial mass of the salt is $m_0 = 3000$. We have $m'(t) = -\frac{3}{1500}m(t) + 20 \cdot 3 = -\frac{1}{500}m(t) + 60$.
- This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-0.002t} + 30,000$. Because $m(0) = 3000$, we have $C = -27,000$. Thus, $m(t) = -27,000e^{-0.002t} + 30,000$ for $t \geq 0$.

9.5.25

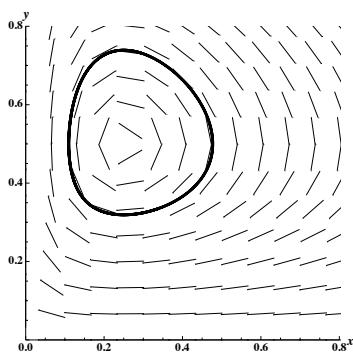
- The initial mass of the sugar is $m_0 = 2000 \cdot 40 = 80,000$. We have $m'(t) = -\frac{10}{2000}m(t) + 10 \cdot 10 = -\frac{1}{200}m(t) + 100$.
- This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-0.005t} + 20,000$. Because $m(0) = 80,000$, we have $C = 60,000$. Thus, $m(t) = 60,000e^{-0.005t} + 20,000$ for $t \geq 0$.

9.5.26

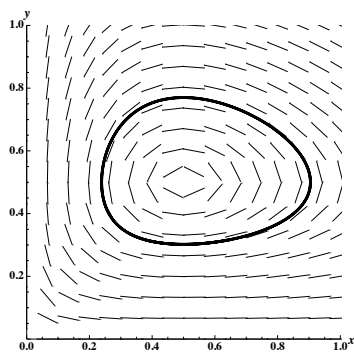
- The initial mass of the pollutant is $m_0 = 1000000 \cdot 20 = 20,000,000$. We have $m'(t) = -\frac{1200}{1000000}m(t) + 0 \cdot 1200 = -\frac{3}{2500}m(t)$.
- This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-0.0012t}$. Because $m(0) = 20,000,000$, we have $C = 20,000,000$. Thus, $m(t) = 20,000,000e^{-0.0012t}$ for $t \geq 0$. The pond will have a mass of 10 percent of the initial value when $0.1 = e^{-0.0012t}$, so $t = -\ln(0.1)/0.0012 \approx 1919$ hours.

9.5.27

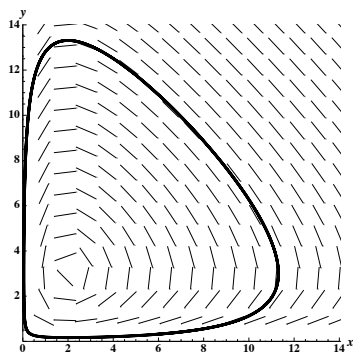
- a. x is the predator and y is the prey.
- b. $-3x + 6xy = 0$ when $x = 0$ or $y = 1/2$. $y - 4xy = 0$ when $y = 0$ or $x = 1/4$.
The desired lines are $x = 1/4$ and $y = 1/2$.
- c. The equilibrium points are where both equations are zero simultaneously, so they are $(0,0)$ and $(1/4, 1/2)$.
- d. Note that $x'(t) = 3x(2y - 1)$ and $y'(t) = y(1 - 4x)$.
- For $0 < x < 1/4$ and $0 < y < 1/2$, we have $x' < 0$ and $y' > 0$.
 - For $0 < x < 1/4$ and $y > 1/2$, we have $x' > 0$ and $y' > 0$.
 - For $x > 1/4$ and $0 < y < 1/2$, we have $x' < 0$ and $y' < 0$.
 - For $x > 1/4$ and $y > 1/2$, we have $x' > 0$ and $y' < 0$.
- e. The direction of the solution is clockwise.

**9.5.28**

- a. x is the prey and y is the predator.
- b. $2x - 4xy = 0$ when $x = 0$ or $y = 1/2$. $-y + 2xy = 0$ when $y = 0$ or $x = 1/2$.
The desired lines are $x = 1/2$ and $y = 1/2$.
- c. The equilibrium points are where both equations are zero simultaneously, so they are $(0,0)$ and $(1/2, 1/2)$.
- d. Note that $x'(t) = 2x(1 - 2y)$ and $y'(t) = y(2x - 1)$.
- For $0 < x < 1/2$ and $0 < y < 1/2$, we have $x' > 0$ and $y' < 0$.
 - For $0 < x < 1/2$ and $y > 1/2$, we have $x' < 0$ and $y' < 0$.
 - For $x > 1/2$ and $0 < y < 1/2$, we have $x' > 0$ and $y' > 0$.
 - For $x > 1/2$ and $y > 1/2$, we have $x' < 0$ and $y' > 0$.
- e. The direction of the solution is counterclockwise.

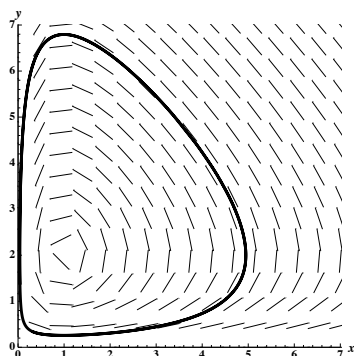
**9.5.29**

- a. x is the predator and y is the prey.
- b. $-3x + xy = 0$ when $x = 0$ or $y = 3$. $2y - xy = 0$ when $y = 0$ or $x = 2$.
The desired lines are $x = 2$ and $y = 3$.
- c. The equilibrium points are where both equations are zero simultaneously, so they are $(0, 0)$ and $(2, 3)$.
- d. Note that $x'(t) = x(y - 3)$ and $y'(t) = y(2 - x)$.
 - For $0 < x < 2$ and $0 < y < 3$, we have $x' < 0$ and $y' > 0$.
 - For $0 < x < 2$ and $y > 3$, we have $x' > 0$ and $y' > 0$.
 - For $x > 2$ and $0 < y < 3$, we have $x' < 0$ and $y' < 0$.
 - For $x > 2$ and $y > 3$, we have $x' > 0$ and $y' < 0$.
- e. The direction of the solution is clockwise.

**9.5.30**

- a. x is the prey and y is the predator.
- b. $2x - xy = 0$ when $x = 0$ or $y = 2$. $-y + xy = 0$ when $y = 0$ or $x = 1$.
The desired lines are $x = 1$ and $y = 2$.
- c. The equilibrium points are where both equations are zero simultaneously, so they are $(0, 0)$ and $(1, 2)$.
- d. Note that $x'(t) = x(2 - y)$ and $y'(t) = y(x - 1)$.
 - For $0 < x < 1$ and $0 < y < 2$, we have $x' > 0$ and $y' < 0$.
 - For $0 < x < 1$ and $y > 2$, we have $x' < 0$ and $y' < 0$.
 - For $x > 1$ and $0 < y < 2$, we have $x' > 0$ and $y' > 0$.

- For $x > 1$ and $y > 2$, we have $x' < 0$ and $y' > 0$.
- e. The direction of the solution is counterclockwise.

**9.5.31**

- a. True. The growth rate function is the derivative, so where it is positive, the population is increasing.
- b. True. In the limit, the solution in the tank is the same as the solution being poured in.
- c. True. In the absence of predators, we assume that the prey population increases exponentially.

9.5.32

- a. $f'(P) = r \left(1 - \frac{P}{K}\right) + rP \left(\frac{-1}{K}\right) = r \left(1 - \frac{2P}{K}\right)$. This is zero when $P = \frac{K}{2}$, and an analysis of the sign of f' shows that $f' > 0$ for $P < \frac{K}{2}$ and $f' < 0$ when $P > \frac{K}{2}$. Thus there is a maximum of $f(K/2) = r(K/2)(1/2) = rK/4$ at $P = K/2$.
- b. $f'(M) = -r \ln(M/K) - rM(K/M)(1/K) = -r(\ln(M/K) + 1)$. This is zero when $\ln(M/K) = -1$, or $M/K = e^{-1}$, which occurs for $M = \frac{K}{e}$. Note that $f(K/e) = -r(K/e) \ln(1/e) = rK/e$. An analysis of the sign of f' shows that $f' > 0$ for $M < \frac{K}{e}$ and $f' < 0$ when $M > \frac{K}{e}$. Thus there is a maximum at $M = \frac{K}{e}$.

9.5.33 We must solve $P'(t) = rP \left(1 - \frac{P}{K}\right)$. We have $\int \frac{KP'(t)}{P(K-P)} dt = \int r dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{K-P(t)} \right) dt = \int r dt$. Thus, $\ln \left(\frac{P(t)}{K-P(t)} \right) = rt + C$. Taking the exponential of both sides and reciprocating gives $\frac{K-P(t)}{P(t)} = Ae^{-rt}$. Because $P(0) = P_0$, we have $A = \frac{K-P_0}{P_0}$. Thus $\frac{K}{P(t)} = \left(\frac{K-P_0}{P_0} \right) e^{-rt} + 1$, so $P(t) = \frac{K}{\frac{K-P_0}{P_0} e^{-rt} + 1} = \frac{KP_0}{(K-P_0)e^{-rt} + P_0}$.

9.5.34 $M(0) = K \left(\frac{M_0}{K} \right)^{e^0} = K \left(\frac{M_0}{K} \right) = M_0$. Because $\lim_{t \rightarrow \infty} e^{-rt} = 0$, we have

$$\lim_{t \rightarrow \infty} M(t) = K \left(\frac{M_0}{K} \right)^0 = K.$$

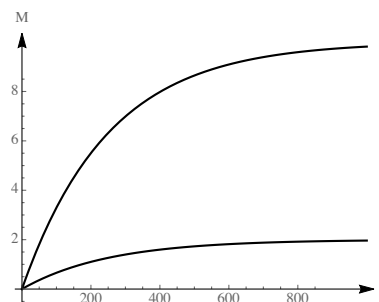
9.5.35

- a. Note that this differential equation is first-order linear with $k = -\frac{R}{V}$ and $b = C_i R$, so the solution is $m(t) = Ce^{(-R/V)t} - \frac{C_i R}{-R/V} = Ce^{(-R/V)t} + C_i V$. Because $m(0) = m_0$, we have $m_0 = C + C_i V$, so $C = m_0 - C_i V$. Thus, the solution is $m(t) = (m_0 - C_i V)e^{-Rt/V} + C_i V$.
- b. $m(0) = (m_0 - C_i V) + C_i V = m_0$.
- c. Note that $\lim_{t \rightarrow \infty} e^{-Rt/V} = 0$, so $\lim_{t \rightarrow \infty} m(t) = C_i V$. In the limit, the solution in the tank is the incoming solution, so the amount of material in the tank is the amount per unit volume times the volume of the tank.
- d. Increasing R causes the graph to approach the asymptote more quickly.

9.5.36

- a. The initial mass of the drug in the body is 0. We have $m'(t) = -\frac{0.06}{4}m(t) + (0.06)(500) = -0.015m(t) + 30$.

- This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-0.015t} + 2000$.
- b. Because $m(0) = 0$, we have $C = -2000$. Thus, $m(t) = -2000e^{-0.015t} + 2000$.



- c. Because $\lim_{t \rightarrow \infty} e^{-0.015t} = 0$, $\lim_{t \rightarrow \infty} m(t) = 2000$.
- d. The drug mass reaches 1800 mg when $1800 = -2000e^{-0.015t} + 2000$, so when $0.1 = e^{-0.015t}$, so $t = -\ln(0.1)/0.015 \approx 153.5$ minutes.

9.5.37

- a. We can write the equation as $I'(t) = -\frac{1}{RC}I(t)$. This is a first-order linear equation with $k = -\frac{1}{RC}$ and $b = 0$. Thus, the solution is $I(t) = Ce^{kt} - \frac{b}{k} = Ae^{-t/RC}$. Because $I(0) = \frac{V}{R}$, we have $A = \frac{V}{R}$. Thus, $I(t) = \frac{V}{R}e^{-t/RC}$. The current decays exponentially with decay constant $-\frac{1}{RC}$.
- b. We can write the equation as $Q'(t) = -\frac{1}{RC}Q + \frac{V}{R}$. This is a first-order linear equation with $k = -\frac{1}{RC}$ and $b = \frac{V}{R}$. Thus, the solution is $Q(t) = Ce^{kt} - \frac{b}{k} = Ae^{-t/RC} - \frac{(V/R)}{(-1/RC)} = Ae^{-t/RC} + VC$. Because $Q(0) = 0$, we have $A = -VC$. Thus,

$$Q(t) = (-VC)e^{-t/RC} + VC.$$

In the long run, the charge has limit VC .

9.5.38

a. $P(t) = P_0 e^{rt}$. $P(0) = P_0 = 296$ and $P(10) = 321$, so $321 = 296e^{10r}$, so $r = \frac{1}{10} \ln \left(\frac{321}{296} \right) \approx 0.0081$.

b. $P(t) = \frac{K}{1 + Ce^{-0.0081t}}$. Because $P(0) = 296 = \frac{K}{1 + C}$, we have $1 + C = \frac{K}{296}$, so $C = \frac{K - 296}{296}$. Then we have

$$P(t) = \frac{K}{1 + \left(\frac{K-296}{296} \right) e^{-0.0081t}} = \frac{296K}{296 + (K - 296)e^{-0.0081t}}.$$

Because $P(45) = 398$, we have

$$398 = \frac{296K}{296 + (K - 296)e^{-0.0081t}},$$

and solving for K gives $K \approx 1838.65$.

c. It will reach 95 percent of 1838.65 when $\frac{296(1838.65)}{296 + (1838.65 - 296)e^{-0.0081t}} = 0.95(1838.65)$. Solving for t gives $t \approx 567.3$ years.

d. Solving $410 = \frac{296}{296 + (K - 296)e^{-0.0081(45)}}$ for K gives $K \approx 3289.9$.

e. Solving $380 = \frac{296}{296 + (K - 296)e^{-0.0081(45)}}$ for K gives $K \approx 1071.22$.

f. Small differences in the 35-year projection result in large differences in the estimated carrying capacity.

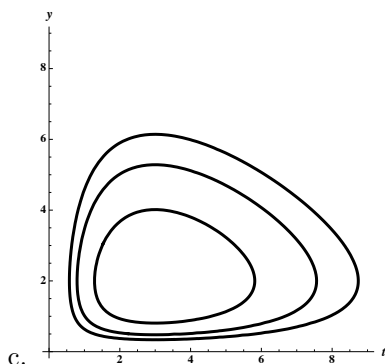
9.5.39

a. We have $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{cy - dxy}{-ax + bxy} = \frac{y(c - dx)}{x(by - a)}$. Thus we can write $\frac{by - a}{y} \frac{dy}{dx} = \frac{c - dx}{x}$, or

$$\left(b - \frac{a}{y} \right) \frac{dy}{dx} = \frac{c}{x} - d.$$

Thus, the equation is separable.

b. We have $\int \left(b - \frac{a}{y} \right) \frac{dy}{dx} dx = \int \left(\frac{c}{x} - d \right) dx$, so $by - a \ln y = c \ln x - dx + K$, so $dx + by = c \ln x + a \ln y + K$. Taking the exponential function of both sides gives $e^{dx+by} = Cx^c y^a$ for an arbitrary constant C .



Chapter Nine Review

1

- a. False. It is a first-order, linear differential equation, but it isn't separable.
- b. False. It is a first-order, separable differential equation, but it isn't linear.
- c. True. Note that $y' = 1 - t^{-2}$, so $ty' = t - t^{-1}$. Thus, $ty' + y = t - t^{-1} + (t + t^{-1}) = 2t$. Also, $y(1) = 2$.
- d. True.
- e. False. In general, Euler's method gives approximate solutions.

2 $y'(t) = -3y$, so $\int \frac{y'(t)}{y} dt = \int (-3) dt$. Thus, $\ln |y| = -3t + K$, and exponentiating both sides gives $y = Ce^{-3t}$.

3 $y'(t) = -2y + 6$, so $\int \frac{y'(t)}{-2y + 6} dt = \int 1 dt$, and therefore $-\frac{1}{2} \ln |-2y + 6| = t + K$. It follows that $|-2y + 6| = Ae^{-2t}$. We can write $-2y + 6 = \pm Ae^{-2t}$, so $y(t) = Ce^{-2t} + 3$.

4 $p'(x) = 4p + 8 = 4(p + 2)$, so $\int \frac{p'(x)}{p + 2} dx = \int 4 dx$, and thus $\ln |p + 2| = 4x + K$. Thus $|p + 2| = Ae^{4x}$, so $p + 2 = \pm Ae^{4x}$, and thus $p(x) = Ce^{4x} - 2$.

5 $y'(t) = 2ty$, so $\int \frac{y'(t)}{y} dt = \int 2t dt$, and therefore, $\ln |y| = t^2 + K$. It follows that $y = Ce^{t^2}$.

6 $y'(t) = \frac{\sqrt{y}}{\sqrt{t}}$, so $\int y^{-1/2} y'(t) dt = \int t^{-1/2} dt$. Integrating gives $2\sqrt{y} = 2\sqrt{t} + K$, so $\sqrt{y} = \sqrt{t} + C$, so $y(t) = (\sqrt{t} + C)^2$.

7 $\frac{y'(t)}{y} = \frac{1}{t^2 + 1}$. Integrating both sides with respect to t gives $\ln |y| = \tan^{-1}(t) + K$, so $y = Ce^{\tan^{-1}(t)}$.

8 $2yy'(x) = \sin x$, so $y^2 = -\cos x + C$, and thus $y = \pm\sqrt{C - \cos x}$.

9 $\int \frac{y'(t)}{y^2 + 1} dt = \int (2t + 1) dt$, so $\tan^{-1}(y) = t^2 + t + C$, and thus $y = \tan(t^2 + t + C)$.

10 $\frac{z'(t)}{z} = \frac{t}{t^2 + 1}$. Integrating both sides with respect to t gives $\ln |z| = \frac{1}{2} \ln |t^2 + 1| + K$. Thus $z = C\sqrt{t^2 + 1}$.

11 $\int y'(t) dt = \int (2t + \cos t) dt$, so $y(t) = t^2 + \sin t + C$. Because $y(0) = 1$, we have $1 = 0 + 0 + C$, so $C = 1$. Thus, $y(t) = t^2 + \sin t + 1$.

12 $y'(t) = -3(y - 3)$, so $\int \frac{y'(t)}{y - 3} dt = \int (-3) dt$, so $\ln |y - 3| = -3t + K$, and thus $y - 3 = Ce^{-3t}$. Because $y(0) = 4$, we have $C = 1$. Thus $y = e^{-3t} + 3$.

13 $\int \frac{Q'(t)}{Q - 8} dt = \int 1 dt$, so $\ln |Q - 8| = t + K$. Thus, $Q - 8 = Ce^t$. Because $Q(1) = 0$, we have $-8 = Ce$, so $C = -\frac{8}{e}$. Thus, $Q = -8e^{t-1} + 8$.

14 $yy'(x) = x$, so $\int yy'(x) dx = \int x dx$, so $y^2/2 = x^2/2 + C$. Because $y(2) = 4$, we have $8 = 2 + C$, so $C = 6$. Thus, $y^2 = x^2 + 12$, and $y = \sqrt{x^2 + 12}$.

15 $u^{-1/3}u'(t) = t^{-1/3}$. Thus $\int u^{-1/3}u'(t) dt = \int t^{-1/3} dt$. We have $\frac{3}{2}u^{2/3} = \frac{3}{2}t^{2/3} + C$. Because $u(1) = 8$, we have $6 = \frac{3}{2} + C$, so $C = \frac{9}{2}$. Thus, $u^{2/3} = t^{2/3} + 3$, and $u = (t^{2/3} + 3)^{3/2}$ for $t > 0$.

16 $\int (\sin y)y'(x) dx = \int 4x dx$, so $-\cos y = 2x^2 + C$. Because $y(0) = \pi/2$, we have $0 = 0 + C$, so $C = 0$. Thus $\cos y = -2x^2$, and $y = \cos^{-1}(-2x^2)$.

17 $\frac{s'(t)}{s} = \frac{1}{t(t^2 + 1)}$, so $\int \frac{ds}{s} = \int \frac{1}{t(t^2 + 1)} dt$, which can be written as $\int \frac{ds}{s} = \int \left(\frac{1}{t} - \frac{t}{t^2 + 1} \right) dt$. Then we have

$$\ln |s| = \ln |t| - \frac{1}{2} \ln(1 + t^2) + C,$$

Because $s(1) = 1$, we have $C = \ln \sqrt{2}$. Then $\ln |s| = \ln \left(\frac{\sqrt{2}|t|}{\sqrt{1 + t^2}} \right)$, and

$$|s| = \frac{\sqrt{2}|t|}{\sqrt{1 + t^2}}.$$

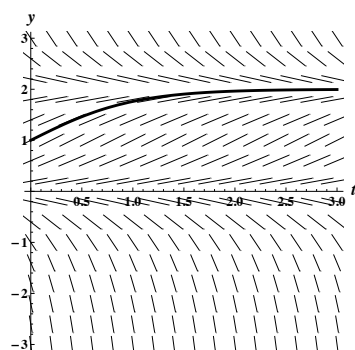
Because $s(1) = 1$, we have

$$s = \frac{\sqrt{2}t}{\sqrt{1 + t^2}}.$$

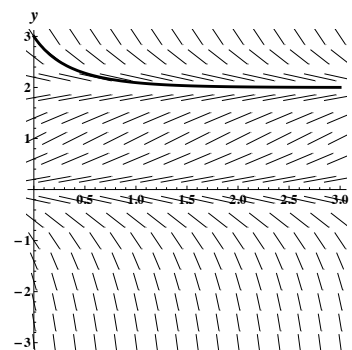
18 $\int (\sec^2 \theta)\theta'(x) dx = \int 4x dx$. Thus, $\tan \theta = 2x^2 + C$. Because $\theta(0) = \pi/4$, we have $1 = 0 + C$, so $C = 1$. It follows that $\theta = \tan^{-1}(2x^2 + 1)$.

19

a.



b.



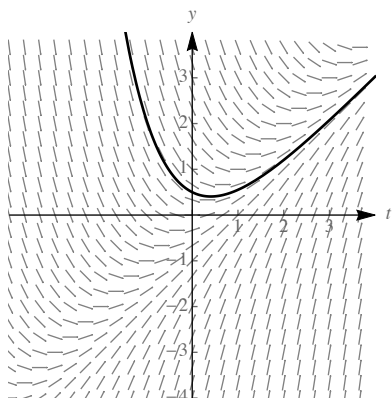
c. The solutions are increasing when $0 < A < 2$.

d. The solutions are decreasing when $A < 0$ or when $A > 2$.

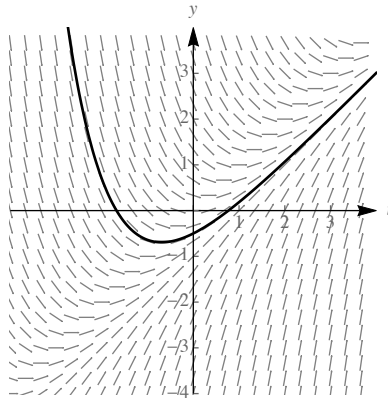
e. The equilibrium solutions are $y = 0$ and $y = 2$.

20

a.



b.



c. The solutions are increasing for $t > y$ and decreasing for $t < y$.

d. $y = t$.

21

a. $u_0 = 1$. $u_1 = 1 + f(0, 1)(0.1) = 1 + 1/20 = 1.05$. Also, $u_2 = 1.05 + f(0.1, 1.05)(0.1) = 1.05 + 0.047619 = 1.09762$. Thus, $y(0.1) \approx 1.05$ and $y(0.2) \approx 1.09762$.

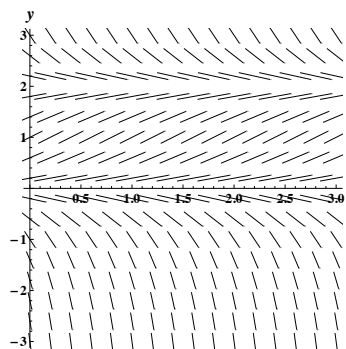
b. $u_0 = 1$. $u_1 = 1 + f(0, 1)(0.05) = 1 + 0.025 = 1.025$. Also, $u_2 = 1.025 + f(0.05, 1.025)(0.05) = 1.04939$. $u_3 = 1.04939 + f(0.1, 1.04939)(.05) = 1.07321$. $u_4 = 1.07321 + f(0.15, 1.07321)(.05) = 1.0961$. Thus, $y(0.1) \approx 1.04939$ and $y(0.2) \approx 1.09651$.

c. For part a, we have $\frac{|1.09762 - \sqrt{1.2}|}{\sqrt{1.2}} = 0.00198539$.

For part b, we have $\frac{|1.09651 - \sqrt{1.2}|}{\sqrt{1.2}} = 0.000972103$. Part b is more accurate.

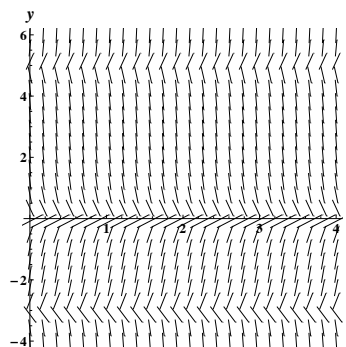
22

The equilibrium solutions are $y = 0$ (unstable) and $y = 2$ (stable).



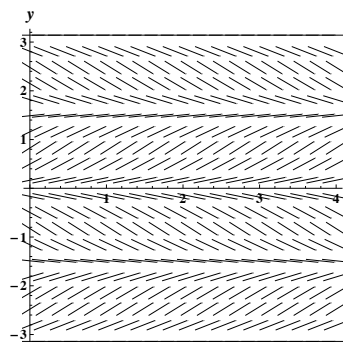
23

The equilibrium solutions are $y = -3$ (unstable) and $y = 0$ (stable), and $y = 5$ (unstable).



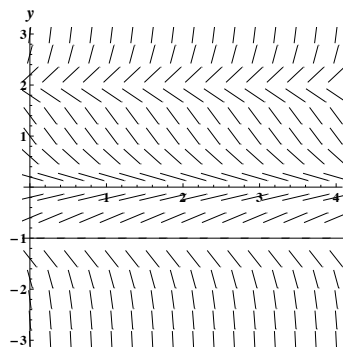
24

The equilibrium solutions are $y = -\pi/2$ (stable) and $y = 0$ (unstable), and $y = \pi/2$ (stable).



25

The equilibrium solutions are $y = -1$ (unstable) and $y = 0$ (stable), and $y = 2$ (unstable).



26

a. The equilibrium solutions occur where $P'(t) = 0$, so they are $P = 0$ and $P = 1200$.

b. We must solve $P'(t) = 0.2P(1 - \frac{P}{1200})$. We have $\int \frac{1200P'(t)}{P(1200 - P)} dt = \int 0.2 dt$, which can be written as $\int \left(\frac{P'(t)}{P(t)} + \frac{P'(t)}{1200 - P(t)} \right) dt = \int 0.2 dt$. Thus, $\ln \left(\frac{P(t)}{1200 - P(t)} \right) = 0.2t + C$. Taking the exponential of both sides and reciprocating gives $\frac{1200 - P(t)}{P(t)} = Ae^{-0.2t}$. Because $P(0) = 50$, we have $A = 23$. Thus $\frac{1200}{P(t)} = 23e^{-0.2t} + 1$, so $P(t) = \frac{1200}{23e^{-0.2t} + 1} = \frac{60000}{1150e^{-0.2t} + 50}$.

- c. The carrying capacity is $\lim_{t \rightarrow \infty} \frac{1200}{23e^{-0.2t} + 1} = 1200$.
- d. $P''(t) = 0.2(1 - (P/1200)) + 0.2P(-1/1200) = 0.2 - 0.2(P/600)$. This is zero when $P = 600$. Also, note that $P'''(600) < 0$, so there is a maximum for P' at 600 by the Second Derivative Test.

27

- a. The logistic equation has solution $P(t) = \frac{KP_0}{(K - P_0)e^{-rt} + P_0} = \frac{32000}{1580e^{-rt} + 20}$. We are seeking r so that $80 = \frac{32000}{1580e^{-20r} + 20}$. Solving for r gives $r = \frac{\ln(79/19)}{20} \approx 0.0713$.
- b. The solution is $P(t) = \frac{32000}{1580e^{-0.0713t} + 20} = \frac{1600}{79e^{-0.0713t} + 1}$ for $t \geq 0$.
- c. We are seeking t so that $800 = \frac{1600}{79e^{-0.0713t} + 1}$. Solving for t gives $t \approx 61$ hours.

28

- a. Assuming $P(t) = 435e^{kt}$ and $P(5) = 435e^{5k} = 487$, we have $k = \ln(487/435)/5 \approx 0.0226$.
- b. The logistic equation has solution $P(t) = \frac{KP_0}{(K - P_0)e^{-rt} + P_0} = \frac{K \cdot 435}{(K - 435)e^{-0.0226t} + 435}$. We are seeking K so that $1570 = \frac{K \cdot 435}{(K - 435)e^{-0.0226 \cdot 90} + 435}$. Solving for K gives $K \approx 2585$.
- c. $P(t) = \frac{2585 \cdot 435}{2150e^{-0.0226t} + 435}$, $t \geq 0$.
- d. Solving $2000 = \frac{2585 \cdot 435}{2150e^{-0.0226t} + 435}$ for t gives $t \approx 125$, which would be in about 2085.
- e. It is very hard to predict the actual carrying capacity. Perhaps the estimate for the population in 2050 isn't accurate. Also, small changes in these kinds of estimates can lead to large changes in the actual population sizes.

29

- a. The initial mass of the sugar is $m_0 = 0$. We have $m'(t) = -\frac{0.5}{100}m(t) + 100 \cdot 0.5 = -\frac{1}{200}m(t) + 10$. This is an equation of the form $m'(t) = km + b$, so the solution is $m(t) = Ce^{kt} - \frac{b}{k} = Ce^{-0.005t} + 2000$. Because $m(0) = 0$, we have $C = -2000$. Thus, $m(t) = -2000e^{-0.005t} + 2000$.
- b. The steady-state mass is $\lim_{t \rightarrow \infty} m(t) = 2000$.
- c. The mass reaches $0.95(2000)$ when $0.05 = e^{-0.005t}$, or $t = \ln(0.05)/(-0.005) \approx 599$ minutes.

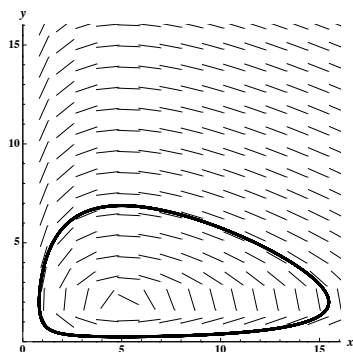
30

- a. $T(t) = (80 - 25)e^{-kt} + 25$. Because $T(5) = 60$, we have $60 = 55e^{-5k} + 25$, so $35/55 = e^{-5k}$, and thus $k = -\ln(35/55)/5 \approx 0.0904$.
- b. $T(t) = 55e^{-0.0904t} + 25$.
- c. The coffee reaches 50 degrees when $50 = 55e^{-0.0904t} + 25$, or when $e^{-0.0904t} = 25/55$. Thus $t = -\ln(5/11)/0.0904 \approx 8.7$ minutes.

31

- a. x represents the predator and y represents the prey.

- b. $x'(t) = 2x(-2 + y)$ is zero when $x = 0$ and when $y = 2$. $y'(t) = y(5 - x)$ is zero when $y = 0$ and when $x = 5$.
- c. The equilibrium points occur when both equations are zero simultaneously, which occurs at $(0, 0)$ and $(5, 2)$.
- d.
- For $0 < x < 5$ and $0 < y < 2$, we have $x' < 0$ and $y' > 0$.
 - For $0 < x < 5$ and $y > 2$, we have $x' > 0$ and $y' > 0$.
 - For $x > 5$ and $0 < y < 2$, we have $x' < 0$ and $y' < 0$.
 - For $x > 5$ and $y > 2$, we have $x' > 0$ and $y' < 0$.
- e. The solution evolves in the clockwise direction.

**32**

- a. $t^2 y'(t) + (2t)y(t) = \frac{d}{dt}(t^2 y(t))$.
- b. $\int (t^2 y'(t) + (2t)y(t)) dt = \int e^{-t} dt$, so $t^2 y(t) = -e^{-t} + C$. Thus, $y(t) = \frac{-e^{-t} + C}{t^2}$, $t > 0$.
- c. If $y(1) = 0$, then $0 = C - 1/e$, so $C = 1/e$. Thus, $y(t) = \frac{-e^{-t} + 1/e}{t^2}$, $t > 0$.

33

- a. If $y(t) = t^p$ is a solution, then $p(p-1)t^p + 2pt^p - 12t^p = 0$, so $p^2 - p + 2p - 12 = 0$, so $p^2 + p - 12 = (p-3)(p+4) = 0$. Thus, $p_1 = 3$ and $p_2 = -4$.
- b. If $y(t) = C_1 t^3 + C_2 t^{-4}$, and if $y(1) = 0$, then $C_1 + C_2 = 0$. Note that $y'(t) = 3C_1 t^2 - 4C_2 t^{-5}$, so if $y'(1) = 7$, we have $7 = 3C_1 - 4C_2$. Solving the system of linear equations in C_1 and C_2 gives $C_1 = 1$ and $C_2 = -1$. Thus the solution we seek is $y(t) = t^3 - t^{-4}$, $t > 0$.

Chapter 10

Sequences and Infinite Series

10.1 An Overview

10.1.1 A *sequence* is an ordered list of numbers $a_1, a_2, a_3, \dots$, often written $\{a_1, a_2, \dots\}$ or $\{a_n\}$. For example, the natural numbers $\{1, 2, 3, \dots\}$ are a sequence where $a_n = n$ for every n .

10.1.2 $a_1 = \frac{1}{1} = 1; a_2 = \frac{1}{2}; a_3 = \frac{1}{3}; a_4 = \frac{1}{4}; a_5 = \frac{1}{5}.$

10.1.3 $a_1 = 1$ (given); $a_2 = 1 \cdot a_1 = 1$; $a_3 = 2 \cdot a_2 = 2$; $a_4 = 3 \cdot a_3 = 6$; $a_5 = 4 \cdot a_4 = 24$.

10.1.4 $a_n = (-1)^n$ for $n = 1, 2, 3, \dots$. Or, we could write $a_n = (-1)^{n+1}$ for $n = 0, 1, 2, \dots$. Other answers are possible.

10.1.5 $a_n = (-1)^{n+1}n$ for $n = 1, 2, 3, \dots$. Or we could write $a_n = (-1)^n(n+1)$ for $n = 0, 1, 2, \dots$. Other answers are possible.

10.1.6 The sequence has limit π .

10.1.7 The sequence has limit e .

10.1.8 No. The terms of the sequence do not approach a unique number.

10.1.9 $S_1 = \sum_{k=1}^1 k^2 = 1$; $S_2 = \sum_{k=1}^2 k^2 = 1 + 4 = 5$; $S_3 = \sum_{k=1}^3 k^2 = 1 + 4 + 9 = 14$; $S_4 = \sum_{k=1}^4 k^2 = 1 + 4 + 9 + 16 = 30$.

10.1.10 $S_1 = \sum_{k=1}^1 k = 1$; $S_2 = \sum_{k=1}^2 k = 1 + 2 = 3$; $S_3 = \sum_{k=1}^3 k = 1 + 2 + 3 = 6$; $S_4 = \sum_{k=1}^4 k = 1 + 2 + 3 + 4 = 10$.

10.1.11 One possible answer is $\sum_{k=1}^{\infty} 10$.

10.1.12 $S_1 = \sum_{k=1}^1 \frac{1}{k} = \frac{1}{1} = 1$; $S_2 = \sum_{k=1}^2 \frac{1}{k} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$; $S_3 = \sum_{k=1}^3 \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$; $S_4 = \sum_{k=1}^4 \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$.

10.1.13 $a_1 = \frac{1}{10}$; $a_2 = \frac{1}{100}$; $a_3 = \frac{1}{1000}$; $a_4 = \frac{1}{10000}$.

10.1.14 $a_1 = 3(1) + 1 = 4$. $a_2 = 3(2) + 1 = 7$, $a_3 = 3(3) + 1 = 10$, $a_4 = 3(4) + 1 = 13$.

$$10.1.15 \quad a_1 = -\frac{1}{2}, a_2 = \frac{1}{2^2} = \frac{1}{4}, a_3 = -\frac{2}{2^3} = -\frac{1}{8}, a_4 = \frac{1}{2^4} = \frac{1}{16}.$$

$$10.1.16 \quad a_1 = 2 - 1 = 1, a_2 = 2 + 1 = 3, a_3 = 2 - 1 = 1, a_4 = 2 + 1 = 3.$$

$$10.1.17 \quad a_1 = \frac{2^2}{2+1} = \frac{4}{3}, a_2 = \frac{2^3}{2^2+1} = \frac{8}{5}, a_3 = \frac{2^4}{2^3+1} = \frac{16}{9}, a_4 = \frac{2^5}{2^4+1} = \frac{32}{17}.$$

$$10.1.18 \quad a_1 = 1 + \frac{1}{1} = 2; a_2 = 2 + \frac{1}{2} = \frac{5}{2}; a_3 = 3 + \frac{1}{3} = \frac{10}{3}; a_4 = 4 + \frac{1}{4} = \frac{17}{4}.$$

$$10.1.19 \quad a_1 = 1 + \sin(\pi/2) = 2; a_2 = 1 + \sin(2\pi/2) = 1 + \sin(\pi) = 1; a_3 = 1 + \sin(3\pi/2) = 0; a_4 = 1 + \sin(4\pi/2) = 1 + \sin(2\pi) = 1.$$

$$10.1.20 \quad a_1 = 1; a_2 = 2(1) = 2; a_3 = 3(2)(1) = 6; a_4 = 4(3)(2)(1) = 24.$$

$$10.1.21 \quad a_1 = 2, a_2 = 2(2) = 4, a_3 = 2(4) = 8, a_4 = 2(8) = 16.$$

$$10.1.22 \quad a_1 = 32, a_2 = 32/2 = 16, a_3 = 16/2 = 8, a_4 = 8/2 = 4.$$

$$10.1.23 \quad a_1 = 10 \text{ (given)}; a_2 = 3 \cdot a_1 - 12 = 30 - 12 = 18; a_3 = 3 \cdot a_2 - 12 = 54 - 12 = 42; a_4 = 3 \cdot a_3 - 12 = 126 - 12 = 114.$$

$$10.1.24 \quad a_1 = 1 \text{ (given)}; a_2 = a_1^2 - 1 = 0; a_3 = a_2^2 - 1 = -1; a_4 = a_3^2 - 1 = 0.$$

$$10.1.25 \quad a_0 = 1 \text{ (given)}; a_2 = \frac{1}{1+1} = \frac{1}{2}; a_3 = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}; a_4 = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}.$$

$$10.1.26 \quad a_0 = 1 \text{ (given)}; a_1 = 1 \text{ (given)}; a_2 = a_1 + a_0 = 2; a_3 = a_2 + a_1 = 3; a_4 = a_3 + a_2 = 5.$$

10.1.27

a. $\frac{1}{32}, \frac{1}{64}.$

b. $a_1 = 1; a_{n+1} = \frac{a_n}{2}.$

c. $a_n = \frac{1}{2^{n-1}}.$

10.1.29

a. 32, 64.

b. $a_1 = 1; a_{n+1} = 2a_n.$

c. $a_n = 2^{n-1}.$

10.1.31

a. 243, 729.

b. $a_1 = 1; a_{n+1} = 3a_n.$

c. $a_n = 3^{n-1}.$

10.1.28

a. 14, 17.

b. $a_1 = 2; a_{n+1} = a_n + 3.$

c. $a_n = -1 + 3n.$

10.1.30

a. 2, 1.

b. $a_1 = 64; a_{n+1} = \frac{a_n}{2}.$

c. $a_n = \frac{64}{2^{n-1}}.$

10.1.32

a. 36, 49.

b. $a_1 = 1; a_{n+1} = (\sqrt{a_n} + 1)^2.$

c. $a_n = n^2.$

10.1.33

- a. $-5, 5$.
- b. $a_1 = -5, a_{n+1} = -a_n$.
- c. $a_n = (-1)^n 5$.

10.1.34

- a. $0, 1$
- b. $a_1 = 1, a_{n+1} = 1 - a_n$ for $n = 1, 2, 3, \dots$
- c. $a_n = \frac{1 - (-1)^n}{2}$, for $n = 1, 2, 3, \dots$

10.1.35 $a_1 = 9, a_2 = 99, a_3 = 999, a_4 = 9999$. This sequence diverges, because the terms get larger without bound.

10.1.36 $a_1 = 1$ (given); $a_2 = \frac{10}{1} = 10$; $a_3 = \frac{10}{10} = 1$; $a_4 = \frac{10}{1} = 10$. This sequence diverges.

10.1.37 $a_1 = \frac{1}{10}, a_2 = \frac{1}{100}, a_3 = \frac{1}{1000}, a_4 = \frac{1}{10,000}$. This sequence converges to zero.

10.1.38 $a_0 = 1$ (given); $a_1 = \frac{1}{10}$; $a_2 = \frac{1}{1/10} = 10$; $a_3 = \frac{1}{10}$; $a_4 = \frac{1}{1/10} = 10$. This sequence diverges.

10.1.39 $a_1 = 3 + \cos \pi = 2$. $a_2 = 3 + \cos 2\pi = 4$. $a_3 = 3 + \cos 3\pi = 2$. $a_4 = 3 + \cos 4\pi = 4$. This sequence diverges.

10.1.40 $a_1 = 0.9, a_2 = 0.99, a_3 = 0.999, a_4 = 0.9999$. This sequence converges to 1.

10.1.41 $a_1 = 1 + 1 = 2, a_2 = 1 + 1 = 2, a_3 = 2, a_4 = 2$. This constant sequence converges to 2.

10.1.42 $a_1 = 1 - \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{3}$. Similarly, $a_2 = a_3 = a_4 = \frac{2}{3}$. This constant sequences converges to $\frac{2}{3}$.

10.1.43 $a_0 = 100, a_1 = 0.5 \cdot 100 + 50 = 100, a_2 = 0.5 \cdot 100 + 50 = 100, a_3 = 0.5 \cdot 100 + 50 = 100, a_4 = 0.5 \cdot 100 + 50 = 100$. This constant sequence converges to 100.

10.1.44 $a_1 = 0 - 1 = -1$. $a_2 = -10 - 1 = -11, a_3 = -110 - 1 = -111, a_4 = -1110 - 1 = -1111$. This sequence diverges.

10.1.45

n	a_n
1	0.83333
2	0.96154
3	0.99206
4	0.99840
5	0.99968
6	0.99994
7	0.99999
8	0.99999
9	0.99999
10	0.99999

This sequence appears to converge to 1.

10.1.46

n	a_n
1	0.95885
2	0.98961
3	0.99740
4	0.9995
5	0.99984
6	0.99996
7	0.99999
8	0.99999
9	0.99999
10	1.0000

This sequence appears to converge to 1.

10.1.47

n	1	2	3	4	5	6	7	8	9	10
a_n	2	6	12	20	30	42	56	72	90	110

This sequence appears to diverge.

10.1.48

n	1	2	3	4	5	6	7	8	9	10
a_n	9.9	9.95	9.9667	9.975	9.98	9.9833	9.9857	9.9875	9.9889	9.99

This sequence appears to converge to 10.

10.1.49

- a. 2.5, 2.25, 2.125, 2.0625.
b. The limit appears to be 2.

10.1.50

- a. 1.33333, 1.125, 1.06667, 1.04167.
b. The limit appears to be 1.

10.1.51

n	1	2	3	4	5	6	7	8	9	10
a_n	3	3.5000	3.7500	3.8750	3.9375	3.9688	3.9844	3.9922	3.9961	3.9980

The limit appears to be 4.

10.1.52

n	0	1	2	3	4	5	6	7	8	9
a_n	1	-2.75	-3.6875	-3.9219	-3.9805	-3.9951	-3.9988	-3.9997	-3.9999	-4.00

The limit appears to be -4 .

10.1.53

n	0	1	2	3	4	5	6	7	8	9
a_n	1	5	21	85	341	1365	5461	21,845	87,381	349,525

This sequence appears to diverge.

10.1.54

n	0	1	2	3	4	5	6	7	8	9	10
a_n	32	16	8	4	2	1	0.5	0.25	0.125	0.0625	0.03125

The limit appears to be 0.

10.1.55

The limit appears to be 4.

n	a_n
1	8
2	4.41421356
3	4.05050149
4	4.00629289
5	4.00078630
6	4.00009828
7	4.00001229
8	4.00000154
9	4.00000019
10	4.00000002

10.1.56

n	a_n
1	10
2	9.4339811
3	9.1908568
4	9.0944292
5	9.0374462
6	9.0166274
7	9.0073869
8	9.0032825
9	9.0014588
10	9.0006483

The limit appears to be 9.

10.1.57

- a. 20, 10, 5, 2.5.
b. $h_n = 20(0.5)^n$.

10.1.58

- a. 10, 9, 8.1, 7.29.
b. $h_n = 10(0.9)^n$.

10.1.59

- a. 30, 7.5, 1.875, 0.46875.
b. $h_n = 30(0.25)^n$.

10.1.60

- a. 20, 15, 11.25, 8.4375
b. $h_n = 20(0.75)^n$.

10.1.61 $S_1 = 0.3$, $S_2 = 0.33$, $S_3 = 0.333$, $S_4 = 0.3333$. It appears that the infinite series has a value of $0.3333\ldots = \frac{1}{3}$.

10.1.62 $S_1 = 0.6$, $S_2 = 0.66$, $S_3 = 0.666$, $S_4 = 0.6666$. It appears that the infinite series has a value of $0.6666\ldots = \frac{2}{3}$.

10.1.63 $S_1 = 4$, $S_2 = 4.9$, $S_3 = 4.99$, $S_4 = 4.999$. The infinite series has a value of $4.999\ldots = 5$.

10.1.64 $S_1 = 10$, $S_2 = 10 + 100 = 110$, $S_3 = 10 + 100 + 1000 = 1110$, $S_4 = 10 + 100 + 1000 + 10000 = 11110$. The infinite series diverges.

10.1.65 $S_1 = \frac{6}{10} = 0.6$, $S_2 = \frac{66}{100} = 0.66$, $S_3 = \frac{666}{1000} = 0.666$, $S_4 = \frac{6666}{10000} = 0.6666$. The limit is $\frac{2}{3}$.

10.1.66 $S_1 = -1$, $S_2 = 0$, $S_3 = -1$, $S_4 = 0$. The limit does not exist.

10.1.67

- a. $S_1 = \frac{2}{3}$, $S_2 = \frac{4}{5}$, $S_3 = \frac{6}{7}$, $S_4 = \frac{8}{9}$.
b. It appears that $S_n = \frac{2n}{2n+1}$. $S_5 = \frac{10}{11}$, $S_6 = \frac{12}{13}$, $S_7 = \frac{14}{15}$, $S_8 = \frac{16}{17}$.
c. The series has a value of 1 (the partial sums converge to 1).

10.1.68

- a. $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$, $S_4 = \frac{15}{16}$.

b. $S_n = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$. $S_5 = \frac{31}{32}$, $S_6 = \frac{63}{64}$, $S_7 = \frac{127}{128}$, $S_8 = \frac{255}{256}$.

c. The partial sums converge to 1, so that is the value of the series.

10.1.69

a. $S_1 = 9$, $S_2 = 9.9$, $S_3 = 9.99$, $S_4 = 9.999$.

b. $S_n = 10 - (0.1)^{n-1}$. $S_5 = 9.9999$, $S_6 = 9.99999$, $S_7 = 9.999999$, $S_8 = 9.9999999$.

c. The partial sums converge to 10, which is the value of the series.

10.1.70

a. $S_1 = 2$, $S_2 = \frac{8}{3}$, $S_3 = \frac{26}{9}$, $S_4 = \frac{80}{27}$.

b. $S_n = \frac{3^n - 1}{3^{n-1}}$. $S_5 = \frac{242}{81} \approx 2.98765$, $S_6 = \frac{728}{243} \approx 2.99588$, $S_7 = \frac{2186}{729} \approx 2.99863$, $S_8 = \frac{6560}{2187} \approx 2.99954$.

c. The partial sums converge to 3, which is the value of the series.

10.1.71

a. True. For example, $S_2 = 1 + 2 = 3$, and $S_4 = a_1 + a_2 + a_3 + a_4 = 1 + 2 + 3 + 4 = 10$.

b. False. For example, $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots$ where $a_n = 1 - \frac{1}{2^n}$ converges to 1, but each term is greater than the previous one.

c. True. In order for the partial sums to converge, they must get closer and closer together. In order for this to happen, the difference between successive partial sums, which is just the value of a_n , must approach zero.

10.1.72

a. $p_0 = 250$, $p_1 = 250 \cdot 1.03 = 258$, $p_2 = 250 \cdot 1.03^2 = 265$, $p_3 = 250 \cdot 1.03^3 = 273$, $p_4 = 250 \cdot 1.03^4 = 281$.

b. The initial population is 250, so that $p_0 = 250$. Then $p_n = 250 \cdot (1.03)^n$, because the population increases by 3 percent each month.

c. $p_{n+1} = p_n \cdot 1.03$.

d. The population increases without bound.

10.1.73

a. $M_0 = 20$, $M_1 = 20 \cdot 0.5 = 10$, $M_2 = 20 \cdot 0.5^2 = 5$, $M_3 = 20 \cdot 0.5^3 = 2.5$, $M_4 = 20 \cdot 0.5^4 = 1.25$

b. $M_n = 20 \cdot 0.5^n$.

c. The initial mass is $M_0 = 20$. We are given that 50% of the mass is gone after each decade, so that $M_{n+1} = 0.5 \cdot M_n$, $n \geq 0$.

d. The amount of material goes to 0.

10.1.74

a. $c_0 = 100$, $c_1 = 103$, $c_2 = 106.09$, $c_3 = 109.27$, $c_4 = 112.55$.

b. $c_n = 100(1.03)^n$, $n \geq 0$.

c. We are given that $c_0 = 100$ (where year 0 is 1984); because it increases by 3% per year, $c_{n+1} = 1.03 \cdot c_n$.

- d. The sequence diverges.

10.1.75

- a. $d_0 = 200$, $d_1 = 200 \cdot .95 = 190$, $d_2 = 200 \cdot .95^2 = 180.5$, $d_3 = 200 \cdot .95^3 = 171.475$, $d_4 = 200 \cdot .95^4 = 162.90125$.

- b. $d_n = 200(0.95)^n$, $n \geq 0$.

- c. We are given $d_0 = 200$; because 5% of the drug is washed out every hour, that means that 95% of the preceding amount is left every hour, so that $d_{n+1} = 0.95 \cdot d_n$.

- d. The sequence converges to 0.

10.1.76 The height at the n^{th} bounce is given by the recurrence $h_n = r \cdot h_{n-1}$; an explicit form for this sequence is $h_n = h_0 \cdot r^n$. The distance traveled by the ball during the n^{th} bounce is thus $2h_n = 2h_0 \cdot r^n$, so that $S_n = \sum_{i=0}^n 2h_0 \cdot r^i$.

- a. Here $h_0 = 20$, $r = 0.5$, so $S_0 = 40$, $S_1 = 40 + 40 \cdot 0.5 = 60$, $S_2 = S_1 + 40 \cdot (0.5)^2 = 70$, $S_3 = S_2 + 40 \cdot (0.5)^3 = 75$, $S_4 = S_3 + 40 \cdot (0.5)^4 = 77.5$

b.

n	0	1	2	3	4	5
a_n	40	60	70	75	77.5	78.75
n	6	7	8	9	10	11
a_n	79.375	79.6875	79.8438	79.9219	79.9609	79.9805
n	12	13	14	15	16	17
a_n	79.9902	79.9951	79.9976	79.9988	79.9994	79.9997
n	18	19	20	21	22	23
a_n	79.9998	79.9999	79.9999	80.0000	80.0000	80.0000

The sequence converges to 80.

10.1.77 Using the work from the previous problem:

- a. Here $h_0 = 20$, $r = 0.75$, so $S_0 = 40$, $S_1 = 40 + 40 \cdot 0.75 = 70$, $S_2 = S_1 + 40 \cdot (0.75)^2 = 92.5$, $S_3 = S_2 + 40 \cdot (0.75)^3 = 109.375$, $S_4 = S_3 + 40 \cdot (0.75)^4 = 122.03125$

b.

n	0	1	2	3	4	5
a_n	40	70	92.5	109.375	122.0313	131.5234
n	6	7	8	9	10	11
a_n	138.6426	143.9819	147.9865	150.9898	153.2424	154.9318
n	12	13	14	15	16	17
a_n	156.1988	157.1491	157.8618	158.3964	158.7973	159.0980
n	18	19	20	21	22	23
a_n	159.3235	159.4926	159.6195	159.715	159.786	159.839

The sequence converges to 160.

10.1.78

- a. Using the recurrence $a_{n+1} = \frac{1}{2} \left(a_n + \frac{10}{a_n} \right)$, we build a table:

n	0	1	2	3	4	5
a_n	10	5.5	3.659090909	3.196005081	3.162455622	3.162277665

The true value is $\sqrt{10} \approx 3.162277660$, so the sequence converges with an error of less than 0.01 after only 4 iterations, and is within 0.0001 after only 5 iterations.

- b. The recurrence is now $a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right)$

n	0	1	2	3	4	5	6
a_n	2	1.5	1.416666667	1.414215686	1.414213562	1.414213562	1.414213562

The true value is $\sqrt{2} \approx 1.414213562$, so the sequence converges with an error of less than 0.01 after 2 iterations, and is within 0.0001 after only 3 iterations.

10.1.79

n	a_n
0	0.800
1	0.697
2	0.767
3	0.720
4	0.752
5	0.730
6	0.745
7	0.735
8	0.742
9	0.737
10	0.740
11	0.738
12	0.740
13	0.739
14	0.739

The limit appears to be 0.739.

10.1.80

n	a_n
0	5.000
1	0.561
2	0.054
3	0.050
4	0.050

The limit appears to be 9.

10.2 Sequences

10.2.1 There are many examples; one is $a_n = \frac{1}{n}$. This sequence is nonincreasing (in fact, it is decreasing) and has a limit of 0.

10.2.2 Again there are many examples; one is $a_n = \ln(n)$. It is increasing, and has no limit.

10.2.3 There are many examples; one is $a_n = \frac{1}{n}$. This sequence is nonincreasing (in fact, it is decreasing), is bounded above by 1 and below by 0, and has a limit of 0.

10.2.4 For example, $a_n = (-1)^n$. For all values of n we have $|a_n| = 1$, so it is bounded. All the odd terms are -1 and all the even terms are 1 , so the sequence does not have a limit.

10.2.5 $\{r^n\}$ converges for $-1 < r \leq 1$. It diverges for all other values of r (see Theorem 10.3).

10.2.6 Because $0.2 < 1$, this sequence converges to 0 . Because $0.2 > 0$, the convergence is monotone.

10.2.7 Because $1.00001 > 1$, the sequence diverges; because $1.00001 > 0$, the divergence is monotone.

10.2.8 Because $|-2.5| > 1$, the sequence diverges; because $-2.5 < 0$, the divergence is not monotone. The sequence diverges by oscillation.

10.2.9 Because $|-0.7| < 1$, the sequence converges to 0 ; because $-0.7 < 0$, it does not do so monotonically. The sequence converges by oscillation.

10.2.10 Because $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1$, we have $\lim_{n \rightarrow \infty} a_n = 1$ by the Squeeze Theorem.

10.2.11 $\{e^{n/100}\}$ grows faster than $\{n^{100}\}$ as $n \rightarrow \infty$.

10.2.12 $\lim_{n \rightarrow \infty} \frac{n^{100}}{n^n} = 0$, because $n^{100} \ll n^n$.

10.2.13 Divide numerator and denominator by n^4 to get $\lim_{n \rightarrow \infty} \frac{1/n}{1 + \frac{1}{n^4}} = 0$.

10.2.14 Divide numerator and denominator by n^{12} to get $\lim_{n \rightarrow \infty} \frac{1}{3 + \frac{4}{n^{12}}} = \frac{1}{3}$.

10.2.15 Divide numerator and denominator by n^3 to get $\lim_{n \rightarrow \infty} \frac{3 - n^{-3}}{2 + n^{-3}} = \frac{3}{2}$.

10.2.16 Divide numerator and denominator by n^3 to get $\lim_{n \rightarrow \infty} \frac{n^2 + 3/n^2}{10 + 1/n^2} = \infty$. The sequence diverges.

10.2.17 Because $\lim_{n \rightarrow \infty} \frac{10n}{10n + 4} = 1$, and because the inverse tangent function is continuous, the given sequence has limit $\tan^{-1} 1 = \pi/4$.

10.2.18 $\lim_{n \rightarrow \infty} \frac{n\pi}{2n + 2} = \frac{\pi}{2}$, using L'Hôpital's rule. Thus the sequence converges to $\cot(\pi/2) = 0$.

10.2.19 $\lim_{n \rightarrow \infty} (1 + \cos(1/n)) = 1 + \cos(0) = 2$.

10.2.20 $\ln(n^3 + 1) - \ln(3n^3 + 10n) = \ln\left(\frac{n^3 + 1}{3n^3 + 10n}\right) = \ln\left(\frac{1 + n^{-3}}{3 + 10n^{-2}}\right)$, so the limit is $\ln(1/3) = -\ln 3$.

10.2.21 $\ln(\sin(1/n)) + \ln n = \ln(n \sin(1/n)) = \ln\left(\frac{\sin(1/n)}{1/n}\right)$. As $n \rightarrow \infty$, $\sin(1/n)/(1/n) \rightarrow 1$, so the limit of the original sequence is $\ln 1 = 0$.

10.2.22 Divide numerator by k and denominator by $k = \sqrt{k^2}$ to get $\lim_{k \rightarrow \infty} \frac{1}{\sqrt{9 + (1/k^2)}} = \frac{1}{3}$.

10.2.23 Divide numerator by $n^2 = \sqrt{n^4}$ and the denominator by n^2 to obtain

$$\lim_{n \rightarrow \infty} \frac{\sqrt{4 + 3/n^3}}{8 + 1/n^2} = \frac{\sqrt{4 + 0}}{8 + 0} = \frac{2}{8} = \frac{1}{4}.$$

10.2.24 Divide numerator and denominator by e^n to get $\lim_{n \rightarrow \infty} \frac{2 + (1/e^n)}{1} = 2$.

10.2.25 The sequence is equivalent to $\left\{ \frac{2 \ln n}{\ln 3 + \ln n} \right\}$. Divide numerator and denominator by $\ln n$ to obtain

$$\lim_{n \rightarrow \infty} \frac{2}{(\ln 3 / \ln n) + 1} = \frac{2}{1} = 2.$$

10.2.26 Because $|-1.01| > 1$, the sequence diverges.

10.2.27 This is the sequence

$$\frac{2^{n+1}}{3^n} = 2 \cdot \left(\frac{2}{3} \right)^n;$$

because $0 < \frac{2}{3} < 1$, the sequence converges monotonically to zero.

10.2.28 $|-0.003| < 1$, so the sequence converges to zero.

10.2.29 Because $|0.5| < 1$ and $|0.75| < 1$, the sequence converges to $0 + 3 \cdot 0 = 0$.

10.2.30 $\lim_{n \rightarrow \infty} \frac{e^n + \pi^n}{e^n} = \lim_{n \rightarrow \infty} \left(1 + \left(\frac{\pi}{e} \right)^n \right) = 1 + \lim_{n \rightarrow \infty} \left(\frac{\pi}{e} \right)^n$ which doesn't exist, because $\left| \frac{\pi}{e} \right| > 1$.

10.2.31 Divide numerator and denominator by 3^n to get $\lim_{n \rightarrow \infty} \frac{3 + (1/3^{n-1})}{1} = 3$.

10.2.32 Divide numerator and denominator by 4^n to get $\lim_{n \rightarrow \infty} \frac{(3/4)^n}{(3/4)^n + 1} = \frac{0}{0 + 1} = 0$.

10.2.33 Because $\frac{(n+1)!}{n!} = n+1$, this sequence diverges.

10.2.34 $\frac{(2n)!n^2}{(2n+2)!} = \frac{(2n)!n^2}{(2n+2)(2n+1)(2n)!} = \frac{n^2}{4n^2 + 6n + 2}$. Now divide numerator and denominator by n^2 to obtain $\lim_{n \rightarrow \infty} \frac{1}{4 + 6/n + 2/n^2} = \frac{1}{4}$.

10.2.35 $\lim_{n \rightarrow \infty} \tan^{-1} n = \frac{\pi}{2}$.

10.2.36 Note that $e^{1/10} = \sqrt[10]{e} \approx 1.1$. Let $r = \frac{e^{1/10}}{2}$ and note that $0 < r < 1$. Thus $\lim_{n \rightarrow \infty} \frac{e^{n/10}}{2^n} = \lim_{n \rightarrow \infty} r^n = 0$.

10.2.37 Multiply by $\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$ to obtain

$$\lim_{n \rightarrow \infty} \left(\sqrt{n^2+1} - n \right) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^2+1} + n} = 0.$$

10.2.38 Let $y = n^{2/n}$. Then $\ln y = \frac{2 \ln n}{n}$. By L'Hôpital's rule we have $\lim_{x \rightarrow \infty} \frac{2 \ln x}{x} = \lim_{x \rightarrow \infty} \frac{2}{x} = 0$, so $\lim_{n \rightarrow \infty} n^{2/n} = e^0 = 1$.

10.2.39 Find the limit of the logarithm of the expression, which is $n \ln \left(1 + \frac{2}{n} \right)$. Using L'Hôpital's rule:

$$\lim_{n \rightarrow \infty} n \ln \left(1 + \frac{2}{n} \right) = \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{2}{n} \right)}{1/n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{1+(2/n)} \left(-\frac{2}{n^2} \right)}{-1/n^2} = \lim_{n \rightarrow \infty} \frac{2}{1 + (2/n)} = 2.$$

Thus the limit of the original expression is e^2 .

10.2.40 Using L'Hôpital's rule, we have $\lim_{n \rightarrow \infty} \frac{\ln n}{n^{1.1}} = \lim_{n \rightarrow \infty} \frac{1/n}{(1.1)n^{1.1}} = \lim_{n \rightarrow \infty} \frac{1}{(1.1)n^{1.1}} = 0$.

10.2.41 Let $y = \sqrt[n]{e^{3n+4}} = e^{\frac{3n+4}{n}}$. Then $\ln y = \frac{3n+4}{n}$. Then $\lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{3n+4}{n} = 3$.
So $\lim_{n \rightarrow \infty} \sqrt[n]{e^{3n+4}} = e^3$.

10.2.42 Take the logarithm of the expression and use L'Hôpital's rule:

$$\lim_{n \rightarrow \infty} n \ln \left(\frac{n}{n+5} \right) = \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{n}{n+5} \right)}{1/n} = \lim_{n \rightarrow \infty} \frac{\frac{n+5}{n} \cdot \frac{-5}{(n+5)^2}}{-1/n^2} = \lim_{n \rightarrow \infty} \frac{-5n}{n+5} = -5.$$

Thus the original limit is e^{-5} .

10.2.43 Take the logarithm of the expression and use L'Hôpital's rule:

$$\lim_{n \rightarrow \infty} \frac{n}{2} \ln \left(1 + \frac{1}{2n} \right) = \lim_{n \rightarrow \infty} \frac{\ln(1 + (1/2n))}{2/n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{1+(1/2n)} \cdot \frac{-1}{2n^2}}{-2/n^2} = \lim_{n \rightarrow \infty} \frac{1}{4(1 + (1/2n))} = \frac{1}{4}.$$

Thus the original limit is $e^{1/4}$.

10.2.44 Find the limit of the logarithm of the expression, which is $3n \ln \left(1 + \frac{4}{n} \right)$. Using L'Hôpital's rule:

$$\lim_{n \rightarrow \infty} 3n \ln \left(1 + \frac{4}{n} \right) = \lim_{n \rightarrow \infty} \frac{3 \ln \left(1 + \frac{4}{n} \right)}{1/n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{1+(4/n)} \left(-\frac{4}{n^2} \right)}{-1/n^2} = \lim_{n \rightarrow \infty} \frac{12}{1 + (4/n)} = 12.$$

Thus the limit of the original expression is e^{12} .

10.2.45 Using L'Hôpital's rule: $\lim_{n \rightarrow \infty} \frac{n}{e^n + 3n} = \lim_{n \rightarrow \infty} \frac{1}{e^n + 3} = 0$.

10.2.46 $\ln \frac{1}{n} = -\ln n$, so this is $-\lim_{n \rightarrow \infty} \frac{\ln n}{n}$. By L'Hôpital's rule, we have $-\lim_{n \rightarrow \infty} \frac{\ln n}{n} = -\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

10.2.47 Taking logs, we have $\lim_{n \rightarrow \infty} \frac{1}{n} \ln(1/n) = \lim_{n \rightarrow \infty} -\frac{\ln n}{n} = \lim_{n \rightarrow \infty} -\frac{1}{n} = 0$ by L'Hôpital's rule. Thus the original sequence has limit $e^0 = 1$.

10.2.48 Find the limit of the logarithm of the expression, which is $n \ln \left(1 - \frac{4}{n} \right)$, using L'Hôpital's rule:

$\lim_{n \rightarrow \infty} n \ln \left(1 - \frac{4}{n} \right) = \lim_{n \rightarrow \infty} \frac{\ln \left(1 - \frac{4}{n} \right)}{1/n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{1-(4/n)} \left(\frac{4}{n^2} \right)}{-1/n^2} = \lim_{n \rightarrow \infty} -\frac{4}{1 - (4/n)} = -4$. Thus the limit of the original expression is e^{-4} .

10.2.49 Except for a finite number of terms, this sequence is just $a_n = ne^{-n}$, so it has the same limit as this sequence. Note that $\lim_{n \rightarrow \infty} \frac{n}{e^n} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0$, by L'Hôpital's rule.

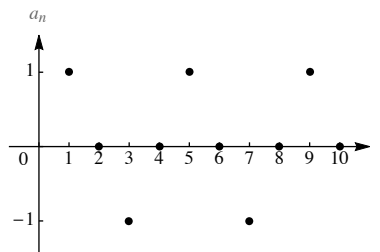
10.2.50 Using L'Hôpital's rule:

$$\lim_{n \rightarrow \infty} n(1 - \cos(1/n)) = \lim_{n \rightarrow \infty} \frac{1 - \cos(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{-\sin(1/n)(-1/n^2)}{-1/n^2} = -\sin(0) = 0.$$

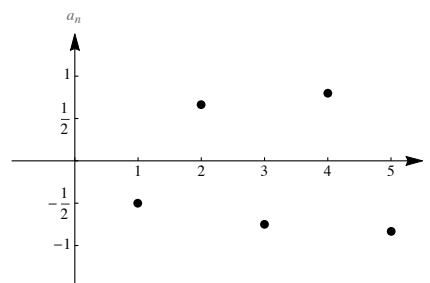
10.2.51 $\lim_{n \rightarrow \infty} n \sin(6/n) = \lim_{n \rightarrow \infty} \frac{\sin(6/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{\frac{-6 \cos(6/n)}{n^2}}{(-1/n^2)} = \lim_{n \rightarrow \infty} 6 \cos(6/n) = 6 \cdot \cos 0 = 6$.

10.2.52 Dividing the numerator and denominator by n^8 gives $a_n = \frac{1 + (1/n)}{(1/n) + \ln n}$. Because $1 + (1/n) \rightarrow 1$ as $n \rightarrow \infty$ and $(1/n) + \ln n \rightarrow \infty$ as $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} a_n = 0$.

10.2.53 When n is an integer, $\sin\left(\frac{n\pi}{2}\right)$ oscillates between the values ± 1 and 0 , so this sequence does not converge.



10.2.54 The even terms form a sequence $a_{2n} = \frac{2n}{2n+1}$, which converges to 1 (e.g. by L'Hôpital's rule); the odd terms form the sequence $a_{2n+1} = -\frac{2n+1}{2n+2}$, which converges to -1 . Thus the sequence as a whole does not converge.



10.2.55 Because

$$-\frac{1}{2^n} \leq \frac{(-1)^n}{2^n} \leq \frac{1}{2^n},$$

and because $\lim_{n \rightarrow \infty} -\frac{1}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$, the given sequence has limit zero by the Squeeze Theorem.

10.2.56 Note that

$$-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}.$$

Because $\lim_{n \rightarrow \infty} -\frac{1}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$, The given sequence has limit zero by the Squeeze Theorem.

10.2.57 Ignoring the factor of $(-1)^n$ for the moment, we see, taking logs, that $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$, so that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = e^0 = 1$. Taking the sign into account, the odd terms converge to -1 while the even terms converge to 1 . Thus the sequence does not converge.

10.2.58 Because $-1 \leq \cos(n\pi/2) \leq 1$ for all n , we have $-\frac{1}{\sqrt{n}} \leq \frac{\cos(n\pi/2)}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$ and because both $\pm \frac{1}{\sqrt{n}} \rightarrow 0$ as $n \rightarrow \infty$, the given sequence converges to 0 as well by the Squeeze Theorem.

10.2.59 This sequence diverges. To see this, call the given sequence a_n , and assume it converges to limit L . Then because the sequence $b_n = \frac{n}{n+1}$ converges to 1, the sequence $c_n = \frac{a_n}{b_n}$ would converge to L as well. But $c_n = \sin^3 \frac{\pi n}{2}$ doesn't converge (because it is $1, -1, 1, -1, \dots$), so the given sequence doesn't converge either.

10.2.60 Because $-\frac{n^2}{2n^3+n} \leq \frac{(-1)^{n+1}n^2}{2n^3+n} \leq \frac{n^2}{2n^3+n}$, and because both $-\frac{n^2}{2n^3+n}$ and $\frac{n^2}{2n^3+n}$ have limit 0 as $n \rightarrow \infty$, the limit of the given sequence is also 0 by the Squeeze Theorem. Note that $\lim_{n \rightarrow \infty} \frac{n^2}{2n^3+n} = \lim_{n \rightarrow \infty} \frac{1/n}{2+1/n^2} = \frac{0}{2} = 0$.

10.2.61 This is the sequence $\frac{\cos n}{e^n}$; the numerator is bounded in absolute value by 1 and the denominator increases without bound, so the limit is zero.

10.2.62 By L'Hôpital's rule we have: $\lim_{n \rightarrow \infty} \frac{e^{-n}}{2 \sin(e^{-n})} = \lim_{n \rightarrow \infty} \frac{-e^{-n}}{2 \cos(e^{-n})(-e^{-n})} = \frac{1}{2 \cos 0} = \frac{1}{2}$.

10.2.63 Because $\lim_{n \rightarrow \infty} \tan^{-1} n = \frac{\pi}{2}$, $\lim_{n \rightarrow \infty} \frac{\tan^{-1} n}{n} = 0$.

10.2.64 $\lim_{n \rightarrow \infty} \frac{(2n^3+n) \tan^{-1} n}{n^3+4} = \lim_{n \rightarrow \infty} \frac{2n^3+n}{n^3+4} \cdot \lim_{n \rightarrow \infty} \tan^{-1} n = 2 \cdot \frac{\pi}{2} = \pi$.

10.2.65 Because $-1 \leq \cos n \leq 1$, we have $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$. Because both $-\frac{1}{n}$ and $\frac{1}{n}$ have limit 0 as $n \rightarrow \infty$, the given sequence does as well.

10.2.66 Because $-1 \leq \sin 6n \leq 1$, we have $-\frac{1}{5n} \leq \frac{\sin 6n}{5n} \leq \frac{1}{5n}$. Because both $-\frac{1}{5n}$ and $\frac{1}{5n}$ have limit 0 as $n \rightarrow \infty$, the given sequence does as well.

10.2.67 Because $-1 \leq \sin n \leq 1$ for all n , the given sequence satisfies $-\frac{1}{2^n} \leq \frac{\sin n}{2^n} \leq \frac{1}{2^n}$, and because both $\pm \frac{1}{2^n} \rightarrow 0$ as $n \rightarrow \infty$, the given sequence converges to zero as well by the Squeeze Theorem.

10.2.68 If $f(t) = \int_1^t x^{-2} dx$, then $\lim_{t \rightarrow \infty} f(t) = \lim_{n \rightarrow \infty} a_n$. But

$$\lim_{t \rightarrow \infty} f(t) = \int_1^\infty x^{-2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) = 1.$$

10.2.69 Evaluate the limit of each term separately: $\lim_{n \rightarrow \infty} \frac{75^{n-1}}{99^n} = \frac{1}{99} \lim_{n \rightarrow \infty} \left(\frac{75}{99} \right)^{n-1} = 0$, while $-\frac{5^n}{8^n} \leq \frac{5^n \sin n}{8^n} \leq \frac{5^n}{8^n}$, so by the Squeeze Theorem, this second term converges to 0 as well. Thus the sum of the terms converges to zero.

10.2.70 Because $\lim_{n \rightarrow \infty} 0.99^n = 0$, and because cosine is continuous, the first term converges to $\cos 0 = 1$. The limit of the second term is $\lim_{n \rightarrow \infty} \frac{7^n + 9^n}{63^n} = \lim_{n \rightarrow \infty} \left(\frac{7}{63} \right)^n + \lim_{n \rightarrow \infty} \left(\frac{9}{63} \right)^n = 0$. Thus the sum converges to 1.

10.2.71

- After the n^{th} dose is given, the amount of drug in the bloodstream is $d_n = 0.5 \cdot d_{n-1} + 80$, because the half-life is one day. The initial condition is $d_1 = 80$.
- The limit of this sequence is 160 mg.
- Let $L = \lim_{n \rightarrow \infty} d_n$. Then from the recurrence relation, we have $d_n = 0.5 \cdot d_{n-1} + 80$, and thus $\lim_{n \rightarrow \infty} d_n = 0.5 \cdot \lim_{n \rightarrow \infty} d_{n-1} + 80$, so $L = 0.5 \cdot L + 80$, and therefore $L = 160$.

10.2.72

a.

$$\begin{aligned}
B_0 &= \$20,000 \\
B_1 &= 1.005 \cdot B_0 - \$200 = \$19,900 \\
B_2 &= 1.005 \cdot B_1 - \$200 = \$19,799.50 \\
B_3 &= 1.005 \cdot B_2 - \$200 = \$19,698.50 \\
B_4 &= 1.005 \cdot B_3 - \$200 = \$19,596.99 \\
B_5 &= 1.005 \cdot B_4 - \$200 = \$19,494.97
\end{aligned}$$

b. $B_n = 1.005 \cdot B_{n-1} - \200

c. Using a calculator or computer program, B_n becomes negative after the 139th payment, so 139 months or almost 11 years.**10.2.73**

a.

$$\begin{aligned}
B_0 &= 0 \\
B_1 &= 1.0075 \cdot B_0 + \$100 = \$100 \\
B_2 &= 1.0075 \cdot B_1 + \$100 = \$200.75 \\
B_3 &= 1.0075 \cdot B_2 + \$100 = \$302.26 \\
B_4 &= 1.0075 \cdot B_3 + \$100 = \$404.52 \\
B_5 &= 1.0075 \cdot B_4 + \$100 = \$507.56
\end{aligned}$$

b. $B_n = 1.0075 \cdot B_{n-1} + \100 .

c. Using a calculator or computer program, $B_n > \$5,000$ during the 43rd month.**10.2.74**

a. Let D_n be the *total number* of liters of alcohol in the mixture after the n^{th} replacement. At the next step, 2 liters of the 100 liters is removed, thus leaving $0.98 \cdot D_n$ liters of alcohol, and then $0.1 \cdot 2 = 0.2$ liters of alcohol are added. Thus $D_n = 0.98 \cdot D_{n-1} + 0.2$. Now, $C_n = D_n/100$, so we obtain a recurrence relation for C_n by dividing this equation by 100: $C_n = 0.98 \cdot C_{n-1} + 0.002$.

$$\begin{aligned}
C_0 &= 0.4 \\
C_1 &= 0.98 \cdot 0.4 + 0.002 = 0.394 \\
C_2 &= 0.98 \cdot C_1 + 0.002 = 0.38812 \\
C_3 &= 0.98 \cdot C_2 + 0.002 = 0.38236 \\
C_4 &= 0.98 \cdot C_3 + 0.002 = 0.37671 \\
C_5 &= 0.98 \cdot C_4 + 0.002 = 0.37118
\end{aligned}$$

The rounding is done to five decimal places.

b. Using a calculator or a computer program, $C_n < 0.15$ after the 89th replacement.c. If the limit of C_n is L , then taking the limit of both sides of the recurrence equation yields $L = 0.98L + 0.002$, so $.02L = .002$, and $L = .1 = 10\%$.**10.2.75** Because $n! \ll n^n$ by Theorem 10.6, we have $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$.

10.2.76 $\{3^n\} \ll \{n!\}$ because $\{b^n\} \ll \{n!\}$ in Theorem 10.6. Thus, $\lim_{n \rightarrow \infty} \frac{3^n}{n!} = 0$.

10.2.77 Theorem 10.6 indicates that $\ln^q n \ll n^p$, so $\ln^{20} n \ll n^{10}$, so $\lim_{n \rightarrow \infty} \frac{n^{10}}{\ln^{20} n} = \infty$.

10.2.78 Theorem 10.6 indicates that $\ln^q n \ll n^p$, so $\ln^{1000} n \ll n^{10}$, so $\lim_{n \rightarrow \infty} \frac{n^{10}}{\ln^{1000} n} = \infty$.

10.2.79 By Theorem 10.6, $n^p \ll b^n$, so $n^{1000} \ll 2^n$, and thus $\lim_{n \rightarrow \infty} \frac{n^{1000}}{2^n} = 0$.

10.2.80 Dividing the numerator and denominator by $n!$ gives $a_n = \frac{(4^n/n!) + 5}{1 + (2^n/n!)}$. By Theorem 10.6, we have $4^n \ll n!$ and $2^n \ll n!$. Thus, $\lim_{n \rightarrow \infty} a_n = \frac{0 + 5}{1 + 0} = 5$.

10.2.81 Dividing the numerator and denominator by 6^n gives $a_n = \frac{1 + (1/2)^n}{1 + (n^{100}/6^n)}$. By Theorem 10.6, $n^{100} \ll 6^n$. Thus $\lim_{n \rightarrow \infty} a_n = \frac{1 + 0}{1 + 0} = 1$.

10.2.82 We can write $a_n = \frac{(7/5)^n}{n^7}$. Theorem 10.6 indicates that $n^7 \ll b^n$ for $b > 1$, so $\lim_{n \rightarrow \infty} a_n = \infty$.

10.2.83

- True. See Theorem 10.2 part 4.
- False. For example, if $a_n = 1/n$ and $b_n = e^n$, then $\lim_{n \rightarrow \infty} a_n b_n = \infty$.
- True. The definition of the limit of a sequence involves only the behavior of the n^{th} term of a sequence as n gets large (see the Definition of Limit of a Sequence). Thus suppose a_n, b_n differ in only finitely many terms, and that M is large enough so that $a_n = b_n$ for $n > M$. Suppose a_n has limit L . Then for $\varepsilon > 0$, if N is such that $|a_n - L| < \varepsilon$ for $n > N$, first increase N if required so that $N > M$ as well. Then we also have $|b_n - L| < \varepsilon$ for $n > N$. Thus a_n and b_n have the same limit. A similar argument applies if a_n has no limit.
- True. Note that a_n converges to zero. Intuitively, the nonzero terms of b_n are those of a_n , which converge to zero. More formally, given ϵ , choose N_1 such that for $n > N_1$, $a_n < \epsilon$. Let $N = 2N_1 + 1$. Then for $n > N$, consider b_n . If n is even, then $b_n = 0$ so certainly $b_n < \epsilon$. If n is odd, then $b_n = a_{(n-1)/2}$, and $(n-1)/2 > ((2N_1 + 1) - 1)/2 = N_1$ so that $a_{(n-1)/2} < \epsilon$. Thus b_n converges to zero as well.
- False. If $\{a_n\}$ happens to converge to zero, the statement is true. But consider for example $a_n = 2 + \frac{1}{n}$. Then $\lim_{n \rightarrow \infty} a_n = 2$, but $(-1)^n a_n$ does not converge (it oscillates between positive and negative values increasingly close to ± 2).
- True. Suppose $\{0.000001a_n\}$ converged to L , and let $\epsilon > 0$ be given. Choose N such that for $n > N$, $|0.000001a_n - L| < \epsilon \cdot 0.000001$. Dividing through by 0.000001, we get that for $n > N$, $|a_n - 1000000L| < \epsilon$, so that a_n converges as well (to $1000000L$).

10.2.84

- The first three terms of the sequence are 1, 2.5, and 3.25. Because we are given that the sequence is monotonic, the sequence is nondecreasing.
- Assuming the sequence converges, let its limit be L . Then $\lim_{n \rightarrow \infty} a_{n+1} = \frac{1}{2} \lim_{n \rightarrow \infty} a_n + 2$, so $L = \frac{1}{2}L + 2$, and thus $\frac{1}{2}L = 2$, so $L = 4$.

10.2.85

- a. The first three terms of the sequence are 0.3, 0.42, and 0.4872. Because we are given that the sequence is monotonic, the sequence is nondecreasing.
- b. Let its limit be L , then $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} (2a_n(1 - a_n)) = 2(\lim_{n \rightarrow \infty} a_n)(1 - \lim_{n \rightarrow \infty} a_n)$, so $L = 2L(1 - L) = 2L - 2L^2$, and thus $2L^2 - L = 0$, so $L = 0, \frac{1}{2}$. Thus the limit appears to be either 0 or $1/2$; with the given initial condition, doing a few iterations by hand (as above) confirms that the sequence converges to $1/2$.

10.2.86

- a. The first 3 terms of the sequence are 2, 1.5, and 1.416667. Because we are given that the sequence is monotonic, it is nonincreasing.
- b. Let the limit be L . Then $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(a_n + \frac{2}{a_n} \right) = \frac{1}{2} \lim_{n \rightarrow \infty} a_n + \frac{1}{\lim_{n \rightarrow \infty} a_n}$, so $L = \frac{1}{2}L + \frac{1}{L}$, and therefore $L^2 = \frac{1}{2}L^2 + 1$. So $L^2 = 2$, and thus $L = \sqrt{2}$.

10.2.87

- a. The first 3 terms of the sequence are 3, 2.236068, and 2.058171. Because we are given that the sequence is monotonic, it is nonincreasing.
- b. Let its limit be L . Then $\lim_{n \rightarrow \infty} a_{n+1} = \sqrt{2 + \lim_{n \rightarrow \infty} a_n}$, so $L = \sqrt{2 + L}$. Thus we have $L^2 = 2 + L$, so $L^2 - L - 2 = 0$, and thus $L = -1, 2$. A square root can never be negative, so this sequence must converge to 2.

10.2.88

- a. Rounded to the nearest fish, the populations are

$$\begin{aligned} F_0 &= 4000 \\ F_1 &= 1.015F_0 - 80 = 3980 \\ F_2 &= 1.015F_1 - 80 \approx 3960 \\ F_3 &= 1.015F_2 - 80 \approx 3939 \\ F_4 &= 1.015F_3 - 80 \approx 3918 \\ F_5 &= 1.015F_4 - 80 \approx 3897 \end{aligned}$$

- b. $F_n = 1.015F_{n-1} - 80$
- c. The population decreases and eventually reaches zero.
- d. With an initial population of 5500 fish, the population increases without bound.
- e. If the initial population is less than 5333 fish, the population will decline to zero. This is essentially because for a population of less than 5333, the natural increase of 1.5% does not make up for the loss of 80 fish.

10.2.89

- a. $d_{n+1} = 0.4d_n + 75$; $d_1 = 75$.

- b. By examining the first few terms of the sequence, it appears to be increasing and bounded above by 125. To confirm this, observe that $d_2 = 105$ and so $d_2 > d_1$. Assume that $d_{k+1} > d_k$, for some positive integer k . Then $d_{k+2} = 0.4d_{k+1} + 75 > 0.4d_k + 75 = d_{k+1}$. Therefore $d_{k+2} > d_{k+1}$. So by mathematical induction, it follows that $\{d_n\}$ is an increasing sequence. Because $\{d_n\}$ is an increasing sequence and $d_1 = 75$, the sequence is bounded below by 75. To show that the sequence is indeed bounded above by 125, first observe that $d_1 = 75 < 125$. Now assume that $d_k < 125$, for some positive integer k for $k \geq 1$. Then $d_{k+1} = 0.4d_k + 75 \leq 0.4(125) + 75 = 125$, and therefore $d_{k+1} < 125$. So by mathematical induction, the sequence is bounded above by 125. Because $\{d_n\}$ is bounded below and above, it is bounded. So by Theorem 10.5, it converges.
- c. Assume that the sequence $\{d_n\}$ converges to L . Taking limits of the recurrence relation, we have $\lim_{n \rightarrow \infty} d_n = 0.4 \lim_{n \rightarrow \infty} d_n + 75 = 0.4L + 75$. So the limit is $L = 75/0.6 = 125$. In the long run there will be approximately 125 mg of medication in the blood

10.2.90

- a. $x_0 = 7$, $x_1 = 6$, $x_2 = 6.5 = \frac{13}{2}$, $x_3 = 6.25$, $x_4 = 6.375 = \frac{51}{8}$, $x_5 = 6.3125 = \frac{101}{16}$, $x_6 = 6.34375 = \frac{203}{32}$.
- b. For the formula given in the problem, we have $x_0 = \frac{19}{3} + \frac{2}{3} \left(-\frac{1}{2}\right)^0 = 7$, $x_1 = \frac{19}{3} + \frac{2}{3} \cdot \frac{-1}{2} = \frac{19}{3} - \frac{1}{3} = 6$, so that the formula holds for $n = 0, 1$. Now assume the formula holds for all integers $\leq k$; then

$$\begin{aligned} x_{k+1} &= \frac{1}{2}(x_k + x_{k-1}) = \frac{1}{2} \left(\frac{19}{3} + \frac{2}{3} \left(-\frac{1}{2}\right)^k + \frac{19}{3} + \frac{2}{3} \left(-\frac{1}{2}\right)^{k-1} \right) \\ &= \frac{1}{2} \left(\frac{38}{3} + \frac{2}{3} \left(-\frac{1}{2}\right)^{k-1} \left(-\frac{1}{2} + 1\right) \right) \\ &= \frac{1}{2} \left(\frac{38}{3} + 4 \cdot \frac{2}{3} \left(-\frac{1}{2}\right)^{k+1} \cdot \frac{1}{2} \right) \\ &= \frac{1}{2} \left(\frac{38}{3} + 2 \cdot \frac{2}{3} \left(-\frac{1}{2}\right)^{k+1} \right) \\ &= \frac{19}{3} + \frac{2}{3} \left(-\frac{1}{2}\right)^{k+1}. \end{aligned}$$

- c. As $n \rightarrow \infty$, $(-1/2)^n \rightarrow 0$, so that the limit is $19/3$, or $6 \frac{1}{3}$.

10.2.91 The approximate first few values of this sequence are:

n	0	1	2	3	4	5	6
c_n	0.7071	0.6325	0.6136	0.6088	0.6076	0.6074	0.6073

The value of the constant appears to be around 0.607.

10.2.92 Note that $1 - \frac{1}{i} = \frac{i-1}{i}$, so that the product is $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots$, so that $a_n = \frac{1}{n}$ for $n \geq 2$. The sequence $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$ has limit zero.

10.2.93

- a. We begin by showing that $a_{n+1} > a_n$, for $n \geq 1$. This inequality holds for $n = 0$ because $a_1 = 7 > a_0$. Assume that $a_{n+1} > a_n$ for a nonnegative integer $n = k$. Then $a_{k+2} = (1/3)a_{k+1} + 6 > (1/3)a_k + 6 = a_{k+1}$. Therefore $a_{n+1} > a_n$ holds for $n = k + 1$. So by mathematical induction, we have established that $a_{n+1} > a_n$, for $n \geq 1$, which implies that $\{a_n\}$ is a monotonic sequence. Next, we establish that $a_n < 9$, for $n \geq 1$. This result holds for $n = 0$ because $a_0 = 3 < 9$. Now assume that $a_k < 9$ for a nonnegative integer k . Then $a_{k+1} = (1/3)a_k + 6 < (1/3)(9) + 6 = 9$. So by mathematical induction, we've shown that $a_n < 9$, for $n \geq 1$. Because $\{a_n\}$ is an increasing sequence and $a_0 = 3$, it follows that $3 \leq a_n < 9$. Therefore $\{a_n\}$ is bounded. Because $\{a_n\}$ is monotonic and bounded, it converges.
- b. Because $\{a_n\}_{n=0}^{\infty}$ is bounded above and is increasing, it converges. Assume that L is the limit of the sequence. Taking the limit of both sides of the recurrence relation gives $L = \frac{1}{3}L + 6$, or $3L = L + 18$, so $L = 9$.

10.2.94

- a. To show the sequence is decreasing, notice that $a_1 = \sqrt{26} < \sqrt{36} = 6 = a_0$. So $a_1 < a_0$. Assume that $a_{k+1} < a_k$ for some nonnegative integer k . Then $a_{k+2} = \sqrt{a_{k+1} + 20} < \sqrt{a_k + 20} = a_{k+1}$. So $a_{k+2} < a_{k+1}$, and so by mathematical induction, it has been shown that $a_{n+1} < a_n$ for $n \geq 0$. Observe that $a_1 = \sqrt{26} > 5$. Assume that $a_k > 5$ for some nonnegative integer k . Then $a_{k+1} = \sqrt{a_k + 20} > \sqrt{5 + 20} = 5$ and so $a_{k+1} > 5$. So by mathematical induction, we've established that $a_n > 5$ for $n \geq 0$. Because the sequence is decreasing, it is bounded above by the first term in the sequence, which is 6. So the sequence is bounded.
- b. Because $\{a_n\}_{n=0}^{\infty}$ is bounded below and is decreasing, it converges. Assume that L is the limit of the sequence. Taking the limit of both sides of the recurrence relation gives $L = \sqrt{L + 20}$, so $L^2 - L - 20 = (L - 5)(L + 4) = 0$, so $L = 5$ or $L = -4$. Because the sequence consists of all positive terms, we must have $L = 5$.

10.2.95

- a. $a_0 = \sqrt{2} \approx 1.41421$, $a_1 = \sqrt{2 + \sqrt{2}} \approx 1.84776$, $a_2 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.96157$,
 $a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.99037$, and $a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.99759$.
- b. We have $a_1 = \sqrt{2 + \sqrt{2}}$, so it is the case that $a_1 > a_0$. Now assume that $a_k > a_{k-1}$ for some $k \geq 1$. Then $a_{k+1} = \sqrt{2 + a_k} > \sqrt{2 + a_{k-1}} = a_k$. So we have $a_{k+1} > a_k$. So by mathematical induction, it follows that a_n is increasing. Because the first term in the sequence is $\sqrt{2}$ and the sequence is increasing, the sequence is bounded below by $\sqrt{2}$. It appears that all of the terms in the sequence are less than 2 and we now use mathematical induction to prove this. We have $a_0 = \sqrt{2} < 2$. Suppose $a_k < 2$ for some $k \geq 1$. Then $a_{k+1} = \sqrt{2 + a_k} < \sqrt{2 + 2} = 2$. So it follows that $\sqrt{2} < a_n < 2$ for $n \geq 0$.
- c. Because $\{a_n\}_{n=0}^{\infty}$ is bounded and increasing, it converges. Assume that L is the limit of the sequence. Taking the limit of both sides of the recurrence relation gives $L = \sqrt{2 + L}$, so $L^2 = 2 + L$, or $L^2 - L - 2 = (L - 2)(L + 1) = 0$. So $L = 2$ or $L = -1$ and because terms of the sequence are all positive, the limit is 2.

10.2.96

- a. With $p = 0.5$ we have for $a_{n+1} = a_n^p$:

n	1	2	3	4	5	6	7
a_n	0.707	0.841	0.971	0.8	0.979	0.989	0.995

Experimenting with recurrence (1) one sees that for $0 < p \leq 1$ the sequence converges to 1, while for $p > 1$ the sequence diverges to ∞ .

- b. With $p = 1.2$ and $a_n = p^{a_{n-1}}$ we obtain

n	1	2	3	4	5	6	7	8	9	10
a_n	1.2	1.2446	1.2547	1.2570	1.2577	1.2577	1.2577	1.2577	1.2577	1.2577

With recurrence (2), in addition to converging for $p < 1$ it also converges for values of p less than approximately 1.444. Here is a table of approximate values for different values of p :

p	1.1	1.2	1.3	1.4	1.44	1.444	1.445
$\lim_{n \rightarrow \infty} a_n$	1.1118	1.25776	1.471	1.887	2.39385	2.587	Diverges

It appears that the upper limit of convergence is about 1.444.

10.2.97

- $f_0 = f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, f_6 = 13, f_7 = 21, f_8 = 34, f_9 = 55, f_{10} = 89$.
- The sequence is clearly not bounded.
- $\frac{f_{10}}{f_9} \approx 1.61818$
- We use induction. Note that

$$\frac{1}{\sqrt{5}} \left(\varphi + \frac{1}{\varphi} \right) = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} + \frac{2}{1 + \sqrt{5}} \right) = \frac{1}{\sqrt{5}} \left(\frac{1 + 2\sqrt{5} + 5 + 4}{2(1 + \sqrt{5})} \right) = 1 = f_1.$$

Also note that

$$\frac{1}{\sqrt{5}} \left(\varphi^2 - \frac{1}{\varphi^2} \right) = \frac{1}{\sqrt{5}} \left(\frac{3 + \sqrt{5}}{2} - \frac{2}{3 + \sqrt{5}} \right) = \frac{1}{\sqrt{5}} \left(\frac{9 + 6\sqrt{5} + 5 - 4}{2(3 + \sqrt{5})} \right) = 1 = f_2.$$

Now note that

$$\begin{aligned} f_{n-1} + f_{n-2} &= \frac{1}{\sqrt{5}} (\varphi^{n-1} - (-1)^{n-1} \varphi^{1-n} + \varphi^{n-2} - (-1)^{n-2} \varphi^{2-n}) \\ &= \frac{1}{\sqrt{5}} ((\varphi^{n-1} + \varphi^{n-2}) - (-1)^n (\varphi^{2-n} - \varphi^{1-n})). \end{aligned}$$

Now, note that $\varphi - 1 = \frac{1}{\varphi}$, so that

$$\varphi^{n-1} + \varphi^{n-2} = \varphi^{n-1} \left(1 + \frac{1}{\varphi} \right) = \varphi^{n-1} \cdot \varphi = \varphi^n$$

and

$$\varphi^{2-n} - \varphi^{1-n} = \varphi^{-n} (\varphi^2 - \varphi) = \varphi^{-n} (\varphi(\varphi - 1)) = \varphi^{-n}.$$

Making these substitutions, we get

$$f_n = f_{n-1} + f_{n-2} = \frac{1}{\sqrt{5}} (\varphi^n - (-1)^n \varphi^{-n})$$

10.2.98

a. We show that the arithmetic mean of any two positive numbers exceeds their geometric mean. Let $a, b > 0$; then $\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a - 2\sqrt{ab} + b) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 \geq 0$. Because in addition $a_0 > b_0$, we have $a_n > b_n$ for all n .

b. To see that $\{a_n\}$ is decreasing, note that

$$a_{n+1} = \frac{a_n + b_n}{2} < \frac{a_n + a_n}{2} = a_n.$$

Similarly,

$$b_{n+1} = \sqrt{a_n b_n} > \sqrt{b_n b_n} = b_n,$$

so that $\{b_n\}$ is increasing.

c. $\{a_n\}$ is monotone and nonincreasing by part (b), and bounded below by part (a) (it is bounded below by any of the b_n), so it converges by the monotone convergence theorem. Similarly, $\{b_n\}$ is monotone and nondecreasing by part (b) and bounded above by part (a), so it too converges.

d.

$$a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{1}{2}(a_n - 2\sqrt{a_n b_n} + b_n) < \frac{1}{2}(a_n - 2\sqrt{b_n^2} + b_n) = \frac{1}{2}(a_n - b_n).$$

Thus the difference between a_{n+1} and b_{n+1} is less than half the difference between a_n and b_n , so that difference goes to zero and the two limits are the same.

e. The AGM of 12 and 20 is approximately 15.745; Gauss' constant is $\frac{1}{\text{AGM}(1, \sqrt{2})} \approx 0.8346$.

10.2.99

a. Define a_n as given in the problem statement. Then we can *define* the value of the continued fraction to be $\lim_{n \rightarrow \infty} a_n$.

b. $a_0 = 1$, $a_1 = 1 + \frac{1}{a_0} = 2$, $a_2 = 1 + \frac{1}{a_1} = \frac{3}{2} = 1.5$, $a_3 = 1 + \frac{1}{a_2} = \frac{5}{3} \approx 1.667$, $a_4 = 1 + \frac{1}{a_3} = \frac{8}{5} = 1.6$, $a_5 = 1 + \frac{1}{a_4} = \frac{13}{8} = 1.625$.

c. From the list above, the values of the sequence alternately decrease and increase, so we would expect that the limit is somewhere between 1.6 and 1.625.

d. Assume that the limit is equal to L . Then from $a_{n+1} = 1 + \frac{1}{a_n}$, we have $\lim_{n \rightarrow \infty} a_{n+1} = 1 + \frac{1}{\lim_{n \rightarrow \infty} a_n}$, so $L = 1 + \frac{1}{L}$, and thus $L^2 - L - 1 = 0$. Therefore, $L = \frac{1 \pm \sqrt{5}}{2}$, and because L is clearly positive, it must be equal to $\frac{1 + \sqrt{5}}{2} \approx 1.618$.

e. Here $a_0 = a$ and $a_{n+1} = a + \frac{b}{a_n}$. Assuming that $\lim_{n \rightarrow \infty} a_n = L$ we have $L = a + \frac{b}{L}$, so $L^2 = aL + b$, and thus $L^2 - aL - b = 0$. Therefore, $L = \frac{a \pm \sqrt{a^2 + 4b}}{2}$, and because $L > 0$ we have $L = \frac{a + \sqrt{a^2 + 4b}}{2}$.

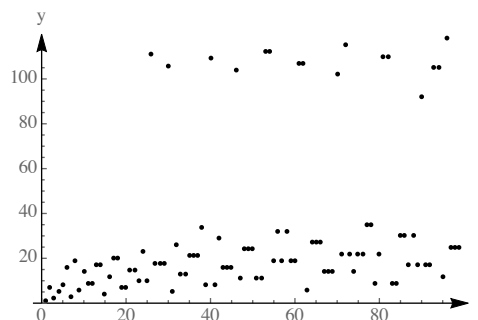
10.2.100

a.

2 : 1
 3 : 10, 5, 16, 8, 4, 2, 1
 4 : 2, 1
 5 : 16, 8, 4, 2, 1
 6 : 3, 10, 5, 16, 8, 4, 2, 1
 7 : 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1
 8 : 4, 2, 1
 9 : 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1
 10 : 5, 16, 8, 4, 2, 1

b. From the above, $H_2 = 1$, $H_3 = 7$, and $H_4 = 2$.

c. This plot is for $1 \leq n \leq 100$. Like hailstones, the numbers in the sequence a_n rise and fall but eventually crash to the earth. The conjecture appears to be true.



10.2.101 Let $\varepsilon > 0$ be given and let N be an integer with $N > \frac{1}{\varepsilon}$. Then if $n > N$, we have

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

10.2.102 Let $\varepsilon > 0$ be given. We wish to find N such that $|(1/n^2) - 0| < \varepsilon$ if $n > N$. This means that $\left| \frac{1}{n^2} - 0 \right| = \frac{1}{n^2} < \varepsilon$. So choose N such that $\frac{1}{N^2} < \varepsilon$, so that $N^2 > \frac{1}{\varepsilon}$, and then $N > \frac{1}{\sqrt{\varepsilon}}$. This shows that such an N always exists for each ε and thus that the limit is zero.

10.2.103 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left| \frac{3n^2}{4n^2 + 1} - \frac{3}{4} \right| = \left| \frac{-3}{4(4n^2 + 1)} \right| = \frac{3}{4(4n^2 + 1)} < \varepsilon$. But this means that $3 < 4\varepsilon(4n^2 + 1)$, or $16\varepsilon n^2 + (4\varepsilon - 3) > 0$. Solving the quadratic, we get $n > \frac{1}{4} \sqrt{\frac{3}{\varepsilon}} - 4$, provided $\varepsilon < 3/4$. So let $N = \frac{1}{4} \sqrt{\frac{3}{\varepsilon}}$ if $\varepsilon < 3/4$ and let $N = 1$ otherwise.

10.2.104 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $|b^{-n} - 0| = b^{-n} < \varepsilon$, so that $-n \ln b < \ln \varepsilon$. So choose N to be any integer greater than $-\frac{\ln \varepsilon}{\ln b}$.

10.2.105 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left| \frac{cn}{bn + 1} - \frac{c}{b} \right| = \left| \frac{-c}{b(bn + 1)} \right| = \frac{c}{b(bn + 1)} < \varepsilon$. But this means that $\varepsilon b^2 n + (b\varepsilon - c) > 0$, so that $N > \frac{c}{b^2 \varepsilon}$ will work.

10.2.106 Let $\varepsilon > 0$ be given. We wish to find N such that for $n > N$, $\left| \frac{n}{n^2+1} - 0 \right| = \frac{n}{n^2+1} < \varepsilon$. Thus we want $n < \varepsilon(n^2+1)$, or $\varepsilon n^2 - n + \varepsilon > 0$. Whenever n is larger than the larger of the two roots of this quadratic, the desired inequality will hold. The roots of the quadratic are $\frac{1 \pm \sqrt{1-4\varepsilon^2}}{2\varepsilon}$, so we choose N to be any integer greater than $\frac{1 + \sqrt{1-4\varepsilon^2}}{2\varepsilon}$.

10.2.107 First note that for $a = 1$ we already know that $\{n^n\}$ grows faster than $\{n!\}$. So if $a > 1$, then $n^{an} \geq n^n$, so that $\{n^{an}\}$ grows faster than $\{n!\}$ for $a > 1$ as well. To settle the case $a < 1$, recall Stirling's formula which states that for large values of n ,

$$n! \sim \sqrt{2\pi n} n^n e^{-n}.$$

Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n!}{n^{an}} &= \lim_{n \rightarrow \infty} \frac{\sqrt{2\pi n} n^n e^{-n}}{n^{an}} \\ &= \sqrt{2\pi} \lim_{n \rightarrow \infty} n^{\frac{1}{2} + (1-a)n} e^{-n} \\ &\geq \sqrt{2\pi} \lim_{n \rightarrow \infty} n^{(1-a)n} e^{-n} \\ &= \sqrt{2\pi} \lim_{n \rightarrow \infty} e^{(1-a)n \ln n} e^{-n} \\ &= \sqrt{2\pi} \lim_{n \rightarrow \infty} e^{((1-a) \ln n - 1)n}. \end{aligned}$$

If $a < 1$ then $(1-a) \ln n - 1 > 0$ for large values of n because $1-a > 0$, so that this limit is infinite. Hence $\{n!\}$ grows faster than $\{n^{an}\}$ exactly when $a < 1$.

10.2.108 $\{2n-3\}_{n=3}^\infty$.

10.2.109 $\{(n-2)^2 + 6(n-2) - 9\}_{n=3}^\infty = \{n^2 + 2n - 17\}_{n=3}^\infty$.

10.2.110 $\{a_n\} \ll \{b_n\}$ means that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$. But $\lim_{n \rightarrow \infty} \frac{ca_n}{db_n} = \frac{c}{d} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, so that $\{ca_n\} \ll \{db_n\}$.

10.2.111 For $b = 2$, $2^3 > 3!$ but $16 = 2^4 < 4! = 24$, so the crossover point is $n = 4$. For e , $e^5 \approx 148.41 > 5! = 120$ while $e^6 \approx 403.4 < 6! = 720$, so the crossover point is $n = 6$. For 10, $24! \approx 6.2 \times 10^{23} < 10^{24}$, while $25! \approx 1.55 \times 10^{25} > 10^{25}$, so the crossover point is $n = 25$.

10.3 Infinite Series

10.3.1 The ratio is the common ratio between successive terms in the sum.

10.3.2 A geometric sum is the sum of a finite number of terms which have a constant ratio; a geometric series is the sum of an infinite number of such terms.

10.3.3 No. For example, the geometric series with $a_n = 3 \cdot 2^n$ does not have a finite sum.

10.3.4 Yes, because there are only a finite number of terms.

10.3.5

a. $a = \frac{2}{3}; r = \frac{1}{5}$.

b. $a = \frac{1}{27}; r = -\frac{1}{3}$.

10.3.6 The series converges if and only if $|r| < 1$.

$$\mathbf{10.3.7} \quad S_n = \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) = \frac{1}{4} - \frac{1}{n+4}. \quad S_{36} = \frac{1}{4} - \frac{1}{40} = \frac{10-1}{40} = \frac{9}{40}.$$

10.3.8

$$\sum_{k=5}^{\infty} \frac{3}{4k^2 - 63} = \sum_{k=1}^{\infty} \frac{3}{4(k+4)^2 - 63} = \sum_{k=1}^{\infty} \frac{3}{4k^2 + 32k + 1}.$$

$$\mathbf{10.3.9} \quad S = 1 \cdot \frac{1 - 3^9}{1 - 3} = \frac{19682}{2} = 9841.$$

$$\mathbf{10.3.10} \quad S = 1 \cdot \frac{1 - (1/4)^{11}}{1 - (1/4)} = \frac{4^{11} - 1}{3 \cdot 4^{10}} = \frac{4194303}{3 \cdot 1048576} = \frac{1398101}{1048576} \approx 1.333.$$

$$\mathbf{10.3.11} \quad S = 1 \cdot \frac{1 - (4/25)^{21}}{1 - 4/25} = \frac{25^{21} - 4^{21}}{25^{21} - 4 \cdot 25^{20}} \approx 1.1905.$$

$$\mathbf{10.3.12} \quad S = 16 \cdot \frac{1 - 2^9}{1 - 2} = 511 \cdot 16 = 8176.$$

$$\mathbf{10.3.13} \quad S = 1 \cdot \frac{1 - (-3/4)^{10}}{1 + 3/4} = \frac{4^{10} - 3^{10}}{4^{10} + 3 \cdot 4^9} = \frac{141361}{262144} \approx 0.5392.$$

$$\mathbf{10.3.14} \quad S = 1 \cdot \frac{1 - \pi^7}{1 - \pi} = \frac{\pi^7 - 1}{\pi - 1} \approx 1409.84.$$

$$\mathbf{10.3.15} \quad \frac{1}{4} \sum_{k=0}^6 \frac{1}{3^k} = \frac{1}{4} \left(\frac{1 - (1/3)^7}{1 - (1/3)} \right) = \frac{1093}{2916}.$$

$$\mathbf{10.3.16} \quad A_{60} = 100 + 100(1.0015) + \cdots + 100(1.0015)^{59} = 100 \left(\frac{1 - 1.0015^{60}}{1 - 1.0015} \right) = 6273.37.$$

$$\mathbf{10.3.17} \quad A_{60} = 250 + 250(1.002) + \cdots + 250(1.002)^{59} = 250 \left(\frac{1 - 1.002^{60}}{1 - 1.002} \right) = 15,920.22.$$

10.3.18

a. The n th partial sum is $S_n = a \left(\frac{1 - r^n}{1 - r} \right) = \frac{1 - (2/5)^n}{1 - (2/5)} = \frac{5}{3} \left(1 - \left(\frac{2}{5} \right)^n \right)$. So the sum of the series is

$$S = \lim_{n \rightarrow \infty} S_n = \frac{5}{3} (1 - 0) = \frac{5}{3}.$$

$$\text{b. } S = \sum_{k=0}^{\infty} \left(\frac{2}{5} \right)^k = \frac{1}{1 - (2/5)} = \frac{5}{3}.$$

10.3.19

a. The n th partial sum is $S_n = a \left(\frac{1 - r^n}{1 - r} \right) = \frac{1 - (-2/7)^n}{1 - (-2/7)} = \frac{7}{9} \left(1 - \left(-\frac{2}{7} \right)^n \right)$. So the sum of the series

$$\text{is } S = \lim_{n \rightarrow \infty} S_n = \frac{7}{9} (1 - 0) = \frac{7}{9}.$$

$$\text{b. } S = \sum_{k=0}^{\infty} \left(-\frac{2}{7} \right)^k = \frac{1}{1 - (-2/7)} = \frac{7}{9}.$$

10.3.20

a. The series can be written as $2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$.

The n th partial sum is $S_n = a \left(\frac{1-r^{n+1}}{1-r}\right) = 2 \left(\frac{1-(2/3)^{n+1}}{1-(2/3)}\right) = 6 \left(1 - \left(\frac{2}{3}\right)^{n+1}\right)$. So the sum of the series is $S = \lim_{n \rightarrow \infty} S_n = 6(1-0) = 6$.

b. $S = 2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{2}{1-(2/3)} = 6$.

10.3.21 $\frac{1}{1-1/4} = \frac{4}{3}$.

10.3.22 $\frac{1}{1-3/5} = \frac{5}{2}$.

10.3.23 $\frac{1}{1+9/10} = \frac{10}{19}$.

10.3.24 $-\frac{2/3}{1+2/3} = -\frac{2}{5}$.

10.3.25 $\frac{1}{1-0.9} = 10$.

10.3.26 $\frac{1}{1-2/7} = \frac{7}{5}$.

10.3.27 Divergent, because $r > 1$.

10.3.28 $\frac{1}{1-1/\pi} = \frac{\pi}{\pi-1}$.

10.3.29 $\frac{e^{-2}}{1-e^{-2}} = \frac{1}{e^2-1}$.

10.3.30 $\frac{5/4}{1-1/2} = \frac{5}{2}$.

10.3.31 $\frac{2^{-3}}{1-2^{-3}} = \frac{1}{7}$.

10.3.32 $\frac{3 \cdot 4^3/7^3}{1-4/7} = \frac{64}{49}$.

10.3.33 $\frac{1/625}{1-1/5} = \frac{1}{500}$.

10.3.34 Note that this is the same as $\sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^k$.

Then $S = \frac{1}{1-3/4} = 4$.

10.3.35 $3 \cdot \frac{1}{1+1/\pi} = \frac{3\pi}{\pi+1}$.

10.3.36 $\sum_{k=1}^{\infty} \left(-\frac{1}{e}\right)^k = -\frac{1/e}{1+1/e} = -\frac{1}{e+1}$.

10.3.37 $\frac{1}{1-e/\pi} = \frac{\pi}{\pi-e}$.

10.3.38 $\frac{1/16}{1-3/4} = \frac{1}{4}$.

10.3.39 $\frac{0.15^2}{1.15} = \frac{9}{460} \approx 0.0196$.

10.3.40 $-\frac{3/8^3}{1+1/8^3} = -\frac{1}{171}$.

10.3.41 $\frac{4/12}{1-(1/2)} = \frac{4/12}{11/12} = \frac{4}{11}$.

10.3.42 $3 \left(\frac{1/e^2}{1-(1/e)}\right) = \frac{3}{e(e-1)}$.

10.3.43 $A_5 = 200 + 200(0.25) + \cdots + 200(0.25)^4 = 200 \left(\frac{1-0.25^5}{1-0.25}\right) = 266.406$.

$A_{10} = 200 + 200(0.25) + \cdots + 200(0.25)^9 = 200 \left(\frac{1-0.25^{10}}{1-0.25}\right) = 266.666$.

$A_{30} = 200 + 200(0.25) + \cdots + 200(0.25)^{29} = 200 \left(\frac{1-0.25^{30}}{1-0.25}\right) = 266.667$.

$\lim_{n \rightarrow \infty} 200 \left(\frac{1}{1-0.25}\right) = 266.\overline{66} \text{ mg.}$

$$\mathbf{10.3.44} \quad A_5 = 150 + 150(0.4) + \cdots + 150(0.4)^4 = 150 \left(\frac{1 - 0.4^5}{1 - 0.4} \right) = 247.44.$$

$$A_{10} = 150 + 150(0.4) + \cdots + 150(0.4)^9 = 150 \left(\frac{1 - 0.4^{10}}{1 - 0.4} \right) = 249.974.$$

$$A_{30} = 150 + 150(0.4) + \cdots + 150(0.4)^{29} = 150 \left(\frac{1 - 0.4^{30}}{1 - 0.4} \right) = 250.$$

$$\lim_{n \rightarrow \infty} A_n = 150 \left(\frac{1}{1 - 0.4} \right) = 250 \text{ mg.}$$

$$\mathbf{10.3.45} \quad 200 + 200(0.5) + 200(0.5)^2 + \cdots = \frac{200}{1 - 0.5} = 400 \text{ mg.}$$

$$\mathbf{10.3.46} \quad 0.\bar{6} = 0.666\ldots = \sum_{k=1}^{\infty} 6(0.1)^k = \frac{0.6}{1 - 0.1} = \frac{0.6}{0.9} = \frac{2}{3}.$$

$$\mathbf{10.3.47} \quad 0.\bar{3} = 0.333\ldots = \sum_{k=1}^{\infty} 3(0.1)^k = \frac{0.3}{1 - 0.1} = \frac{0.3}{0.9} = \frac{1}{3}.$$

$$\mathbf{10.3.48} \quad 0.\overline{09} = 0.0909\ldots = \sum_{k=1}^{\infty} 9(0.01)^k = \frac{0.09}{1 - 0.01} = \frac{0.09}{.99} = \frac{9}{99} = \frac{1}{11}.$$

$$\mathbf{10.3.49} \quad 0.\overline{037} = 0.037037037\ldots = \sum_{k=1}^{\infty} 37(0.001)^k = \frac{0.037}{1 - 0.001} = \frac{0.037}{0.999} = \frac{1}{27}.$$

$$\mathbf{10.3.50} \quad 1.\overline{25} = 1.252525\ldots = 1 + \sum_{k=0}^{\infty} 0.25 \cdot 10^{-2k} = 1 + \frac{0.25}{1 - 1/100} = 1 + \frac{25}{99} = \frac{124}{99}.$$

$$\mathbf{10.3.51} \quad 0.\overline{456} = 0.456456456\ldots = \sum_{k=0}^{\infty} 0.456 \cdot 10^{-3k} = \frac{0.456}{1 - 1/1000} = \frac{456}{999} = \frac{152}{333}.$$

$$\mathbf{10.3.52} \quad 5.1\overline{283} = 5.12838383\ldots = 5.12 + \sum_{k=0}^{\infty} 0.0083 \cdot 10^{-2k} = 5.12 + \frac{0.0083}{1 - 1/100} = \frac{512}{100} + \frac{0.83}{99} = \frac{128}{25} + \frac{83}{9900} = \frac{50771}{9900}.$$

$$\mathbf{10.3.53} \quad 0.00\overline{952} = 0.00952952\ldots = \sum_{k=0}^{\infty} 0.00952 \cdot 10^{-3k} = \frac{0.00952}{1 - 1/1000} = \frac{9.52}{999} = \frac{952}{99900} = \frac{238}{24975}.$$

$$\mathbf{10.3.54} \quad \text{The second part of each term cancels with the first part of the succeeding term, so } S_n = \frac{1}{1+2} - \frac{1}{n+3} = \frac{n}{3n+9}, \text{ and } \lim_{n \rightarrow \infty} \frac{n}{3n+9} = \frac{1}{3}.$$

$$\mathbf{10.3.55} \quad \text{The second part of each term cancels with the first part of the succeeding term, so } S_n = \frac{1}{1+1} - \frac{1}{n+2} = \frac{n}{2n+4}, \text{ and } \lim_{n \rightarrow \infty} \frac{n}{2n+4} = \frac{1}{2}.$$

$$\mathbf{10.3.56} \quad \frac{20}{25k^2 + 15k - 4} = \frac{20}{(5k-1)(5k+4)} = \frac{4}{5k-1} - \frac{4}{5k+4}, \text{ so the series given is the same as } \sum_{k=1}^{\infty} \left(\frac{4}{5k-1} - \frac{4}{5k+4} \right). \text{ In that series, the second part of each term cancels with the first part of the succeeding term, so } S_n = 1 - \frac{4}{5n+4}. \text{ Thus } \lim_{n \rightarrow \infty} S_n = 1.$$

10.3.57 $\frac{1}{(k+6)(k+7)} = \frac{1}{k+6} - \frac{1}{k+7}$, so the series given is the same as $\sum_{k=1}^{\infty} \left(\frac{1}{k+6} - \frac{1}{k+7} \right)$. In that series, the second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{1}{1+6} - \frac{1}{n+7}$. Thus $\lim_{n \rightarrow \infty} S_n = \frac{1}{7}$.

10.3.58

$$\frac{10}{4k^2 + 32k + 63} = \frac{10}{(2k+7)(2k+9)}.$$

Our series can be written as $\sum_{k=-3}^{\infty} \frac{10}{(2k+7)(2k+9)} = \sum_{k=1}^{\infty} \frac{10}{(2k-1)(2k+1)}$ by reindexing. Then, using a partial fraction decomposition, we have $\sum_{k=1}^{\infty} \left(\frac{5}{2k-1} - \frac{5}{2k+1} \right)$. In this last series, the second part of each term cancels with the first part of the succeeding term, so $S_n = \frac{5}{1} - \frac{5}{2n+1}$. Thus $\lim_{n \rightarrow \infty} S_n = 5$.

10.3.59 Note that $\frac{4}{(4k-3)(4k+1)} = \frac{1}{4k-3} - \frac{1}{4k+1}$.

Thus the given series is the same as $\sum_{k=3}^{\infty} \left(\frac{1}{4k-3} - \frac{1}{4k+1} \right) = \sum_{k=1}^{\infty} \left(\frac{1}{4k+5} - \frac{1}{4k+9} \right)$. In this last series, the second part of each term cancels with the first part of the succeeding term, so we have $S_n = \frac{1}{9} - \frac{1}{4n+9}$, and thus $\lim_{n \rightarrow \infty} S_n = \frac{1}{9}$.

10.3.60 $\frac{1}{(3k+1)(3k+4)} = \frac{1}{3} \left(\frac{1}{3k+1} - \frac{1}{3k+4} \right)$, so the series given can be written

$\frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{1}{3k+1} - \frac{1}{3k+4} \right) = \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{1}{3k-2} - \frac{1}{3k+1} \right)$. In this last series, the second part of each term

cancels with the first part of the succeeding term, so we are left with $S_n = \frac{1}{3} \left(\frac{1}{1} - \frac{1}{(3n-1)} \right)$ and

$$\lim_{n \rightarrow \infty} S_n = \frac{1}{3} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n-1} \right) = \frac{1}{3}.$$

10.3.61 $\ln \left(\frac{k+1}{k} \right) = \ln(k+1) - \ln k$, so the series given is the same as $\sum_{k=1}^{\infty} (\ln(k+1) - \ln k)$, in which the first part of each term cancels with the second part of the next term, so we have $S_n = \ln(n+1) - \ln 1 = \ln(n+1)$, and thus the series diverges.

10.3.62 Note that $\frac{2}{(2k-1)(2k+1)} = \frac{1}{2k-1} - \frac{1}{2k+1}$.

Thus the given series is the same as $\sum_{k=3}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) = \sum_{k=1}^{\infty} \left(\frac{1}{2k+3} - \frac{1}{2k+5} \right)$. In that series, the second part of each term cancels with the first part of the succeeding term, so we have $S_n = \frac{1}{5} - \frac{1}{2n+5}$. Thus, $\lim_{n \rightarrow \infty} S_n = \frac{1}{5}$.

10.3.63 $\frac{1}{(k+p)(k+p+1)} = \frac{1}{k+p} - \frac{1}{k+p+1}$, so that $\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+p+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k+p} - \frac{1}{k+p+1} \right)$

and this series telescopes to give $S_n = \frac{1}{p+1} - \frac{1}{n+p+1} = \frac{n}{n(p+1) + (p+1)^2}$ so that $\lim_{n \rightarrow \infty} S_n = \frac{1}{p+1}$.

10.3.64 $\frac{1}{(ak+1)(ak+a+1)} = \frac{1}{a} \left(\frac{1}{ak+1} - \frac{1}{ak+a+1} \right)$, so that $\sum_{k=1}^{\infty} \frac{1}{(ak+1)(ak+a+1)} = \frac{1}{a} \sum_{k=1}^{\infty} \left(\frac{1}{ak+1} - \frac{1}{ak+a+1} \right)$. This series telescopes - the second term of each summand cancels with the first term of the succeeding term - so that $S_n = \frac{1}{a} \left(\frac{1}{a+1} - \frac{1}{an+a+1} \right)$, and thus the limit of the sequence is $\frac{1}{a(a+1)}$.

10.3.65 Let $a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+3}}$. Then the second term of a_n cancels with the first term of a_{n+2} , so the series telescopes and $S_n = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{n-1+3}} - \frac{1}{\sqrt{n+3}}$ and thus the sum of the series is the limit of S_n , which is $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}$.

10.3.66 $\frac{6}{k(k+2)} = \frac{3}{k} - \frac{3}{k+2}$. So our series can be written as $\sum_{k=1}^{\infty} \left(\frac{3}{k} - \frac{3}{k+2} \right)$. Starting with S_3 the second part of each term cancels the first part of the term that is two ahead, so

$$S_n = \frac{3}{1} + \frac{3}{2} - \frac{3}{n+1} - \frac{3}{n+2}.$$

So $\lim_{n \rightarrow \infty} S_n = 3 + \frac{3}{2} = \frac{9}{2}$.

10.3.67 $\frac{3}{k^2+5k+4} = \frac{3}{(k+1)(k+4)} = \frac{1}{k+1} - \frac{1}{k+4}$. So our series can be written as

$$\sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+4} \right).$$

Starting with S_4 , the second part of each term cancels the first part of the term that is 3 terms ahead, so

$$S_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4},$$

and $\lim_{n \rightarrow \infty} S_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$.

10.3.68 Note that $S_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{n+1} - \sqrt{n})$. The second part of each term cancels with the first part of the previous term. Thus, $S_n = \sqrt{n+1} - 1$. and because $\lim_{n \rightarrow \infty} \sqrt{n+1} - 1 = \infty$, the series diverges.

10.3.69 This series telescopes to give

$$S_n = \tan^{-1}(n+1) - \tan^{-1}(1) = \tan^{-1}(n+1) - \frac{\pi}{4}.$$

Then because $\lim_{n \rightarrow \infty} \tan^{-1}(n+1) = \frac{\pi}{2}$, the sum of the series is equal to $\lim_{n \rightarrow \infty} S_n = \frac{\pi}{4}$.

10.3.70

- a. Because the first part of each term cancels the second part of the previous term, the n th partial sum telescopes to be $S_n = \frac{1}{2} - \frac{1}{2^{n+1}}$. Thus, the sum of the series is $\lim_{n \rightarrow \infty} S_n = \frac{1}{2}$.

$$\text{b. } \sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}} \right) = \sum_{k=1}^{\infty} \frac{1}{2^k} - \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1/2}{1-1/2} - \frac{1/4}{1-1/2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

10.3.71

a. Because the first part of each term cancels the second part of the previous term, the n th partial sum telescopes to be $S_n = \frac{4}{3} - \frac{4}{3^{n+1}}$. Thus, the sum of the series is $\lim_{n \rightarrow \infty} S_n = \frac{4}{3}$.

$$\text{b. } \sum_{k=1}^{\infty} \left(\frac{4}{3^k} - \frac{4}{3^{k+1}} \right) = \sum_{k=1}^{\infty} \frac{4}{3^k} - \sum_{k=1}^{\infty} \frac{4}{3^{k+1}} = \frac{4/3}{1-1/3} - \frac{4/9}{1-1/3} = 2 - \frac{2}{3} = \frac{4}{3}.$$

$$\text{10.3.72 } \frac{1}{9k^2 + 39k + 40} = \frac{1}{(3k+5)(3k+8)} = \frac{1/3}{3k+5} - \frac{1/3}{3k+8}. \text{ We have}$$

$$S_n = \frac{1}{3} \left(\left(\frac{1}{8} - \frac{1}{11} \right) + \left(\frac{1}{11} - \frac{1}{14} \right) + \cdots + \left(\frac{1}{3n+5} - \frac{1}{3n+8} \right) \right) = \frac{1}{3} \left(\frac{1}{8} - \frac{1}{3n+8} \right).$$

$$\text{Thus } S = \lim_{n \rightarrow \infty} S_n = \frac{1}{24}.$$

10.3.73 $16k^2 + 8k - 3 = (4k+3)(4k-1)$, so $\frac{1}{16k^2 + 8k - 3} = \frac{1}{(4k+3)(4k-1)} = \frac{1}{4} \left(\frac{1}{4k-1} - \frac{1}{4k+3} \right)$. Thus the series given is equal to $\frac{1}{4} \sum_{k=0}^{\infty} \left(\frac{1}{4k-1} - \frac{1}{4k+3} \right)$. This series telescopes, so $S_n = \frac{1}{4} \left(-1 - \frac{1}{4n-1} \right)$, so the sum of the series is equal to $\lim_{n \rightarrow \infty} S_n = -\frac{1}{4}$.

10.3.74 The first term of the k^{th} summand is $\sin \left(\frac{k\pi}{2k-1} \right)$; the second term of the $(k+1)^{\text{st}}$ summand is $-\sin \left(\frac{k\pi}{2k-1} \right)$; these two are equal except for sign, so they cancel. Thus

$$S_n = -\sin 0 + \sin \left(\frac{n\pi}{2n-3} \right) = \sin \left(\frac{n\pi}{2n-3} \right).$$

Because $\frac{n\pi}{2n-3}$ has limit $\frac{\pi}{2}$ as $n \rightarrow \infty$, and because the sine function is continuous, $\lim_{n \rightarrow \infty} S_n$ is $\sin \left(\frac{\pi}{2} \right) = 1$.

$$\text{10.3.75 } \sum_{k=0}^{\infty} \left(\frac{1}{4} \right)^k 5^{3-k} = 5^3 \sum_{k=0}^{\infty} \left(\frac{1}{20} \right)^k = 5^3 \cdot \frac{1}{1-1/20} = \frac{5^3 \cdot 20}{19} = \frac{2500}{19}.$$

$$\text{10.3.76 } \frac{3^6/8^6}{1-(3/8)^3} = \frac{729}{248320}$$

10.3.77 This can be written as $\frac{1}{3} \sum_{k=1}^{\infty} \left(-\frac{2}{3} \right)^k$. This is a geometric series with ratio $r = -\frac{2}{3}$ so the sum is

$$\frac{1}{3} \cdot \frac{-2/3}{1-(-2/3)} = \frac{1}{3} \cdot \left(-\frac{2}{5} \right) = -\frac{2}{15}.$$

10.3.78 We have

$$S_n = \left(\sin^{-1} 1 - \sin^{-1} \frac{1}{2} \right) + \left(\sin^{-1} \frac{1}{2} - \sin^{-1} \frac{1}{3} \right) + \cdots + \left(\sin^{-1} \frac{1}{n} - \sin^{-1} \frac{1}{n+1} \right).$$

Note that the first part of each term cancels the second part of the previous term, so the n th partial sum telescopes to be $\sin^{-1} 1 - \sin^{-1} \frac{1}{n+1}$. Because $\sin^{-1} 1 = \frac{\pi}{2}$ and $\lim_{n \rightarrow \infty} \sin^{-1} \frac{1}{n+1} = \sin^{-1} 0 = 0$, we have

$$\lim_{n \rightarrow \infty} S_n = \frac{\pi}{2}.$$

10.3.79 Note that

$$\frac{\ln((k+1)k^{-1})}{(\ln k)\ln(k+1)} = \frac{\ln(k+1)}{(\ln k)\ln(k+1)} - \frac{\ln k}{(\ln k)\ln(k+1)} = \frac{1}{\ln k} - \frac{1}{\ln(k+1)}.$$

In the partial sum S_n , the first part of each term cancels the second part of the preceding term, so we have $S_n = \frac{1}{\ln 2} - \frac{1}{\ln(n+2)}$. Thus we have $\lim_{n \rightarrow \infty} S_n = \frac{1}{\ln 2}$.

10.3.80 This can be written as $\frac{1}{e} \sum_{k=1}^{\infty} \left(\frac{\pi}{e}\right)^k$. This is a geometric series with $r = \frac{\pi}{e} > 1$, so the series diverges.

$$\mathbf{10.3.81} \quad \sum_{k=0}^{\infty} \left(3 \left(\frac{2}{5} \right)^k - 2 \left(\frac{5}{7} \right)^k \right) = 3 \sum_{k=0}^{\infty} \left(\frac{2}{5} \right)^k - 2 \sum_{k=0}^{\infty} \left(\frac{5}{7} \right)^k = 3 \left(\frac{1}{3/5} \right) - 2 \left(\frac{1}{2/7} \right) = 5 - 7 = -2.$$

$$\mathbf{10.3.82} \quad \sum_{k=1}^{\infty} \left(2 \left(\frac{3}{5} \right)^k + 3 \left(\frac{4}{9} \right)^k \right) = 2 \sum_{k=1}^{\infty} \left(\frac{3}{5} \right)^k + 3 \sum_{k=1}^{\infty} \left(\frac{4}{9} \right)^k = 2 \left(\frac{3/5}{2/5} \right) + 3 \left(\frac{4/9}{5/9} \right) = 3 + \frac{12}{5} = \frac{27}{5}.$$

$$\mathbf{10.3.83} \quad \sum_{k=1}^{\infty} \left(\frac{1}{3} \left(\frac{5}{6} \right)^k + \frac{3}{5} \left(\frac{7}{9} \right)^k \right) = \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{5}{6} \right)^k + \frac{3}{5} \sum_{k=1}^{\infty} \left(\frac{7}{9} \right)^k = \frac{1}{3} \left(\frac{5/6}{1/6} \right) + \frac{3}{5} \left(\frac{7/9}{2/9} \right) = \frac{5}{3} + \frac{21}{10} = \frac{113}{30}.$$

$$\mathbf{10.3.84} \quad \sum_{k=0}^{\infty} \left(\frac{1}{2} (0.2)^k + \frac{3}{2} (0.8)^k \right) = \frac{1}{2} \sum_{k=0}^{\infty} (0.2)^k + \frac{3}{2} \sum_{k=0}^{\infty} (0.8)^k = \frac{1}{2} \left(\frac{1}{0.8} \right) + \frac{3}{2} \left(\frac{1}{0.2} \right) = \frac{5}{8} + \frac{15}{2} = \frac{65}{8}.$$

$$\mathbf{10.3.85} \quad \sum_{k=1}^{\infty} \left(\left(\frac{1}{6} \right)^k + \left(\frac{1}{3} \right)^{k-1} \right) = \sum_{k=1}^{\infty} \left(\frac{1}{6} \right)^k + \sum_{k=1}^{\infty} \left(\frac{1}{3} \right)^{k-1} = \frac{1/6}{5/6} + \frac{1}{2/3} = \frac{17}{10}.$$

$$\mathbf{10.3.86} \quad \sum_{k=0}^{\infty} \frac{2-3^k}{6^k} = \sum_{k=0}^{\infty} \left(\frac{2}{6^k} - \frac{3^k}{6^k} \right) = 2 \sum_{k=0}^{\infty} \left(\frac{1}{6} \right)^k - \sum_{k=0}^{\infty} \left(\frac{1}{2} \right)^k = 2 \left(\frac{1}{5/6} \right) - \frac{1}{1/2} = \frac{2}{5}.$$

10.3.87

- True. The two series differ by a finite amount $\left(\sum_{k=1}^9 a_k \right)$, so if one converges, so does the other.
- True. The same argument applies as in part (a).
- False. If $\sum a_k$ converges, then $a_k \rightarrow 0$ as $k \rightarrow \infty$, so that $a_k + 0.0001 \rightarrow 0.0001$ as $k \rightarrow \infty$, so that $\sum (a_k + 0.0001)$ cannot converge.
- False. Suppose $p = -1.0001$. Then $\sum p^k$ diverges but $p + .0001 = -0.9991$ so that $\sum (p + .0001)^k$ converges.
- True. $\left(\frac{\pi}{e} \right)^{-k} = \left(\frac{e}{\pi} \right)^k$; because $e < \pi$, this is a geometric series with ratio less than 1.
- False. For example, let $0 < a < 1$ and $b > 1$.
- True. Suppose $a > \frac{1}{2}$. Then we want $a = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. Solving for r gives $r = 1 - \frac{1}{a}$. Because $a > 0$ we have $r < 1$; because $a > \frac{1}{2}$ we have $r > 1 - \frac{1}{1/2} = -1$. Thus $|r| < 1$ so that $\sum_{k=0}^{\infty} r^k$ converges, and it converges to a .

$$\mathbf{10.3.88} \quad 0.010101\ldots = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = \frac{1/4}{1 - 1/4} = \frac{1}{3}.$$

10.3.89

$$\text{a. } 0.00110011\ldots = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^{11}} + \frac{1}{2^{12}} + \cdots = \frac{3}{2^4} + \frac{3}{2^8} + \frac{3}{2^{12}} + \cdots = \frac{3/2^4}{1 - 1/2^4} = \frac{3}{15} = \frac{1}{5}.$$

$$\text{b. } 0.0011001100110011 = \frac{3}{2^4} + \frac{3}{2^8} + \frac{3}{2^{12}} + \frac{3}{2^{16}} \approx 0.19999695.$$

$$\mathbf{10.3.90} \quad \text{We seek } r \text{ so that } \frac{1}{1-r} = 10, \text{ so } r-1 = -\frac{1}{10}, \text{ so } r = \frac{9}{10}.$$

10.3.91 We seek r so that $\frac{r^6}{1-r^2} = 10$, so $r^6 = 10 - 10r^2$, so $r^6 + 10r^2 - 10 = 0$. Using a computer algebra system, we obtain $r \approx 0.96$.

10.3.92

$$\text{a. Note that } \frac{3^k}{(3^{k+1}-1)(3^k-1)} = \frac{1}{2} \cdot \left(\frac{1}{3^k-1} - \frac{1}{3^{k+1}-1} \right). \text{ Then}$$

$$\sum_{k=1}^{\infty} \frac{3^k}{(3^{k+1}-1)(3^k-1)} = \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{3^k-1} - \frac{1}{3^{k+1}-1} \right).$$

This series telescopes to give $S_n = \frac{1}{2} \left(\frac{1}{3-1} - \frac{1}{3^{n+1}-1} \right)$, so that the sum of the series is $\lim_{n \rightarrow \infty} S_n = \frac{1}{4}$.

b. We mimic the above computations. First, $\frac{a^k}{(a^{k+1}-1)(a^k-1)} = \frac{1}{a-1} \cdot \left(\frac{1}{a^k-1} - \frac{1}{a^{k+1}-1} \right)$, so we see that we cannot have $a = 1$, because the fraction would then be undefined. Continuing, we obtain $S_n = \frac{1}{a-1} \left(\frac{1}{a-1} - \frac{1}{a^{n+1}-1} \right)$. Now, $\lim_{n \rightarrow \infty} \frac{1}{a^{n+1}-1}$ converges if and only if the denominator grows without bound; this happens if and only if $|a| > 1$. Thus, the original series converges for $|a| > 1$, when it converges to $\frac{1}{(a-1)^2}$. Note that this is valid even for a negative.

10.3.93

$$\begin{aligned} A_n &= A_{n-1}(1+r) - M \\ &= (A_{n-2}(1+r) - M)(1+r) - M \\ &= (A_{n-3}(1+r) - M)(1+r)^2 - (M + M(1+r)) \\ &= A_{n-3}(1+r)^3 - (M + M(1+r) + M(1+r)^2) = \cdots \\ &= A_0(1+r)^n - (M + M(1+r) + M(1+r)^2 + \cdots + M(1+r)^{n-1}) \\ &= A_0(1+r)^n - M \left(\frac{1 - (1+r)^n}{1 - (1+r)} \right) \\ &= A_0(1+r)^n - \frac{M}{r} ((1+r)^n - 1) \end{aligned}$$

10.3.94 $20000(1.0075)^n - \frac{600}{0.0075}(1.0075^n - 1) = 0$, so $20000(1.0075)^n - 80000((1.0075)^n - 1) = 0$, so $-60000(1.0075)^n + 80000 = 0$. Then $(1.0075)^n = \frac{80000}{60000} = \frac{4}{3}$, so $n = \frac{\ln(4/3)}{\ln 1.0075} \approx 38.5$. Rounding up, we have 39 payments.

10.3.95 $180000(1.005)^n - \frac{1000}{0.005}((1.005)^n - 1) = 0$, so $-20000(1.005)^n + 200000 = 0$. Then $(1.005)^n = 10$, so $n = \frac{\ln 10}{\ln 1.005} \approx 461.67$. Rounding up we have 462 months.

10.3.96 $\sum_{k=1}^{\infty} (a_k \pm b_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n (a_k \pm b_k) = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n a_k \pm \sum_{k=1}^n b_k \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k \pm \lim_{n \rightarrow \infty} \sum_{k=1}^n b_k = A \pm B$.

10.3.97 $\sum_{k=1}^{\infty} ca_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n ca_k = \lim_{n \rightarrow \infty} c \sum_{k=1}^n a_k = c \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$, so that one sum diverges if and only if the other one does.

10.3.98 It will take Achilles $1/5$ hour to cover the first mile. At this time, the tortoise has gone $1/5$ mile more, and it will take Achilles $1/25$ hour to reach this new point. At that time, the tortoise has gone another $1/25$ of a mile, and it will take Achilles $1/125$ hour to reach this point. Adding the times up, we have

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{1/5}{1 - 1/5} = \frac{1}{4},$$

so it will take Achilles $1/4$ of an hour (15 minutes) to catch the tortoise.

10.3.99 At the n^{th} stage, there are 2^{n-1} triangles of area $A_n = \frac{1}{8}A_{n-1} = \frac{1}{8^{n-1}}A_1$, so the total area of the triangles formed at the n^{th} stage is $\frac{2^{n-1}}{8^{n-1}}A_1 = \left(\frac{1}{4}\right)^{n-1}A_1$. Thus the total area under the parabola is

$$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1} A_1 = A_1 \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1} = A_1 \frac{1}{1 - 1/4} = \frac{4}{3}A_1.$$

10.3.100 Let L_n be the amount of light transmitted through the window the n^{th} time the beam hits the second pane. Then the amount of light that was available before the beam went through the pane was $\frac{L_n}{1-p}$, so $\frac{pL_n}{1-p}$ is reflected back to the first pane, and $\frac{p^2L_n}{1-p}$ is then reflected back to the second pane. Of that, a fraction equal to $1-p$ is transmitted through the window. Thus

$$L_{n+1} = (1-p)\frac{p^2L_n}{1-p} = p^2L_n.$$

The amount of light transmitted through the window the first time is $(1-p)^2$. Thus the total amount is

$$\sum_{i=0}^{\infty} p^{2n}(1-p)^2 = \frac{(1-p)^2}{1-p^2} = \frac{1-p}{1+p}.$$

10.3.101

a. I_{n+1} is obtained by I_n by dividing each edge into three equal parts, removing the middle part, and adding two parts equal to it. Thus 3 equal parts turn into 4, so $L_{n+1} = \frac{4}{3}L_n$. This is a geometric sequence with a ratio greater than 1, so the n^{th} term grows without bound.

b. As the result of part (a), I_n has $3 \cdot 4^n$ sides of length $\frac{1}{3^n}$; each of those sides turns into an added triangle in I_{n+1} of side length 3^{-n-1} . Thus the added area in I_{n+1} consists of $3 \cdot 4^n$ equilateral triangles with side 3^{-n-1} . The area of an equilateral triangle with side x is $\frac{x^2\sqrt{3}}{4}$. Thus $A_{n+1} = A_n + 3 \cdot 4^n \cdot \frac{3^{-2n-2}\sqrt{3}}{4} = A_n + \frac{\sqrt{3}}{12} \cdot \left(\frac{4}{9}\right)^n$, and $A_0 = \frac{\sqrt{3}}{4}$. Thus $A_{n+1} = A_0 + \sum_{i=0}^n \frac{\sqrt{3}}{12} \cdot \left(\frac{4}{9}\right)^i$, so that

$$A_{\infty} = A_0 + \frac{\sqrt{3}}{12} \sum_{i=0}^{\infty} \left(\frac{4}{9}\right)^i = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \frac{1}{1 - 4/9} = \frac{\sqrt{3}}{4} \left(1 + \frac{3}{5}\right) = \frac{2}{5}\sqrt{3}.$$

10.3.102

a. Clearly for $k < n$, h_k is a leg of a right triangle whose hypotenuse is r_k and whose other leg is formed where the vertical line (in the picture) meets a diameter of the next smaller sphere; thus the other leg of the triangle is r_{k+1} . The Pythagorean theorem then implies that $h_k^2 = r_k^2 - r_{k+1}^2$.

b. The height is $H_n = \sum_{i=1}^n h_i = r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2}$ by part (a).

c. From part (b), because $r_i = a^{i-1}$,

$$\begin{aligned} H_n &= r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2} = a^{n-1} + \sum_{i=1}^{n-1} \sqrt{a^{2i-2} - a^{2i}} \\ &= a^{n-1} + \sum_{i=1}^{n-1} a^{i-1} \sqrt{1 - a^2} = a^{n-1} + \sqrt{1 - a^2} \sum_{i=1}^{n-1} a^{i-1} \\ &= a^{n-1} + \sqrt{1 - a^2} \left(\frac{1 - a^{n-1}}{1 - a} \right). \end{aligned}$$

$$\text{d. } \lim_{n \rightarrow \infty} H_n = \lim_{n \rightarrow \infty} a^{n-1} + \sqrt{1 - a^2} \lim_{n \rightarrow \infty} \frac{1 - a^{n-1}}{1 - a} = 0 + \sqrt{1 - a^2} \left(\frac{1}{1 - a} \right) = \sqrt{\frac{1 - a^2}{(1 - a)(1 - a)}} = \sqrt{\frac{1 + a}{1 - a}}.$$

10.3.103 $|S - S_n| = \left| \sum_{i=n}^{\infty} r^k \right| = \left| \frac{r^n}{1 - r} \right|$ because the latter sum is simply a geometric series with first term r^n and ratio r .

10.3.104

a. Solve $\frac{0.6^n}{0.4} < 10^{-6}$ for n to get $n = 29$.

b. Solve $\frac{0.15^n}{0.85} < 10^{-6}$ for n to get $n = 8$.

10.3.105

a. Solve $\left| \frac{(-0.8)^n}{1.8} \right| = \frac{0.8^n}{1.8} < 10^{-6}$ for n to get $n = 60$.

b. Solve $\frac{0.2^n}{0.8} < 10^{-6}$ for n to get $n = 9$.

10.3.106

a. Solve $\frac{0.72^n}{0.28} < 10^{-6}$ for n to get $n = 46$.

b. Solve $\left| \frac{(-0.25)^n}{1.25} \right| = \frac{0.25^n}{1.25} < 10^{-6}$ for n to get $n = 10$.

10.3.107

a. Solve $\frac{1/\pi^n}{1 - 1/\pi} < 10^{-6}$ for n to get $n = 13$.

b. Solve $\frac{1/e^n}{1 - 1/e} < 10^{-6}$ for n to get $n = 15$.

10.3.108

a. $f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$; because f is represented by a geometric series, $f(x)$ exists only for $|x| < 1$.

Then $f(0) = 1$, $f(0.2) = \frac{1}{0.8} = 1.25$, $f(0.5) = \frac{1}{1-0.5} = 2$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

10.3.109

a. $f(x) = \sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x}$; because f is a geometric series, $f(x)$ exists only when the ratio, $-x$, is

such that $|-x| = |x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{1.2} = \frac{5}{6}$, $f(0.5) = \frac{1}{1+.05} = \frac{2}{3}$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

10.3.110

a. $f(x) = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$. f is a geometric series, so $f(x)$ is defined only when the ratio, x^2 , is less

than 1, which means $|x| < 1$. Then $f(0) = 1$, $f(0.2) = \frac{1}{1-.04} = \frac{25}{24}$, $f(0.5) = \frac{1}{1-0.25} = \frac{4}{3}$. Neither $f(1)$ nor $f(1.5)$ exists.

b. The domain of f is $\{x : |x| < 1\}$.

10.3.111 $f(x)$ is a geometric series with ratio $\frac{1}{1+x}$; thus $f(x)$ converges when $\left| \frac{1}{1+x} \right| < 1$. For $x > -1$, $\left| \frac{1}{1+x} \right| = \frac{1}{1+x}$ and $\frac{1}{1+x} < 1$ when $1 < 1+x$, $x > 0$. For $x < -1$, $\left| \frac{1}{1+x} \right| = \frac{1}{-1-x}$, and this is less than 1 when $1 < -1-x$, i.e. $x < -2$. So $f(x)$ converges for $x > 0$ and for $x < -2$. When $f(x)$ converges, its value is $\frac{1}{1-\frac{1}{1+x}} = \frac{1+x}{x}$, so $f(x) = 3$ when $1+x = 3x$, $x = \frac{1}{2}$.

10.4 The Divergence and Integral Tests

10.4.1 The series diverges.

10.4.2 No. For example, the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges although $\frac{1}{k} \rightarrow 0$ as $k \rightarrow \infty$.

10.4.3 We can be sure that $\lim_{k \rightarrow \infty} a_k = 0$.

10.4.4 It converges for $p > 1$, and diverges for all other values of p .

10.4.5 For the same values of p as in the previous problem – it converges for $p > 1$, and diverges for all other values of p .

10.4.6 Let S_n be the partial sums. Then $S_{n+1} - S_n = a_{n+1} > 0$ because $a_{n+1} > 0$. Thus the sequence of partial sums is increasing.

10.4.7 The remainder of an infinite series is the error in approximating a convergent infinite series by a finite number of terms.

10.4.8 Yes. Suppose $\sum a_k$ converges to S , and let the sequence of partial sums be $\{S_n\}$. Then for any $\epsilon > 0$ there is some N such that for any $n > N$, $|S - S_n| < \epsilon$. But $|S - S_n|$ is simply the remainder R_n when the series is approximated to n terms. Thus $R_n \rightarrow 0$ as $n \rightarrow \infty$.

10.4.9 $a_k = \frac{k}{2k+1}$ and $\lim_{k \rightarrow \infty} a_k = \frac{1}{2}$, so the series diverges.

10.4.10 $a_k = \frac{k}{k^2+1}$ and $\lim_{k \rightarrow \infty} a_k = 0$, so the divergence test is inconclusive.

10.4.11 $a_k = \frac{1}{1000+k}$ and $\lim_{k \rightarrow \infty} a_k = 0$, so the divergence test is inconclusive.

10.4.12 $a_k = \frac{k^3}{k^3+1}$ and $\lim_{k \rightarrow \infty} a_k = 1$, so the series diverges.

10.4.13 $a_k = \frac{k}{\ln k}$ and $\lim_{k \rightarrow \infty} a_k = \infty$, so the series diverges.

10.4.14 $a_k = \frac{k^2}{2^k}$ and $\lim_{k \rightarrow \infty} a_k = 0$, so the divergence test is inconclusive.

10.4.15 $a_k = \frac{\sqrt{k}}{\ln^{10} k}$ and $\lim_{k \rightarrow \infty} a_k = \infty$, so the series diverges.

10.4.16 By Theorem 10.6 $k^3 \ll k!$, so $\lim_{k \rightarrow \infty} \frac{k^3}{k!} = 0$. The divergence test is inconclusive.

10.4.17 Let $f(x) = \frac{1}{\sqrt{x+8}}$. $f(x)$ is obviously continuous and decreasing for $x \geq 1$. Because

$$\int_1^{\infty} \frac{1}{\sqrt{x+8}} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{\sqrt{x+8}} dx = \lim_{b \rightarrow \infty} 2\sqrt{x+8} \Big|_1^b = \lim_{b \rightarrow \infty} (2\sqrt{b+8} - 6) = \infty,$$

the series diverges.

10.4.18 The function $f(x) = \frac{1}{(2x+4)^2}$ is positive, decreasing, and continuous for $x \geq 1$. The relevant integral is

$$\int_1^{\infty} \frac{dx}{(2x+4)^2} = \lim_{b \rightarrow \infty} -\frac{1}{2(2x+4)} \Big|_1^b = -\frac{1}{2} \lim_{b \rightarrow \infty} \left(\frac{1}{2b+4} - \frac{1}{6} \right) = \frac{1}{12}.$$

Because the integral converges, the given series also converges by the Integral Test.

10.4.19 The function $f(x) = \frac{1}{\sqrt[3]{5x+3}}$ is positive, decreasing, and continuous for $x \geq 1$. The relevant integral is

$$\int_1^{\infty} \frac{dx}{\sqrt[3]{5x+3}} = \lim_{b \rightarrow \infty} \frac{3}{10} (5x+3)^{2/3} \Big|_1^b = \frac{3}{10} \lim_{b \rightarrow \infty} ((5b+3)^{2/3} - 4) = \infty.$$

Because the integral diverges, the given series also diverges by the Integral Test.

10.4.20 The function $f(x) = \frac{e^x}{1+e^{2x}}$ is positive and continuous for $x \geq 1$. Note that $f'(x) = \frac{e^x(1-e^{2x})}{(e^{2x}+1)^2} \leq 0$ for $x \geq 1$, so f is decreasing. The relevant integral is

$$\int_1^{\infty} \frac{e^x}{1+e^{2x}} dx = \lim_{b \rightarrow \infty} \tan^{-1} e^x \Big|_1^b = \lim_{b \rightarrow \infty} (\tan^{-1} e^b - \tan^{-1} e) = \frac{\pi}{2} - \tan^{-1} e.$$

Because the integral converges, the given series also converges by the Integral Test.

10.4.21 Let $f(x) = x \cdot e^{-2x^2}$. This function is continuous for $x \geq 1$. Its derivative is $e^{-2x^2}(1 - 4x^2) < 0$ for $x \geq 1$, so $f(x)$ is decreasing. The relevant integral is

$$\int_1^\infty x \cdot e^{-2x^2} dx = -\frac{1}{4} \lim_{b \rightarrow \infty} e^{-2x^2} \Big|_1^b = -\frac{1}{4} \lim_{b \rightarrow \infty} (e^{-2b^2} - e^{-2}) = \frac{1}{4e^2}.$$

Because the integral converges, the given series also converges by the Integral Test.

10.4.22 Let $f(x) = \frac{1}{x(\ln x)^2}$. $f(x)$ is continuous and decreasing for $x \geq 2$. The relevant integral is

$$\int_2^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x(\ln x)^2} = \lim_{b \rightarrow \infty} -\frac{1}{\ln x} \Big|_2^b = \lim_{b \rightarrow \infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln b} \right) = \frac{1}{\ln 2}.$$

Because the integral converges, the given series also converges by the Integral Test.

10.4.23 $\sum_{k=1}^\infty \frac{1}{k^{1/5}}$ is a p -series with $p = \frac{1}{5} < 1$. The series diverges by the p -series test.

10.4.24 $\sum_{k=1}^\infty \frac{1}{k^{3/2}}$ is a p -series with $p = \frac{3}{2} > 1$. The series converges by the p -series test.

10.4.25 We will use the Integral Test. $f(x) = \frac{x^3}{e^{x^4}}$ is continuous and positive, and $f'(x) = \frac{x^2(3 - 4x^4)}{e^{x^4}} < 0$ for $x > 1$. Thus f is decreasing for $x > 1$. The relevant integral is

$$\int_1^\infty \frac{x^3}{e^{x^4}} dx = \lim_{b \rightarrow \infty} \left(-\frac{e^{-x^4}}{4} \right) \Big|_1^b = \frac{1}{4e}.$$

Because the integral converges, the given series also converges by the Integral Test.

10.4.26 We will use the Integral Test. $f(x) = \frac{x^2}{x^3 + 5}$ is continuous and positive, and $f'(x) = \frac{x(10 - x^3)}{(x^3 + 5)^2} < 0$ for $x \geq 3$. Thus f is decreasing for $x \geq 3$. The relevant integral is

$$\int_1^\infty \frac{x^2}{x^3 + 5} dx = \frac{1}{3} \lim_{b \rightarrow \infty} (\ln(x^3 + 5)) \Big|_1^b = \infty.$$

Because the integral diverges, the given series also diverges by the Integral Test.

10.4.27 $a_k = k^{1/k}$. In order to compute $\lim_{k \rightarrow \infty} a_k$, we let $y_k = \ln a_k = \frac{\ln k}{k}$. By Theorem 10.6, (or by L'Hôpital's rule) $\lim_{k \rightarrow \infty} y_k = 0$, so $\lim_{k \rightarrow \infty} a_k = e^0 = 1$. The given series thus diverges by the Divergence Test.

10.4.28 $a_k = \frac{\sqrt{k^2 + 1}}{k}$ and $\lim_{k \rightarrow \infty} a_k = 1$, so the series diverges by the Divergence Test.

10.4.29 This is a p -series with $p = 10$, so this series converges by the p -series test.

10.4.30 $\sum_{k=2}^\infty \frac{k^e}{k^\pi} = \sum_{k=2}^\infty \frac{1}{k^{\pi-e}}$. Note that $\pi - e \approx 3.1416 - 2.71828 < 1$, so this series diverges by the p -series test.

10.4.31 $\sum_{k=3}^\infty \frac{1}{(k-2)^4} = \sum_{k=1}^\infty \frac{1}{k^4}$, which is a p -series with $p = 4$, thus convergent by the p -series test.

10.4.32 $\sum_{k=1}^{\infty} 2k^{-3/2} = 2 \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a p -series with $p = 3/2$, thus convergent by the p -series test.

10.4.33 Let $f(x) = \frac{x}{e^x}$. $f(x)$ is clearly continuous for $x > 1$, and its derivative, $f'(x) = \frac{e^x - xe^x}{e^{2x}} = (1-x)\frac{e^x}{e^{2x}}$, is negative for $x > 1$ so that $f(x)$ is decreasing. Because $\int_1^{\infty} f(x) dx = 2e^{-1}$, the series converges by the Integral Test.

10.4.34 Let $f(x) = \frac{1}{x \cdot \ln x \cdot \ln \ln x}$. $f(x)$ is continuous and decreasing for $x > 3$, and

$$\int_3^{\infty} \frac{1}{x \cdot \ln x \cdot \ln \ln x} dx = \lim_{b \rightarrow \infty} \int_3^b \frac{1}{x \cdot \ln x \cdot \ln \ln x} dx = \lim_{b \rightarrow \infty} (\ln \ln \ln b - \ln \ln \ln 3) = \infty.$$

The given series therefore diverges by the Integral Test.

10.4.35 Because $\lim_{k \rightarrow \infty} \left(\frac{k}{k+10} \right)^k = \lim_{k \rightarrow \infty} \frac{1}{(1+(10/k))^k} = \frac{1}{e^{10}} \neq 0$, the series diverges by the Divergence Test.

10.4.36 We use the Integral Test. Note that $f(x) = \frac{x}{(x^2+1)^3}$ is continuous and positive on $(1, \infty)$. Also, $f'(x) = \frac{1-5x^2}{(x^2+1)^4} < 0$ for $x > 1$, so is decreasing. The relevant integral is

$$\int_1^{\infty} \frac{x}{(x^2+1)^3} dx = -\frac{1}{4} \lim_{b \rightarrow \infty} \frac{1}{(x^2+1)^2} \Big|_1^b = -\frac{1}{4} \lim_{b \rightarrow \infty} \left(\frac{1}{(b^2+1)^2} - \frac{1}{4} \right) = \frac{1}{16}.$$

Thus, the given series converges by the Integral Test.

10.4.37 $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$ is a p -series with $p = 1/3$, thus divergent by the p -series test.

10.4.38 $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{27k^2}} = \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$ is a p -series with $p = 2/3$, thus divergent by the p -series test.

10.4.39

a. $S \approx S_2 = 1 + \frac{1}{2^7} = 1.0078125.$

b. $R_2 < \int_2^{\infty} \frac{dx}{x^7} = \lim_{b \rightarrow \infty} \int_2^b x^{-7} dx = \lim_{b \rightarrow \infty} -\frac{1}{6x^6} \Big|_2^b = \lim_{b \rightarrow \infty} -\left(\frac{1}{6b^6} - \frac{1}{384} \right) \approx 0.0026042.$

c.

$$\begin{aligned} L_2 &= S_2 + \int_3^{\infty} \frac{dx}{x^7} = S_2 + \lim_{b \rightarrow \infty} \int_3^b x^{-7} dx \\ &= S_2 + \lim_{b \rightarrow \infty} -\frac{1}{6x^6} \Big|_3^b = S_2 + \lim_{b \rightarrow \infty} -\left(\frac{1}{6b^6} - \frac{1}{4374} \right) \\ &\approx 1.0078125 + 0.000228624 \approx 1.0080411. \end{aligned}$$

$$\begin{aligned} U_2 &= S_2 + \int_2^{\infty} \frac{dx}{x^7} = S_2 + \lim_{b \rightarrow \infty} \int_2^b x^{-7} dx \\ &= S_2 + \lim_{b \rightarrow \infty} -\frac{1}{6x^6} \Big|_2^b = S_2 + \lim_{b \rightarrow \infty} -\left(\frac{1}{6b^6} - \frac{1}{384} \right) \\ &\approx 1.0078125 + 0.0026042 \approx 1.0104167. \end{aligned}$$

So

$$1.0080411 \leq S \leq 1.0104167.$$

10.4.40

a. $S \approx S_5 = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} \approx 1.185662.$

b. $R_5 < \int_5^\infty \frac{dx}{x^3} = \lim_{b \rightarrow \infty} \int_5^b x^{-3} dx = \lim_{b \rightarrow \infty} -\frac{1}{2x^2} \Big|_5^b = \lim_{b \rightarrow \infty} -\left(\frac{1}{2b^2} - \frac{1}{2(25)}\right) = \frac{1}{50} = 0.02.$

c.

$$\begin{aligned} L_5 &= S_5 + \int_6^\infty \frac{dx}{x^3} = S_5 + \lim_{b \rightarrow \infty} \int_6^b x^{-3} dx \\ &= S_5 + \lim_{b \rightarrow \infty} -\frac{1}{2x^2} \Big|_6^b = S_5 + \lim_{b \rightarrow \infty} -\left(\frac{1}{2b^2} - \frac{1}{72}\right) \\ &\approx 1.185662 + 0.013888889 \approx 1.1995509 \end{aligned}$$

$$\begin{aligned} U_5 &= S_5 + \int_5^\infty \frac{dx}{x^3} = S_5 + \lim_{b \rightarrow \infty} -\frac{1}{2x^2} \Big|_5^b \\ &= S_5 + \lim_{b \rightarrow \infty} -\left(\frac{1}{2b^2} - \frac{1}{2(25)}\right) = S_5 + \frac{1}{50} \\ &\approx 1.1856620 + 0.02 \approx 1.2056620. \end{aligned}$$

So

$$1.1995509 \leq S \leq 1.205662.$$

10.4.41

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x^6} dx = \frac{1}{5n^5}.$

b. We solve $\frac{1}{5n^5} < 10^{-3}$ to get $n = 3.$

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x^6} dx = S_n + \frac{1}{5(n+1)^5},$ and $U_n = S_n + \int_n^\infty \frac{1}{x^6} dx = S_n + \frac{1}{5n^5}.$

d. $S_{10} \approx 1.017341512,$ so $L_{10} \approx 1.017341512 + \frac{1}{5 \cdot 11^5} \approx 1.017342754,$ and $U_{10} \approx 1.017341512 + \frac{1}{5 \cdot 10^5} \approx 1.017343512.$

10.4.42

a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{x \ln^2 x} dx = \frac{1}{\ln n}.$

b. We solve $\frac{1}{\ln n} < 10^{-3}$ to get $n = e^{1000} \approx 10^{434}.$

c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{x \ln^2 x} dx = S_n + \frac{1}{\ln(n+1)},$ and $U_n = S_n + \int_n^\infty \frac{1}{x \ln^2 x} dx = S_n + \frac{1}{\ln n}.$

d. $S_{10} = \sum_{k=2}^{11} \frac{1}{k \ln^2 k} \approx 1.700396385,$ so $L_{10} \approx 1.700396385 + \frac{1}{\ln 12} \approx 2.102825989,$ and $U_{10} \approx 1.700396385 + \frac{1}{\ln 11} \approx 2.117428776.$

10.4.43

- a. The remainder R_n is bounded by $\int_n^\infty \frac{1}{3^x} dx = \frac{1}{3^n \ln(3)}$.
- b. We solve $\frac{1}{3^n \ln(3)} < 10^{-3}$ to obtain $n = 7$.
- c. $L_n = S_n + \int_{n+1}^\infty \frac{1}{3^x} dx = S_n + \frac{1}{3^{n+1} \ln(3)}$, and $U_n = S_n + \int_n^\infty \frac{1}{3^x} dx = S_n + \frac{1}{3^n \ln(3)}$.
- d. $S_{10} \approx 0.4999915325$, so $L_{10} \approx 0.4999915325 + \frac{1}{3^{11} \ln 3} \approx 0.4999966708$, and $U_{10} \approx 0.4999915325 + \frac{1}{3^{10} \ln 3} \approx 0.5000069475$.

10.4.44

- a. The remainder R_n is bounded by $\int_n^\infty x e^{-x^2} dx = \frac{1}{2e^{n^2}}$.
- b. We solve $\frac{1}{2e^{n^2}} < 10^{-3}$ to get $n = 3$.
- c. $L_n = S_n + \int_{n+1}^\infty x e^{-x^2} dx = S_n + \frac{1}{2e^{(n+1)^2}}$, and $U_n = S_n + \int_n^\infty x e^{-x^2} dx = S_n + \frac{1}{2e^{n^2}}$.
- d. $S_{10} \approx 0.4048813986$, so $L_{10} \approx 0.4048813986 + \frac{1}{2e^{11^2}} \approx 0.4048813986$, and $U_{10} \approx 0.4048813986 + \frac{1}{2e^{10^2}} \approx 0.4048813986$.

10.4.45 We first try to determine what n will suffice. We need $R_n < \frac{1}{10^4}$. We have $R_n = \int_n^\infty \frac{dx}{x^7} = \frac{1}{6n^6}$. Setting this less than $\frac{1}{10^4}$ and solving for n we have $n > \left(\frac{10^4}{6}\right)^{1/6} \approx 3.44$, so $n = 4$ should suffice. We have

$$S_4 = 1 + \frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{4^7} \approx 1.0083.$$

10.4.46 We first try to determine what n will suffice. We need $R_n < \frac{1}{10^3}$. We have $R_n = \int_n^\infty \frac{dx}{(3x+2)^2} = \frac{1}{9n+6}$. Setting this less than $\frac{1}{10^3}$ and solving for n we have $9n+6 > 10^3$, so $n > \frac{10^3-6}{9} \approx 110.4$. So we need $n = 111$.

We have

$$S_{111} = \sum_{k=1}^{111} \frac{1}{(3k+2)^2} \approx 0.08944$$

10.4.47

- a. False. This is a geometric series.
- b. True. If we reindex the series, we obtain $\sum_{k=1}^\infty \frac{1}{\sqrt{k}}$, which is a p -series with $p = \frac{1}{2}$.
- c. False. Both series converge, but not necessarily to the same value.
- d. True. The partial sums are an increasing sequence.
- e. False. Let $p = 1.0005$; then $-p + .001 = -(p - .001) = -0.9995$, so that $\sum k^{-p}$ converges (p -series) but $\sum k^{-p+.001}$ diverges.

f. False. Let $a_k = \frac{1}{k}$, the harmonic series.

10.4.48 $\lim_{k \rightarrow \infty} \ln \left(\frac{2k^6}{1+k^6} \right) = \ln 2 \neq 0$, so the given series diverges by the Divergence Test.

10.4.49 This is a geometric series with $r = \frac{1}{e}$ and $\left| \frac{1}{e} \right| < 1$, so the series converges the Geometric Series Test.

10.4.50 Diverges by the Divergence Test because $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \sqrt{\frac{k+1}{k}} = 1 \neq 0$.

10.4.51 Converges by the Integral Test because $\int_1^\infty \frac{1}{(3x+1)(3x+4)} dx = \int_1^\infty \frac{1}{3(3x+1)} - \frac{1}{3(3x+4)} dx = \lim_{b \rightarrow \infty} \int_1^b \left(\frac{1}{3(3x+1)} - \frac{1}{3(3x+4)} \right) dx = \lim_{b \rightarrow \infty} \frac{1}{9} \left(\ln \left(\frac{3x+1}{3x+4} \right) \right) \Big|_1^b = \lim_{b \rightarrow \infty} = -\frac{1}{9} \cdot \ln(4/7) \approx 0.06217 < \infty$.

10.4.52 Converges by the Integral Test because $\int_0^\infty \frac{10}{x^2+9} dx = \frac{10}{3} \lim_{b \rightarrow \infty} \left(\tan^{-1}(x/3) \Big|_0^b \right) = \frac{10}{3} \frac{\pi}{2} \approx 5.236 < \infty$.

10.4.53 Diverges by the Divergence Test because $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \frac{k}{\sqrt{k^2+1}} = 1 \neq 0$.

10.4.54 Converges because it is the sum of two geometric series. In fact, $\sum_{k=1}^\infty \frac{2^k+3^k}{4^k} = \sum_{k=1}^\infty (2/4)^k + \sum_{k=1}^\infty (3/4)^k = \frac{1/2}{1-(1/2)} + \frac{3/4}{1-(3/4)} = 1+3=4$.

10.4.55 We use the Integral Test. The function $f(x) = \frac{4}{x\sqrt{\ln x}}$ is positive, decreasing, and continuous for $x \geq 3$. The relevant integral is

$$\lim_{b \rightarrow \infty} \int_3^b \frac{4}{x\sqrt{\ln x}} dx = 8 \lim_{b \rightarrow \infty} \sqrt{\ln x} \Big|_3^b = 8 \lim_{b \rightarrow \infty} (\sqrt{\ln b} - \sqrt{\ln 3}) = \infty.$$

The given series therefore diverges by the Integral Test.

10.4.56 Note that

$$\frac{1}{k^2+7k+12} = \frac{1}{(k+3)(k+4)} = \frac{1}{k+3} - \frac{1}{k+4}.$$

The sequence of partial sums telescopes:

$$S_n = \left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) + \cdots + \left(\frac{1}{n+3} - \frac{1}{n+4} \right) = \frac{1}{4} - \frac{1}{n+4}.$$

The series is convergent with sum $S = \lim_{n \rightarrow \infty} \left(\frac{1}{4} - \frac{1}{n+4} \right) = \frac{1}{4}$.

10.4.57 The series can be written as $\sum_{k=1}^\infty \left(\frac{6}{5} \right)^k$, which is a divergent geometric series because $r = \frac{6}{5} > 1$.

10.4.58 The series diverges by the Divergence Test. We have

$$\lim_{k \rightarrow \infty} \left(\frac{k+6}{k} \right)^k = \lim_{k \rightarrow \infty} \left(1 + \frac{6}{k} \right)^k = e^6 \neq 0.$$

10.4.59 Reindexing gives $\sum_{k=4}^{\infty} \frac{3}{(k-3)^4} = 3 \sum_{k=1}^{\infty} \frac{1}{k^4}$, which is convergent because $\sum_{k=1}^{\infty} \frac{1}{k^4}$ is a convergent p -series with $p = 4$ by the p -series test.

10.4.60 The series diverges by the Divergence Test. Note that

$$\lim_{k \rightarrow \infty} \frac{3^k}{k^2 + 1} = \lim_{k \rightarrow \infty} \frac{3^k \ln 3}{2k} = \lim_{k \rightarrow \infty} \frac{3^k \ln^2 3}{2} = \infty \neq 0.$$

10.4.61 This series can be written as $2 \sum_{k=4}^{\infty} \frac{1}{k^2}$, so it converges, because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series with $p = 2$.

10.4.62 This series can be written as $\sum_{k=1}^{\infty} \frac{1}{2k+1}$. We use the Integral Test. Note that $f(x) = \frac{1}{2x+1}$ is positive, continuous, and decreasing for $x \geq 1$. The relevant integral is

$$\int_1^{\infty} \frac{dx}{2x+1} = \frac{1}{2} \lim_{b \rightarrow \infty} \ln(2x+1) \Big|_1^b = \frac{1}{2} \lim_{b \rightarrow \infty} (\ln(2b+1) - \ln 3) = \infty.$$

The given integral therefore diverges by the Integral Test.

10.4.63 This can be written in the form $9 \sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k$ so it is convergent because $\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k$ is a convergent geometric series. In fact, its value is $9 \left(\frac{3/5}{1-3/5}\right) = 9 \left(\frac{3}{2}\right) = \frac{27}{2}$.

10.4.64

a. In order for the series to converge, the integral $\int_2^{\infty} \frac{1}{x(\ln x)^p} dx$ must exist. But

$$\int \frac{1}{x(\ln x)^p} dx = \frac{1}{1-p} (\ln x)^{1-p},$$

so in order for this improper integral to exist, we must have that $1-p < 0$ or $p > 1$.

b. The series converges faster for $p = 3$ because the terms of the series get smaller faster.

10.4.65

a. Note that $\int \frac{1}{x \ln x (\ln \ln x)^p} dx = \frac{1}{1-p} (\ln \ln x)^{1-p}$, and thus the improper integral with bounds n and ∞ exists only if $p > 1$ because $\ln \ln x > 0$ for $x > e$. So this series converges for $p > 1$.

b. For large values of z , clearly $\sqrt{z} > \ln z$, so that $z > (\ln z)^2$. Write $z = \ln x$; then for large x , $\ln x > (\ln \ln x)^2$; multiplying both sides by $x \ln x$ we have that $x \ln^2 x > x \ln x (\ln \ln x)^2$, so that the first series converges faster because the terms get smaller faster.

10.4.66 $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges by the Integral Test, because $\int_2^{\infty} \frac{1}{x \ln x} = \lim_{b \rightarrow \infty} \left(\ln \ln x \Big|_2^b \right) = \infty$.

10.4.67 To approximate the sequence for $\zeta(x)$, note that the remainder R_n after n terms is bounded by

$$\int_n^{\infty} \frac{1}{z^x} dz = \frac{1}{x-1} n^{1-x}.$$

For $x = 3$, if we wish to approximate the value to within 10^{-3} , we must solve $\frac{1}{2}n^{-2} < 10^{-3}$, so that $n = 23$,

and $\sum_{k=1}^{23} \frac{1}{k^3} \approx 1.201151926$. The true value is ≈ 1.202056903 .

For $x = 5$, if we wish to approximate the value to within 10^{-3} , we must solve $\frac{1}{4}n^{-4} < 10^{-3}$, so that $n = 4$,

and $\sum_{k=1}^4 \frac{1}{k^5} \approx 1.036341789$. The true value is ≈ 1.036927755 .

10.4.68

- a. Starting with $\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x$, substitute $k\theta$ for x :

$$\begin{aligned}\cot^2(k\theta) &< \frac{1}{k^2\theta^2} < 1 + \cot^2(k\theta), \\ \sum_{k=1}^n \cot^2(k\theta) &< \sum_{k=1}^n \frac{1}{k^2\theta^2} < \sum_{k=1}^n (1 + \cot^2(k\theta)), \\ \sum_{k=1}^n \cot^2(k\theta) &< \frac{1}{\theta^2} \sum_{k=1}^n \frac{1}{k^2} < n + \sum_{k=1}^n \cot^2(k\theta).\end{aligned}$$

Note that the identity is valid because we are only summing for k up to n , so that $k\theta < \frac{\pi}{2}$.

- b. Substitute $\frac{n(2n-1)}{3}$ for the sum, using the identity:

$$\begin{aligned}\frac{n(2n-1)}{3} &< \frac{1}{\theta^2} \sum_{k=1}^n \frac{1}{k^2} < n + \frac{n(2n-1)}{3}, \\ \theta^2 \frac{n(2n-1)}{3} &< \sum_{k=1}^n \frac{1}{k^2} < \theta^2 \frac{n(2n+2)}{3}, \\ \frac{n(2n-1)\pi^2}{3(2n+1)^2} &< \sum_{k=1}^n \frac{1}{k^2} < \frac{n(2n+2)\pi^2}{3(2n+1)^2}.\end{aligned}$$

- c. By the Squeeze Theorem, if the expressions on either end have equal limits as $n \rightarrow \infty$, the expression in the middle does as well, and its limit is the same. The expression on the left is

$$\pi^2 \frac{2n^2 - n}{12n^2 + 12n + 3} = \pi^2 \frac{2 - n^{-1}}{12 + 12n^{-1} + 3n^{-2}},$$

which has a limit of $\frac{\pi^2}{6}$ as $n \rightarrow \infty$. The expression on the right is

$$\pi^2 \frac{2n^2 + 2n}{12n^2 + 12n + 3} = \pi^2 \frac{2 + 2n^{-1}}{12 + 12n^{-1} + 3n^{-3}},$$

which has the same limit. Thus $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

10.4.69 $\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$, splitting the series into even and odd terms. But $\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}$. Thus $\frac{\pi^2}{6} = \frac{1}{4} \cdot \frac{\pi^2}{6} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$, so that the sum in question is $\frac{3\pi^2}{24} = \frac{\pi^2}{8}$.

10.4.70

- a. The lateral surface area of the cake is

$$2\pi + 2\pi \left(\frac{1}{2}\right) + 2\pi \left(\frac{1}{3}\right) + \cdots = 2\pi \sum_{k=1}^{\infty} \frac{1}{k},$$

which diverges because the Harmonic Series diverges.

- b. The volume of the cake is

$$\pi + \pi \left(\frac{1}{2}\right)^2 + \pi \left(\frac{1}{3}\right)^2 + \cdots = \pi \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^3}{6}.$$

- c. The cake has infinite surface area and yet it has finite volume.

10.4.71 Let $S_n = \sum_{k=1}^n \frac{1}{\sqrt{k}}$. Then this looks like a left Riemann sum for the function $y = \frac{1}{\sqrt{x}}$ on $[1, n+1]$.

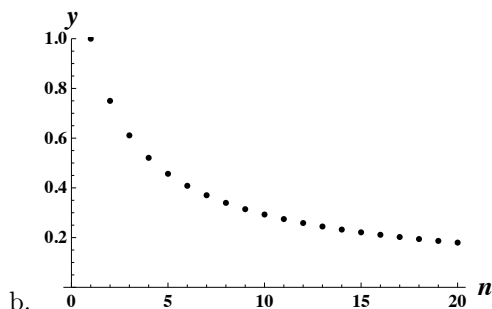
Because each rectangle lies above the curve itself, we see that S_n is bounded below by the integral of $\frac{1}{\sqrt{x}}$ on $[1, n+1]$. Now,

$$\int_1^{n+1} \frac{1}{\sqrt{x}} dx = \int_1^{n+1} x^{-1/2} dx = 2\sqrt{x} \Big|_1^{n+1} = 2\sqrt{n+1} - 2.$$

This integral diverges as $n \rightarrow \infty$, so the series does as well by the bound above.

10.4.72

- a. $\{F_n\}$ is a decreasing sequence because each term in F_n is smaller than the corresponding term in F_{n-1} and thus the sum of terms in F_n is smaller than the sum of terms in F_{n-1} .



- c. It appears that $\lim_{n \rightarrow \infty} F_n = 0$.

10.4.73

a. $x_1 = \sum_{k=2}^2 \frac{1}{k} = \frac{1}{2}$, $x_2 = \sum_{k=3}^4 \frac{1}{k} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, $x_3 = \sum_{k=4}^6 \frac{1}{k} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60}$.

- b. x_n has n terms. Each term is bounded below by $\frac{1}{2n}$ and bounded above by $\frac{1}{n+1}$. Thus $x_n \geq n \cdot \frac{1}{2n} = \frac{1}{2}$, and $x_n \leq n \cdot \frac{1}{n+1} < n \cdot \frac{1}{n} = 1$.

- c. The right Riemann sum for $\int_1^2 \frac{dx}{x}$ using n subintervals has n rectangles of width $\frac{1}{n}$; the right edges of those rectangles are at $1 + \frac{i}{n} = \frac{n+i}{n}$ for $i = 1, 2, \dots, n$. The height of such a rectangle is the value of $\frac{1}{x}$ at the right endpoint, which is $\frac{n}{n+i}$. Thus the area of the rectangle is $\frac{1}{n} \cdot \frac{n}{n+i} = \frac{1}{n+i}$. Adding up over all the rectangles gives x_n .

- d. The limit $\lim_{n \rightarrow \infty} x_n$ is the limit of the right Riemann sum as the width of the rectangles approaches zero.

This is precisely $\int_1^2 \frac{dx}{x} = \ln x \Big|_1^2 = \ln 2$.

10.4.74

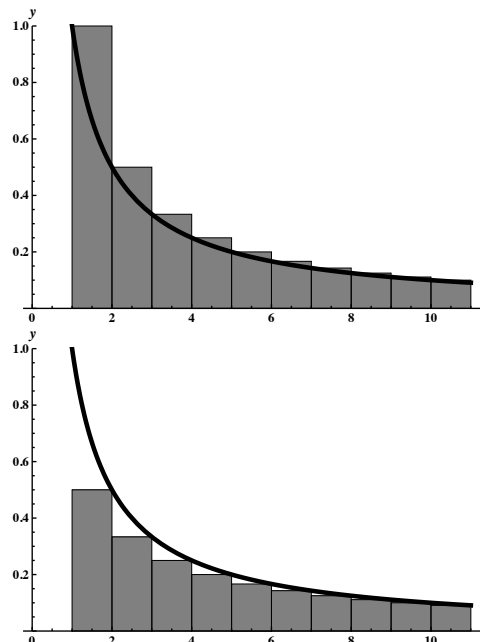
The first diagram is a left Riemann sum for $f(x) = \frac{1}{x}$ on the interval $[1, 11]$ (we assume $n = 10$ for purposes of drawing a graph). The area under the curve is $\int_1^{n+1} \frac{1}{x} dx = \ln(n+1)$, and the sum of the areas of the rectangles is obviously $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Thus

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

- a. The second diagram is a right Riemann sum for the same function on the same interval. Considering only $[1, n]$, we see that, comparing the area under the curve and the sum of the areas of the rectangles, that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n$$

Adding 1 to both sides gives the desired inequality.



- b. According to part (a), $\ln(n+1) < S_n$ for $n = 1, 2, 3, \dots$, so that $E_n = S_n - \ln(n+1) > 0$.
- c. Using the second figure above and assuming $n = 9$, the final rectangle corresponds to $\frac{1}{n+1}$, and the area under the curve between $n+1$ and $n+2$ is clearly $\ln(n+2) - \ln(n+1)$.
- d. $E_{n+1} - E_n = S_{n+1} - \ln(n+2) - (S_n - \ln(n+1)) = \frac{1}{n+1} - (\ln(n+2) - \ln(n+1))$. But this is positive because of the bound established in part (c).
- e. Using part (a), $E_n = S_n - \ln(n+1) < 1 + \ln(n) - \ln(n+1) < 1$.
- f. E_n is a monotone (increasing) sequence that is bounded, so it has a limit.
- g. The first ten values (E_1 through E_{10}) are

$$0.3068528194, 0.401387711, 0.447038972, 0.473895421, 0.491573864, \\ 0.504089851, 0.513415601, 0.520632565, 0.526383161, 0.531072981.$$

$$E_{1000} \approx 0.576716082.$$

- h. For $S_n > 10$ we need $10 - 0.5772 = 9.4228 > \ln(n+1)$. Solving for n gives $n \approx 12366.16$, so $n = 12367$.

10.4.75

- a. Dividing both sides of the recurrence equation by f_n gives $\frac{f_{n+1}}{f_n} = 1 + \frac{f_{n-1}}{f_n}$. Let the limit of the ratio of successive terms be L . Taking the limit of the previous equation gives $L = 1 + \frac{1}{L}$. Thus $L^2 = L + 1$, so $L^2 - L - 1 = 0$. The quadratic formula gives $L = \frac{1 \pm \sqrt{1 - 4 \cdot (-1)}}{2}$, but we know that all the terms are positive, so we must have $L = \frac{1 + \sqrt{5}}{2} = \phi \approx 1.618$.

b. Write the recurrence in the form $f_{n-1} = f_{n+1} - f_n$ and divide both sides by f_{n+1} . Then we have $\frac{f_{n-1}}{f_{n+1}} = 1 - \frac{f_n}{f_{n+1}}$. Taking the limit gives $1 - \frac{1}{\phi}$ on the right-hand side.

c. Consider the harmonic series with the given groupings, and compare it with the sum of $\frac{f_{k-1}}{f_{k+1}}$ as shown.

The first three terms match exactly. The sum of the next two are $\frac{1}{4} + \frac{1}{5} > \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$. The sum of the next three are $\frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$. The sum of the next five are $\frac{1}{9} + \cdots + \frac{1}{13} > 5 \cdot \frac{1}{13} = \frac{5}{13}$.

Thus the harmonic series is bounded below by the series $\sum_{k=1}^{\infty} \frac{f_{k-1}}{f_{k+1}}$.

d. The result above implies that the harmonic series diverges, because the series $\sum_{k=1}^{\infty} \frac{f_{k-1}}{f_{k+1}}$ diverges, since its general term has limit $1 - \frac{1}{\phi} \neq 0$.

10.4.76

a. Note that the center of gravity of any stack of dominoes is the average of the locations of their centers. Define the midpoint of the zeroth (top) domino to be $x = 0$, and stack additional dominoes down and to its right (to increasingly positive x -coordinates.) Let $m(n)$ be the x -coordinate of the midpoint of the n^{th} domino. Then in order for the stack not to fall over, the left edge of the n^{th} domino must be placed directly under the center of gravity of dominos 0 through $n - 1$, which is $\frac{1}{n} \sum_{i=0}^{n-1} m(i)$, so

that $m(n) = 1 + \frac{1}{n} \sum_{i=0}^{n-1} m(i)$. Claim that in fact $m(n) = \sum_{k=1}^n \frac{1}{k}$. Use induction. This is certainly true for $n = 1$. Note first that $m(0) = 0$, so we can start the sum at 1 rather than at 0. Now, $m(n) = 1 + \frac{1}{n} \sum_{i=1}^{n-1} m(i) = 1 + \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j=1}^i \frac{1}{j}$. Now, 1 appears $n - 1$ times in the double sum, 2 appears

$n - 2$ times, and so forth, so we can rewrite this sum as $m(n) = 1 + \frac{1}{n} \sum_{i=1}^{n-1} \frac{n-i}{i} = 1 + \frac{1}{n} \sum_{i=1}^{n-1} \left(\frac{n}{i} - 1 \right) = 1 + \frac{1}{n} \left(n \sum_{i=1}^{n-1} \frac{1}{i} - (n-1) \right) = \sum_{i=1}^{n-1} \frac{1}{i} + 1 - \frac{n-1}{n} = \sum_{i=1}^n \frac{1}{i}$, and we are done by induction (noting that

the statement is clearly true for $n = 0, n = 1$). Thus the maximum overhang is $\sum_{k=2}^n \frac{1}{k}$.

b. For an infinite number of dominos, because the overhang is the harmonic series, the distance is potentially infinite.

10.5 Comparison Tests

10.5.1 Given a series of positive terms $\sum a_k$ that you suspect converges, find a series $\sum b_k$ that you know converges, for which $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = L$ where $L \geq 0$ is a finite number. If you are successful, you will have shown that the series $\sum a_k$ converges.

Given a series of positive terms $\sum a_k$ that you suspect diverges, find a series $\sum b_k$ that you know diverges, for which $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = L$ where $L > 0$ (including the case $L = \infty$). If you are successful, you will have shown that $\sum a_k$ diverges.

10.5.2 The difference between successive partial sums is a term in the sequence. Because the terms are positive, differences between successive partial sums are as well, so the sequence of partial sums is increasing.

10.5.3 $\sum_{k=1}^{\infty} \frac{1}{k^2}.$

10.5.4 No. The problem is that $k^4 - 1 < k^4$, so $\frac{1}{k^4 - 1} > \frac{1}{k^4}$. Having series whose terms are smaller that converges doesn't tell you anything about the series whose terms are bigger.

10.5.5 $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k.$

10.5.6 Note that $\sqrt{k} - 1 < \sqrt{k}$ for $k \geq 2$, so $\frac{1}{\sqrt{k} - 1} > \frac{1}{\sqrt{k}}$ for $k \geq 2$. Because $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is a divergent p -series, $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}}$ diverges and it follows that $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k} - 1}$ diverges by the Comparison Theorem.

10.5.7 $\sum_{k=1}^{\infty} \frac{1}{k}.$

10.5.8 We have $\lim_{k \rightarrow \infty} \frac{\frac{k+1}{3k^3+2}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{k^3+k^2}{3k^3+2} = \frac{1}{3}$. Because $0 < \frac{1}{3} < \infty$, and because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series, it follows that the series $\sum_{k=1}^{\infty} \frac{k+1}{3k^3+2}$ converges by the Limit Comparison Test.

10.5.9 $\frac{1}{k^2+4} < \frac{1}{k^2}$, and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$ converges as well, by the Comparison Test.

10.5.10 Use the Limit Comparison Test with $\left\{\frac{1}{k^2}\right\}$. The ratio of the terms of the two series is $\frac{k^4+k^3-k^2}{k^4+4k^2-3}$ which has limit 1 as $k \rightarrow \infty$. Because the comparison series converges, the given series does as well.

10.5.11 Use the Limit Comparison Test with $\left\{\frac{1}{k}\right\}$. The ratio of the terms of the two series is $\frac{k^3-k}{k^3+4}$ which has limit 1 as $k \rightarrow \infty$. Because the comparison series diverges, the given series does as well.

10.5.12 Use the Limit Comparison Test with $\left\{\frac{1}{k}\right\}$. The ratio of the terms of the two series is $\frac{0.0001k}{k+4}$ which has limit 0.0001 as $k \rightarrow \infty$. Because the comparison series diverges, the given series does as well.

10.5.13 Because $\frac{\sqrt{k}}{k^2+3} < \frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a convergent p -series, $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+3}$ converges by the Comparison Test.

10.5.14 Because $\frac{1}{5^k+3} < \frac{1}{5^k}$ and $\sum_{k=1}^{\infty} \frac{1}{5^k}$ is a convergent geometric series, $\sum_{k=1}^{\infty} \frac{1}{5^k+3}$ converges by the Comparison Test.

10.5.15 We use the Limit Comparison Test with the convergent series $\sum_{k=1}^{\infty} \left(\frac{4}{5}\right)^k$. We have

$$\lim_{k \rightarrow \infty} \frac{4^k}{\frac{5^k - 3}{\left(\frac{4}{5}\right)^k}} = \lim_{k \rightarrow \infty} \frac{5^k}{5^k - 3} = \lim_{k \rightarrow \infty} \frac{1}{1 - \frac{3}{5^k}} = 1.$$

Because $0 < 1 < \infty$, we have that the given series is convergent by the Limit Comparison Test.

10.5.16 Use the Comparison Test. Note that $\sin^2 k \leq 1$ for all k , so $\frac{\sin^2 k}{k^2} \leq \frac{1}{k^2}$ for all k . Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so does the given series.

10.5.17 For all k , $\frac{1}{k^{3/2} + 1} < \frac{1}{k^{3/2}}$. The series whose terms are $\frac{1}{k^{3/2}}$ is a p -series which converges, so the given series converges as well by the Comparison Test.

10.5.18 Note that $k \cdot 10^k > 10^k$ for $k \geq 1$. Taking reciprocals of each side, we have $\frac{1}{k \cdot 10^k} \leq \frac{1}{10^k}$ for $k \geq 1$. Because the series $\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k$ converges (it is a geometric series with $r = \frac{1}{10}$), the given series converges as well by the Comparison Test.

10.5.19 Note that $\frac{1 + \cos^2 k}{k - 3} > \frac{1}{k - 3} > \frac{1}{k}$ for $k \geq 4$. Because $\sum_{k=1}^{\infty} \frac{1}{k}$ is the divergent harmonic series, we have that $\sum_{k=4}^{\infty} \frac{1 + \cos^2 k}{k - 3}$ diverges as well by the Comparison Test.

10.5.20 We use the Limit Comparison Test with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{6^k}$. We have

$$\lim_{k \rightarrow \infty} \frac{\frac{k^2 + k + 2}{6^k(k^2 + 1)}}{\frac{1}{6^k}} = \lim_{k \rightarrow \infty} \frac{k^2 + k + 2}{k^2 + 1} = 1.$$

Because $0 < 1 < \infty$, the given series converges by the Limit Comparison Test.

10.5.21 We use the Limit Comparison Test with the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{k \rightarrow \infty} \frac{\frac{21k^5 + 3k^3 + 28k^2 + 4}{8k^6 + 2k^3 - 1}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{21k^6 + 3k^4 + 28k^3 + 4k}{8k^6 + 2k^3 - 1} = \frac{21}{8}.$$

Because $0 < \frac{21}{8} < \infty$, the given series diverges by the Limit Comparison Test.

10.5.22 Use the Limit Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{k}$. The ratio of the terms of the two series is $k\sqrt{\frac{k}{k^3 + 1}} = \sqrt{\frac{k^3}{k^3 + 1}}$, which has limit 1 as $k \rightarrow \infty$. Because the comparison series diverges, the given series does as well by the Limit Comparison Test.

10.5.23 $\sin(1/k) > 0$ for $k \geq 1$, so we can apply the Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{k^2}$. $\sin(1/k) < 1$, so $\frac{\sin(1/k)}{k^2} < \frac{1}{k^2}$. Because the comparison series converges, the given series converges as well by the Comparison Test.

10.5.24 Use the Limit Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{3^k}$. The ratio of the terms of the two series is $\frac{3^k}{3^k - 2^k} = \frac{1}{1 - \left(\frac{2^k}{3^k}\right)}$, which has limit 1 as $k \rightarrow \infty$. Because the comparison series converges, the given series does as well by the Limit Comparison Test.

10.5.25 Use the Limit Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{k}$. The ratio of the terms of the two series is $\frac{k}{2k - \sqrt{k}} = \frac{1}{2 - 1/\sqrt{k}}$, which has limit $1/2$ as $k \rightarrow \infty$. Because the comparison series diverges, the given series does as well by the Limit Comparison Test.

10.5.26 $\frac{1}{k\sqrt{k+2}} < \frac{1}{k\sqrt{k}} = \frac{1}{k^{3/2}}$. Because the series whose terms are $\frac{1}{k^{3/2}}$ is a p -series with $p > 1$, it converges. Because the comparison series converges, the given series converges as well by the Comparison Test.

10.5.27 Note that $2 + (-1)^k \leq 3$ for $k \geq 1$. It follows that $\frac{2 + (-1)^k}{k^2} \leq \frac{3}{k^2}$ for $k \geq 1$. Because $\sum_{k=1}^{\infty} \frac{3}{k^2} = 3 \sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series, $\sum_{k=1}^{\infty} \frac{3}{k^2}$ converges. It follows that the given series converges by the Comparison Test.

10.5.28 Note that $2 + \sin k > 1$ for $k \geq 1$. It follows that $\frac{2 + \sin k}{k} > \frac{1}{k}$. Because $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, it follows that the given series diverges by the Comparison Test.

10.5.29 Use the Limit Comparison Test with $\frac{k^{2/3}}{k^{3/2}}$. The ratio of corresponding terms of the two series is

$$\frac{\sqrt[3]{k^2 + 1}}{\sqrt{k^3 + 1}} \cdot \frac{k^{3/2}}{k^{2/3}} = \frac{\sqrt[3]{k^2 + 1}}{\sqrt[3]{k^2}} \cdot \frac{\sqrt{k^3}}{\sqrt{k^3 + 1}},$$

which has limit 1 as $k \rightarrow \infty$. The comparison series is the series whose terms are $k^{2/3-3/2} = k^{-5/6}$, which is a p -series with $p < 1$, so it, and the given series, both diverge.

10.5.30 For all $k \geq 3$, $\frac{1}{(k \ln k)^2} < \frac{1}{k^2}$. Because the series whose terms are $\frac{1}{k^2}$ converges, the given series converges as well.

10.5.31 We use the Limit Comparison Test with the divergent series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{20}{\sqrt[3]{k} + \sqrt{k}}}{\frac{1}{\sqrt{k}}} = \lim_{k \rightarrow \infty} \frac{20k^{1/2}}{k^{1/3} + k^{1/2}} = \lim_{k \rightarrow \infty} \frac{20}{(1/k^{1/6}) + 1} = 20.$$

Because $0 < L < \infty$, we have that the given series diverges by the Limit Comparison Test.

10.5.32 We have $\frac{\sqrt{\ln k}}{k} \geq \frac{1}{k}$ for $k \geq 3$ and because $\sum_{k=1}^{\infty} \frac{1}{k}$ is the divergent harmonic series, $\sum_{k=1}^{\infty} \frac{\sqrt{\ln k}}{k}$ diverges as well by the Comparison Test.

10.5.33 We use the Limit Comparison Test with the divergent series $\sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{1}{2^{\ln k}}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{k}{2^{\ln k}} = \lim_{k \rightarrow \infty} \frac{e^{\ln k}}{2^{\ln k}} = \lim_{k \rightarrow \infty} \left(\frac{e}{2}\right)^{\ln k} = \infty.$$

Because $L = \infty$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, then $\sum_{k=1}^{\infty} \frac{1}{2^{\ln k}}$ diverges by part 3 of the Limit Comparison Test.

10.5.34 We use the Limit Comparison Test with the convergent p -series $\sum_{k=2}^{\infty} \frac{1}{k^3}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{\ln^2 k}{k^4}}{\frac{1}{k^3}} = \lim_{k \rightarrow \infty} \frac{\ln^2 k}{k} = \lim_{k \rightarrow \infty} \frac{2 \ln k}{k} = \lim_{k \rightarrow \infty} \frac{2}{k} = 0.$$

Because $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, the given series converges by part 2 of the Limit Comparison Test.

10.5.35 We use the Limit Comparison Test with the convergent p -series $\sum_{k=2}^{\infty} \frac{1}{k^{5/4}}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{1}{4^{\ln k}}}{\frac{1}{k^{5/4}}} = \lim_{k \rightarrow \infty} \frac{k^{5/4}}{4^{\ln k}} = \lim_{k \rightarrow \infty} \frac{e^{\frac{5}{4} \ln k}}{4^{\ln k}} = \lim_{k \rightarrow \infty} \left(\frac{e^{5/4}}{4}\right)^{\ln k} = 0.$$

The last limit follows from the fact that $\frac{e^{5/4}}{4} \approx 0.87 < 1$. Because $L = 0$ and $\sum_{k=1}^{\infty} \frac{1}{k^{5/4}}$ converges, the given series converges by the Limit Comparison Test.

10.5.36 We use the Limit Comparison Test with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{\pi^k}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{5}{\pi^k + e^k - 2}}{\frac{1}{\pi^k}} = \lim_{k \rightarrow \infty} \frac{5\pi^k}{\pi^k + e^k - 2} = \lim_{k \rightarrow \infty} \frac{5}{1 + (e/\pi)^k - 2/\pi^k} = 5.$$

Because $0 < L < \infty$, the given series converges by the Limit Comparison Test.

10.5.37

- False. For example, let $\{a_k\}$ be all zeros, and $\{b_k\}$ be all 1's.
- True. This is a result of the Comparison Test.
- True. Both of these statements follow from the Comparison Test.

d. True. The limit of the ratio is 1 in this case, so the two series both converge.

10.5.38

Method 1: We use the Comparison Test with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{(e^2)^k}$. Note that $\frac{e^{2k}}{e^{4k} + 1} \leq \frac{e^{2k}}{e^{4k}} = \frac{1}{e^{2k}}$. So $\sum_{k=1}^{\infty} \frac{e^{2k}}{e^{4k} + 1}$ converges by the Comparison Test because $\sum_{k=1}^{\infty} \frac{1}{(e^2)^k}$ converges.

Method 2: Let $f(x) = \frac{e^{2x}}{e^{4x} + 1}$. Then f is continuous and positive for $x \geq 1$, and $f'(x) = \frac{2e^{2x}(1 - e^{4x})}{(e^{4x} + 1)^2} < 0$ for $x \geq 1$, so f is decreasing for $x \geq 1$. We have

$$\int_1^{\infty} \frac{e^{2x}}{e^{4x} + 1} dx = \lim_{b \rightarrow \infty} \frac{1}{2} \tan^{-1} e^{2x} \Big|_1^b = \frac{1}{2} \left(\frac{\pi}{2} - \tan^{-1} e^2 \right).$$

Because the integral converges, the given series also converges by the Integral Test.

10.5.39 Method 1: We use the Comparison Test with the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Note that

$\frac{1}{k^2 + 2k + 1} \leq \frac{1}{k^2}$. Thus the given series converges by the Comparison Test.

Method 2: We rewrite the summand. We have $\sum_{k=1}^{\infty} \frac{1}{k^2 + 2k + 1} = \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} = \sum_{k=2}^{\infty} \frac{1}{k^2}$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent series, $\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges as well.

10.5.40 Use the Divergence Test: $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \left(1 - \frac{1}{k}\right)^k = \frac{1}{e} \neq 0$, so the given series diverges.

10.5.41 Use the Divergence Test: $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \left(1 + \frac{2}{k}\right)^k = e^2 \neq 0$, so the given series diverges.

10.5.42 We use the Limit Comparison Test with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{(e^3)^k}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{e^{2k} + k}{e^{5k} - k^2}}{\frac{1}{e^{3k}}} = \lim_{k \rightarrow \infty} \frac{e^{5k} + ke^{3k}}{e^{5k} - k^2} = \lim_{k \rightarrow \infty} \frac{1 + \frac{k}{e^{2k}}}{1 - \frac{k^2}{e^{5k}}} = 1.$$

Thus the given series converges by the Limit Comparison Test.

10.5.43 Use the Divergence Test: $\lim_{k \rightarrow \infty} \frac{k^2 + 2k + 1}{3k^2 + 1} = \frac{1}{3} \neq 0$, so the given series diverges.

10.5.44 Use the Limit Comparison Test with the series whose k th term is $\left(\frac{2}{e}\right)^k$. Note that

$$L = \lim_{k \rightarrow \infty} \frac{2^k}{e^k - 1} \cdot \frac{e^k}{2^k} = \lim_{k \rightarrow \infty} \frac{e^k}{e^k - 1} = 1.$$

The given series thus converges because $\sum_{k=1}^{\infty} \left(\frac{2}{e}\right)^k$ converges (because it is a geometric series with $r = \frac{2}{e} < 1$).

10.5.45 Use the Limit Comparison Test with the harmonic series. Note that $\lim_{k \rightarrow \infty} \frac{\frac{1}{\ln k}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{k}{\ln k} = \infty$, and because the harmonic series diverges, the given series does as well.

10.5.46 Use the Limit Comparison Test with the series whose k th term is $\frac{1}{5^k}$. Note that $\lim_{k \rightarrow \infty} \frac{1}{5^k - 1} \cdot \frac{5^k}{1} = 1$, and the series $\sum_{k=1}^{\infty} \frac{1}{5^k}$ converges because it is a geometric series with $r = \frac{1}{5}$. Thus, the given series also converges.

10.5.47 Use the Limit Comparison Test with the series whose k th term is $\frac{1}{k^{3/2}}$. Note that $\lim_{k \rightarrow \infty} \frac{1}{\sqrt{k^3 - k + 1}} \cdot \frac{\sqrt{k^3}}{1} = \lim_{k \rightarrow \infty} \sqrt{\frac{k^3}{k^3 - k + 1}} = \sqrt{1} = 1$, and the series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges because it is a p -series with $p = \frac{3}{2}$. Thus, the given series also converges.

10.5.48 Use the Limit Comparison Test with the series whose k th term is $\frac{1}{5^k}$. Note that $\lim_{k \rightarrow \infty} \frac{1}{5^k - 3^k} \cdot \frac{5^k}{1} = \lim_{k \rightarrow \infty} \frac{1}{1 - (3/5)^k} = 1$, and the series $\sum_{k=3}^{\infty} \frac{1}{5^k}$ converges because it is a geometric series with $r = \frac{1}{5}$. Thus, the given series also converges.

10.5.49 Use the Comparison Test. Note that $\frac{1}{k} + 2^{-k} > \frac{1}{k}$ for $k \geq 1$. Because the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so does the given series.

10.5.50 Use the Comparison Test with $\sum_{k=1}^{\infty} \frac{5}{k} = 5 \sum_{k=1}^{\infty} \frac{1}{k}$. Note that $\frac{5 \ln k}{k} > \frac{5}{k}$ for $k \geq 3$. Because $\sum_{k=3}^{\infty} \frac{1}{k}$ diverges, so does $5 \sum_{k=1}^{\infty} \frac{1}{k}$, and therefore so does the given series.

10.5.51 Use the Limit Comparison Test with the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^3}$. The ratio of corresponding terms is $\frac{k^{11}}{k^{11} + 3}$, which has limit 1 as $k \rightarrow \infty$. Therefore the given series converges by the Limit Comparison Test.

10.5.52 Use the Comparison Test:: $\frac{1}{k^2 \ln k} < \frac{1}{k^2}$ for $k \geq 3$. Because $\sum_{k=3}^{\infty} \frac{1}{k^2}$ converges, so does $\sum_{k=3}^{\infty} \frac{1}{k^2 \ln k}$, and therefore so does $\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$.

10.5.53 This is a p -series with exponent greater than 1, so it converges.

10.5.54 We use the Limit Comparison Test with the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{\tan^{-1} k}{k^2}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \tan^{-1} k = \frac{\pi}{2}.$$

Because $0 < L < \infty$, the given series is convergent.

10.5.55 $\ln\left(\frac{k+2}{k+1}\right) = \ln(k+2) - \ln(k+1)$, so this series telescopes. We get $\sum_{k=1}^n \ln\left(\frac{k+2}{k+1}\right) = \ln(n+2) - \ln 2$. Because $\lim_{n \rightarrow \infty} (\ln(n+2) - \ln 2) = \infty$, the sequence of partial sums diverges, so the given series is divergent.

10.5.56 Use the Divergence Test. Note that $\lim_{k \rightarrow \infty} k^{-1/k} = \lim_{k \rightarrow \infty} \frac{1}{\sqrt[k]{k}} = 1 \neq 0$, so the given series diverges.

10.5.57 For $k > 7$, $\ln k > 2$ so note that $\frac{1}{k^{\ln k}} < \frac{1}{k^2}$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series converges as well by the Comparison Test.

10.5.58 Use the Limit Comparison Test with the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Note that

$\frac{\sin^2(1/k)}{1/k^2} = \left(\frac{\sin(1/k)}{1/k}\right)^2$. Because $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, the limit of this expression is $1^2 = 1$ as $k \rightarrow \infty$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series does as well.

10.5.59 Use the Limit Comparison Test with the harmonic series. $\frac{\tan(1/k)}{1/k}$ has limit 1 as $k \rightarrow \infty$ because $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$. Thus the original series diverges.

10.5.60 We use the Limit Comparison Test with the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\sqrt{\frac{1}{2^{2k}} + \frac{1}{k^2}}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \sqrt{\frac{k^2}{4^k} + 1} = 1.$$

Therefore the given series diverges by the Limit Comparison Test.

10.5.61 Note that $\frac{1}{(2k+1) \cdot (2k+3)} = \frac{1}{2} \left(\frac{1}{2k+1} - \frac{1}{2k+3} \right)$. Thus this series telescopes.

$$\sum_{k=0}^n \frac{1}{(2k+1)(2k+3)} = \frac{1}{2} \sum_{k=0}^n \left(\frac{1}{2k+1} - \frac{1}{2k+3} \right) = \frac{1}{2} \left(-\frac{1}{2n+3} + 1 \right),$$

so the given series converges to $1/2$, because that is the limit of the sequence of partial sums.

10.5.62 This series is $\sum_{k=1}^{\infty} \frac{k-1}{k^2} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k^2} \right)$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, if the original series also converged, we would have that $\sum_{k=1}^{\infty} \frac{1}{k}$ converged, which is false. Thus the original series diverges.

10.5.63 Use the Limit Comparison Test: $\lim_{k \rightarrow \infty} \frac{a_k^2}{a_k} = \lim_{k \rightarrow \infty} a_k = 0$, because $\sum a_k$ converges. By the Limit Comparison Test part 2, the series $\sum a_k^2$ must converge as well.

10.5.64

- a. Use the Limit Comparison Test with the divergent harmonic series. Note that $\lim_{k \rightarrow \infty} \frac{\sin(1/k)}{1/k} = 1$, because $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. Because the comparison series diverges, the given series does as well.
- b. We use the Limit Comparison Test with the convergent series $\sum \frac{1}{k^2}$. Note that $\lim_{k \rightarrow \infty} \frac{(1/k) \sin(1/k)}{1/k^2} = \lim_{k \rightarrow \infty} \frac{\sin(1/k)}{1/k} = 1$, so the given series converges.

10.5.65 To prove case (2), assume $L = 0$ and that $\sum b_k$ converges. Because $L = 0$, for every $\varepsilon > 0$, there is some N such that for all $n > N$, $|\frac{a_k}{b_k}| < \varepsilon$. Take $\varepsilon = 1$; this then says that there is some N such that for all $n > N$, $0 < a_k < b_k$. By the Comparison Test, because $\sum b_k$ converges, so does $\sum a_k$. To prove case (3), because $L = \infty$, then $\lim_{k \rightarrow \infty} \frac{b_k}{a_k} = 0$, so by the argument above, we have $0 < b_k < a_k$ for sufficient large k . But $\sum b_k$ diverges, so by the Comparison Test, $\sum a_k$ does as well.

10.5.66

- a. Let P_n be the n^{th} partial product of the a_k : $P_n = \prod_{k=1}^n a_k$.

We have

$$\prod_{k=1}^n \frac{k}{k+1} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n}{n+1} = \frac{1}{n+1}.$$

This has limit 0 as $n \rightarrow \infty$.

- b. Because $\sum_{k=1}^{\infty} \ln a_k = L$, the n th partial sum S_n of $\sum_{k=1}^{\infty} \ln a_k$ converges to L . Then

$$e^{S_n} = e^{\ln a_1 + \ln a_2 + \cdots + \ln a_n} = e^{\ln a_1} e^{\ln a_2} \cdots e^{\ln a_n} = a_1 a_2 \cdots a_n.$$

Therefore $\lim_{n \rightarrow \infty} (a_1 a_2 \cdots a_n) = \lim_{n \rightarrow \infty} e^{S_n} = e^L$.

- c. Because $\sum_{k=0}^{\infty} \ln e^{1/2^k} = \sum_{k=1}^{\infty} \ln e^{1/2^{k-1}} = \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = \frac{1}{1 - \frac{1}{2}} = 2$, we must have $\prod_{k=0}^{\infty} e^{1/2^k} = e^2$.

10.5.67 The sum on the left is simply the left Riemann sum over n equal intervals between 0 and 1 for $f(x) = x^p$. The limit of the sum is thus $\int_0^1 x^p dx = \frac{1}{p+1} x^{p+1} \Big|_0^1 = \frac{1}{p+1}$, because p is positive.

10.6 Alternating Series

10.6.1 Because $S_{n+1} - S_n = a_{n+1}$ alternates signs.

10.6.2 Check that the terms of the series are nonincreasing in magnitude after some finite number of terms, and that the $\lim_{k \rightarrow \infty} a_k = 0$.

10.6.3 Because

$$S = S_{2n+1} + (a_{2n} - a_{2n+1}) + (a_{2n+2} - a_{2n+3}) + \cdots$$

and each term of the form $a_{2k} - a_{2k+1} > 0$, so that $S_{2n+1} < S$. Also

$$S = S_{2n} + (-a_{2n+1} + a_{2n+2}) + (-a_{2n+3} + a_{2n+4}) + \cdots$$

and each term of the form $-a_{2k+1} + a_{2k+2} < 0$, so that $S < S_{2n}$. Thus the sum of the series is trapped between the odd partial sums and the even partial sums.

10.6.4 The difference between L and S_n is bounded in magnitude by a_{n+1} .

10.6.5 The remainder is less than the first neglected term because

$$S - S_n = (-1)^{n+1}(a_{n+1} + (-a_{n+2} + a_{n+3}) + \cdots)$$

so that the sum of the series *after* the first disregarded term has the opposite sign from the first disregarded term.

10.6.6 The alternating harmonic series $\sum (-1)^k \frac{1}{k}$ converges, but not absolutely.

10.6.7 No. If the terms are positive, then the absolute value of each term is the term itself, so convergence and absolute convergence would mean the same thing in this context.

10.6.8 Yes. The series $\sum_{k=0}^{\infty} |(-0.5)^k| = \sum_{k=0}^{\infty} 0.5^k$ is also a convergent geometric series.

10.6.9 Yes. For example, $\sum \frac{(-1)^k}{k^3}$ converges absolutely and thus not conditionally (see the definition).

10.6.10 The series $\sum_{k=0}^{\infty} \left| \frac{(-1)^k}{k^p} \right| = \sum_{k=0}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$, which implies that $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^p}$ converges absolutely for $p > 1$. If $0 < p < 1$, the sequence $\left\{ \frac{1}{k^p} \right\}$ is decreasing and converges to 0, and $\sum_{k=0}^{\infty} \frac{1}{k^p}$ is a divergent p -series for $p \leq 1$. So $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^p}$ is conditionally convergent for $0 < p \leq 1$.

10.6.11 The terms of the series decrease in magnitude, and $\lim_{k \rightarrow \infty} \frac{1}{2k+1} = 0$, so the given series converges.

10.6.12 The terms of the series decrease in magnitude, and $\lim_{k \rightarrow \infty} \frac{1}{\sqrt{k}} = 0$, so the given series converges.

10.6.13 $\lim_{k \rightarrow \infty} \frac{k}{3k+2} = \frac{1}{3} \neq 0$, so the given series diverges.

10.6.14 $\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k} \right)^k = e \neq 0$, so the given series diverges.

10.6.15 The terms of the series decrease in magnitude, and $\lim_{k \rightarrow \infty} \frac{1}{k^3} = 0$, so the given series converges.

10.6.16 The terms of the series decrease in magnitude, and $\lim_{k \rightarrow \infty} \frac{1}{k^2+10} = 0$, so the given series converges.

10.6.17 The terms of the series decrease in magnitude, and $\lim_{k \rightarrow \infty} \frac{k^2}{k^3+1} = \lim_{k \rightarrow \infty} \frac{1/k}{1+1/k^3} = 0$, so the given series converges. In order to see that the terms decrease in magnitude, consider $f(x) = \frac{x^2}{x^3+1}$. Then $f'(x) = \frac{x(2-x^3)}{(1+x^3)^2}$, which is negative for $x \geq 2$.

10.6.18 The terms of the series eventually decrease in magnitude, because if $f(x) = \frac{\ln(x)}{x^2}$, then $f'(x) = \frac{x(1-2\ln x)}{x^4} = \frac{1-2\ln x}{x^3}$, which is negative for large enough x . Further, $\lim_{k \rightarrow \infty} \frac{\ln k}{k^2} = \lim_{k \rightarrow \infty} \frac{1/k}{2k} = \lim_{k \rightarrow \infty} \frac{1}{2k^2} = 0$. Thus the given series converges.

10.6.19 $\lim_{k \rightarrow \infty} \frac{k^2-1}{k^2+3} = 1$, so the terms of the series do not tend to zero and thus the given series diverges.

10.6.20 $\sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{5}\right)^k$. Note that $(1/5)^k$ is decreasing, and tends to zero as $k \rightarrow \infty$, so the given series converges.

10.6.21 $\lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right) = 1$, so the given series diverges.

10.6.22 Note that $\cos(\pi k) = (-1)^k$, and so the given series is alternating. Because $\lim_{k \rightarrow \infty} \frac{1}{k^2} = 0$ and $\frac{1}{k^2}$ is decreasing, the given series is convergent.

10.6.23 Because $\lim_{k \rightarrow \infty} \frac{k^{11}+2k^5+1}{4k(k^{10}+1)} = \frac{1}{4}$, $\lim_{k \rightarrow \infty} \left((-1)^{k+1} \frac{k^{11}+2k^5+1}{4k(k^{10}+1)}\right) \neq 0$. So the given series diverges by the Divergence Test.

10.6.24 Clearly $\frac{1}{k \ln^2 k}$ is nonincreasing, and $\lim_{k \rightarrow \infty} \frac{1}{k \ln^2 k} = 0$, so the given series converges.

10.6.25 $\lim_{k \rightarrow \infty} k^{1/k} = 1$ (for example, take logs and apply L'Hôpital's rule), so the given series diverges by the Divergence Test.

10.6.26 $a_{k+1} < a_k$ because $\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} = \left(\frac{k}{k+1}\right)^k < 1$. Additionally, $\frac{k!}{k^k} \rightarrow 0$ as $k \rightarrow \infty$, so the given series converges.

10.6.27 $\frac{1}{\sqrt{k^2+4}}$ is decreasing and tends to zero as $k \rightarrow \infty$, so the given series converges.

10.6.28 $S_3 = \frac{1}{2} - \frac{1}{9} + \frac{1}{28} \approx 0.424603$. $|S - S_3| \leq |a_4| = \frac{1}{65} \approx 0.0153846$.

10.6.29 $S_4 = \frac{-}{1} + \frac{1}{16} - \frac{1}{81} + \frac{1}{256} \approx -0.945939$. $|S - S_4| \leq |a_5| = \frac{1}{625} \approx 0.0016$.

10.6.30 $S_2 = 1 - \frac{1}{256} \approx 0.996094$. $|S - S_2| \leq |a_2| = \frac{1}{2401} \approx 0.000416$.

10.6.31 $S_5 = 1 - \frac{1}{3} + \frac{1}{21} - \frac{1}{91} + \frac{1}{273} \approx 0.70696$. $|S - S_5| \leq |a_5| = \frac{1}{25 + 625 + 1} \approx 0.001536$.

10.6.32 $S_3 = 1 - \frac{1}{6} + \frac{1}{120} \approx 0.841667$. $|S - S_3| = |a_3| = \frac{1}{7!} \approx 0.000198$.

10.6.33 We want $\frac{1}{n+1} < 10^{-4}$, or $n+1 > 10^4$, so $n = 10^4$.

10.6.34 The series starts with $k = 0$, so we want $\frac{1}{n!} < 10^{-4}$, or $n! > 10^4 = 10000$. This happens for $n = 8$.

10.6.35 The series starts with $k = 0$, so we want $\frac{1}{2n+1} < 10^{-4}$, or $2n+1 > 10^4$, $n = 5000$.

10.6.36 We want $\frac{1}{(n+1)^2} < 10^{-4}$, or $(n+1)^2 > 10^4$, so $n = 100$.

10.6.37 We want $\frac{1}{(n+1)^4} < 10^{-4}$, or $(n+1)^4 > 10^4$, so $n = 10$.

10.6.38 The series starts with $k = 0$, so we want $\frac{1}{(2n+1)^3} < 10^{-4}$, or $2n+1 > 10^{4/3}$, so $n = 11$.

10.6.39 To figure out how many terms we need to sum, we must find n such that $\frac{1}{(n+1)^5} < 10^{-3}$, so that $(n+1)^5 > 1000$; this occurs first for $n = 3$. Thus $-\frac{1}{1} + \frac{1}{2^5} - \frac{1}{3^5} \approx -0.972865$.

10.6.40 To figure out how many terms we need to sum, we must find n such that $\frac{1}{(2(n+1)+1)^3} < 10^{-3}$, or $(2n+3)^3 > 10^3$, so $2n+3 > 10$ and $n = 4$. Thus the approximation is $\sum_{k=1}^4 \frac{(-1)^n}{(2n+1)^3} \approx -0.3058074682$.

10.6.41 To figure out how many terms we need to sum, we must find n so that $\frac{n}{n^2+1} < 10^{-3}$, so that $\frac{n^2+1}{n} = n + (1/n) > 1000$. This occurs first for $n = 1000$. We have $\sum_{k=1}^{1000} \frac{(-1)^k k}{k^2+1} \approx -0.269111$.

10.6.42 To figure out how many terms we need to sum, we must find n such that $\frac{n}{n^4+1} < 10^{-3}$, so that $\frac{n^4+1}{n} = n^3 + 1/n > 1000$, which occurs for $n = 10$. We have $\sum_{k=1}^{10} \frac{(-1)^k k}{k^4+1} \approx -0.408068$.

10.6.43 To figure how many terms we need to sum, we must find n such that $\frac{1}{(n+1)^{n+1}} < 10^{-3}$, or $(n+1)^{n+1} > 1000$, so $n = 4$ ($5^5 = 3125$). Thus the approximation is $\sum_{k=1}^4 \frac{(-1)^n}{n^n} \approx -0.7831307870$.

10.6.44 To figure how many terms we need to sum, we must find n such that $\frac{1}{(2(n+1)+1)!} < 10^{-3}$, or $(2n+3)! > 1000$, so $2n+3 \geq 7$ and $n = 2$. The approximation is $\sum_{k=1}^2 \frac{(-1)^{n+1}}{(2n+1)!} \approx 0.1583333333$.

10.6.45 The series of absolute values is a p -series with $p = 2/3$, so it diverges. The given alternating series does converge, though, by the Alternating Series Test. Thus, the given series is conditionally convergent.

10.6.46 Because $\lim_{k \rightarrow \infty} \frac{2k}{\sqrt{k^2+9}} = \lim_{k \rightarrow \infty} 2\sqrt{\frac{k^2}{k^2+9}} = 2\sqrt{1} = 2 \neq 0$, the given series diverges by the Divergence Test.

10.6.47 Note that the absolute value of the k th term is $\frac{1}{(k+1)!}$, and also note that for $k \geq 1$, $(k+1)! > k^2$, and thus $\frac{1}{(k+1)!} < \frac{1}{k^2}$. Now because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series, the given series converges absolutely by the Comparison Test.

10.6.48 The series of absolute values is $\sum_{k=1}^{\infty} \frac{1}{3^k}$, which converges, so the series converges absolutely.

10.6.49 This series of all positive terms is convergent, because it is a geometric series with $r = \frac{3}{4}$. Therefore it converges absolutely.

10.6.50 Note that $\sum_{k=1}^{\infty} \frac{1}{k^{0.99}}$ is a divergent p -series because $p < 1$. However, $\left\{ \frac{1}{k^{0.99}} \right\}$ is a decreasing sequence with limit 0, so the given series converges by the Alternating Series Test. Therefore the given series is conditionally convergent.

10.6.51 The series of absolute values is $\sum_{k=1}^{\infty} \frac{|\cos(k)|}{k^3}$, which converges by the Comparison Test because $\frac{|\cos(k)|}{k^3} \leq \frac{1}{k^3}$. Thus the series converges absolutely.

10.6.52 The series of absolute values is $\sum_{k=1}^{\infty} \frac{k^2}{\sqrt{k^6 + 1}}$. The limit comparison test with $\sum_{k=1}^{\infty} \frac{1}{k}$ gives $\lim_{k \rightarrow \infty} \frac{k^3}{\sqrt{k^6 + 1}} = \lim_{k \rightarrow \infty} \sqrt{\frac{k^6}{k^6 + 1}} = 1$. Because the comparison series diverges, so does the series of absolute values. The original series converges conditionally, however, because the terms are nonincreasing and $\lim_{k \rightarrow \infty} \frac{k^2}{\sqrt{k^6 + 1}} = \lim_{k \rightarrow \infty} \sqrt{\frac{k^4}{k^6 + 1}} = 0$.

10.6.53 The absolute value of the k th term of this series has limit $\pi/2$ as $k \rightarrow \infty$, so the given series is divergent by the Divergence Test.

10.6.54 The series of absolute values is a geometric series with $r = \frac{1}{e}$ and $|r| < 1$, so the given series converges absolutely.

10.6.55 Note that the absolute value of the k th term is $\frac{1}{2\sqrt{k} - 1}$. We will use the Limit Comparison Test with the divergent p -series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. We have

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{2\sqrt{k} - 1}}{\frac{1}{\sqrt{k}}} = \lim_{k \rightarrow \infty} \frac{\sqrt{k}}{2\sqrt{k} - 1} = \frac{1}{2}.$$

Therefore the given series is not absolutely convergent. However, it is the case that $\left\{ \frac{1}{2\sqrt{k} - 1} \right\}$ is decreasing and has limit 0, so by the Alternating Series Test, the given series is convergent. Therefore, the given series is conditionally convergent.

10.6.56 The absolute value of the k th term is $\frac{1}{3^k - 1}$. We use the Limit Comparison Test with the convergent geometric series $\sum_{k=1}^{\infty} \frac{1}{3^k}$. We have

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{3^k - 1}}{\frac{1}{3^k}} = \lim_{k \rightarrow \infty} \frac{3^k}{3^k - 1} = 1.$$

So the given series is absolutely convergent.

10.6.57 The series of absolute values is $\sum_{k=1}^{\infty} \frac{k}{2k + 1}$, but $\lim_{k \rightarrow \infty} \frac{k}{2k + 1} = \frac{1}{2}$, so by the Divergence Test, this series diverges. The original series does not converge conditionally, either, because $\lim_{k \rightarrow \infty} a_k = \frac{1}{2} \neq 0$.

10.6.58 The series of absolute values is $\sum_{k=3}^{\infty} \frac{1}{\ln k}$. Note that for $k \geq 3$, $\ln k \leq k$, so $\frac{1}{\ln k} > \frac{1}{k}$. Therefore

$\sum_{k=3}^{\infty} \frac{1}{\ln k}$ diverges by the Comparison Test. However, because $\lim_{k \rightarrow \infty} \frac{1}{\ln k} = 0$ and the terms are nonincreasing, the series does converge by the Alternating Series Test, so the series converges conditionally.

10.6.59 The series of absolute values is $\sum_{k=1}^{\infty} \frac{\tan^{-1}(k)}{k^3}$, which converges by the Comparison Test because

$\frac{\tan^{-1}(k)}{k^3} < \frac{\pi}{2} \cdot \frac{1}{k^3}$, and $\sum_{k=1}^{\infty} \frac{\pi}{2} \cdot \frac{1}{k^3}$ converges because it is a constant multiple of a convergent p -series. So the original series converges absolutely.

10.6.60 Note that the absolute value of the k th term is $\frac{k^2 + 1}{3k^4 + 3}$. We use the Limit Comparison Test with

the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{k^2 + 1}{3k^4 + 3}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{k^4 + k^2}{3k^4 + 3} = \frac{1}{3}.$$

Because $0 < L < \infty$, the given series converges absolutely by the Limit Comparison Test.

10.6.61 The absolute value of the k th term is $\frac{k^2 + 1}{k^3 - 1}$. We use the Limit Comparison Test with the divergent harmonic series. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{k^2 + 1}{k^3 - 1}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{k^3 + k}{k^3 - 1} = 1.$$

The given series therefore does not converge absolutely. However, the given series does converge by the Alternating Series Test. Note that if $f(x) = \frac{x^2 + 1}{x^3 - 1}$, then $f'(x) = \frac{-(2x + 3x^2 + x^4)}{(x^3 - 1)^2}$ which is less than 0 for $x \geq 1$. Therefore the absolute value of the sequence of terms of the original series is decreasing and has limit 0, so the given series is conditionally convergent.

10.6.62 Note that $\left| \frac{\sin k}{3^k + 4^k} \right| \leq \left| \frac{1}{3^k + 4^k} \right| < \frac{1}{3^k}$. Also note that $\sum_{k=1}^{\infty} \frac{1}{3^k}$ is a convergent geometric series. The given series is therefore absolutely convergent by the Comparison Test.

10.6.63 For $k \geq 3$,

$$\frac{k!}{k^k} = \frac{1 \cdot 2 \cdot 3 \cdots k}{k \cdot k \cdot k \cdots k} = \frac{1}{k} \cdot \frac{2}{k} \cdot \frac{3}{k} \cdots \frac{k-1}{k} \cdot \frac{k}{k} \leq \frac{1}{k} \cdot \frac{2}{k} \cdot \frac{k}{k} \cdot \frac{k}{k} \cdots \frac{k}{k} = \frac{2}{k^2}.$$

Because the series $\sum_{k=3}^{\infty} \frac{2}{k^2}$ is a convergent series (it is a constant multiple of a p -series), it follows that the given series converges absolutely by the Comparison Test.

10.6.64 Neither condition is satisfied. $\frac{a_{k+1}}{a_k} = \frac{(k+1)(2k+1)}{(2k+3)k} = \frac{2k^2 + 3k + 1}{2k^2 + 3k} > 1$, and $\lim_{k \rightarrow \infty} a_k = \frac{1}{2}$.

10.6.65

- a. False. For example, consider the alternating harmonic series.
- b. True. This is part of Theorem 10.21.
- c. True. This statement is simply saying that a convergent series converges.
- d. True. This is part of Theorem 10.21.
- e. False. Let $a_k = \frac{1}{k}$.
- f. True. Use the Comparison Test: $\lim_{k \rightarrow \infty} \frac{a_k^2}{a_k} = \lim_{k \rightarrow \infty} a_k = 0$ because $\sum a_k$ converges, so $\sum a_k^2$ and $\sum a_k$ converge or diverge together. Because the latter converges, so does the former.
- g. True, by definition. If $\sum |a_k|$ converged, the original series would converge absolutely, not conditionally.

10.6.66 Let $S = 1 - \frac{1}{2} + \frac{1}{3} - \dots$. Then

$$\begin{aligned} S &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \dots \\ \frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots \end{aligned}$$

Add these two series together to get

$$\frac{3}{2}S = \frac{3}{2} \ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \dots$$

To see that the results are as desired, consider a collection of four terms:

$$\begin{aligned} \dots + \left(\frac{1}{4k+1} - \frac{1}{4k+2}\right) + \left(\frac{1}{4k+3} - \frac{1}{4k+4}\right) + \dots \\ \dots + \frac{1}{4k+2} - \frac{1}{4k+4} + \dots \end{aligned}$$

Adding these results in the desired sign pattern. This repeats for each group of four elements.

10.6.67 $\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 2 \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = 2 \cdot \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}$, and thus $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{6} - \frac{1}{2} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{12}$.

10.6.68 $\sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = 2 \sum_{k=1}^{\infty} \frac{1}{(2k)^4} = 2 \cdot \frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k^4}$, and thus $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = \frac{\pi^4}{90} - \frac{1}{8} \cdot \frac{\pi^4}{90} = \frac{7\pi^4}{720}$.

10.6.69 Both series diverge, so comparisons of their values are not meaningful.

10.6.70

- a. The sequences $\{a_{2n}\} = \left\{\frac{1}{n}\right\}$ and $\{a_{2n-1}\} = \left\{\frac{2}{n}\right\}$ both converge to 0, so $\{a_n\}$ converges to 0.
- b. $S_{2n} = (2-1) + \left(1 - \frac{1}{2}\right) + \left(\frac{2}{3} - \frac{1}{3}\right) + \dots + \left(\frac{2}{n} - \frac{1}{n}\right) = \sum_{k=1}^n \frac{1}{k}$. Because the harmonic series diverges to ∞ , $\lim_{n \rightarrow \infty} S_{2n} = \infty$.
3. Because $\{S_{2n}\}$ diverges, $\{S_n\}$ diverges. This implies that $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ diverges. To apply the Alternating Series Test, it must be the case that $\{a_n\}$ is nonincreasing and converges to 0. Observe that

$$a_{2n} - a_{2n+1} = \frac{2}{2n} - \frac{4}{2n+2} = \frac{1-n}{n(n+1)} < 0$$

for $n \geq 2$. So it is the case that $a_{2n+1} > a_{2n}$ for $n \geq 2$, so $\{a_n\}$ is not a nonincreasing sequence. So the conditions of the Alternating Series Test are violated.

10.6.71

a. The sequences $\{a_{2n}\} = \left\{\frac{1}{n}\right\}$ and $\{a_{2n-1}\} = \left\{\frac{1}{n^2}\right\}$ both converge to 0, so $\{a_n\}$ converges to 0.

b.

$$S_{2n} = \left(\frac{1}{1^2} - \frac{1}{1}\right) + \left(\frac{1}{2^2} - \frac{1}{2}\right) + \left(\frac{1}{3^2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n^2} - \frac{1}{n}\right) = \sum_{k=1}^{\infty} \left(\frac{1}{k^2} - \frac{1}{k}\right).$$

c. Recall that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges while $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. Therefore, $S_{2n} = \sum_{k=1}^{\infty} \left(\frac{1}{k^2} - \frac{1}{k}\right)$ diverges. So $\{S_n\}$ diverges, which means that the given alternating series diverges. This does not violate the Alternating Series Test because the sequence $\{a_n\}$ is not a nonincreasing sequence.

10.7 The Ratio and Root Tests

10.7.1 Given a series $\sum a_k$ of terms, compute $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$ and call it r . If $r < 1$, the given series converges absolutely. If $r > 1$ (including $r = \infty$), the given series diverges. If $r = 1$, the test is inconclusive.

10.7.2 Given a series $\sum a_k$ of terms, compute $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}$ and call it ρ . If $\rho < 1$, the given series converges absolutely. If $\rho > 1$ (including $\rho = \infty$), the given series diverges. If $\rho = 1$, the test is inconclusive.

$$\mathbf{10.7.3} \quad \frac{1000!}{998!} = \frac{1000(999)(998!)}{998!} = 1000(999) = 999,000.$$

$$\mathbf{10.7.4} \quad \frac{(100!)^2}{(99!)^2} = \frac{100(99!)(100)(99!)}{(99!)(99!)} = 100^2 = 10,000.$$

$$\mathbf{10.7.5} \quad \frac{k!}{(k+2)!} = \frac{k!}{(k+2)(k+1)(k!)} = \frac{1}{(k+2)(k+1)}.$$

$$\mathbf{10.7.6} \quad \frac{(2k+1)!}{(2k-1)!} = \frac{(2k+1)(2k)((2k-1)!)}{(2k-1)!} = (2k+1)(2k).$$

10.7.7 The Ratio Test.

10.7.8 No. These tests only indicate whether the series converges absolutely or diverges.

10.7.9 The absolute value of the ratio between successive terms is $\left| \frac{a_{k+1}}{a_k} \right| = \frac{1}{(k+1)!} \cdot \frac{(k)!}{1} = \frac{1}{k+1}$, which goes to zero as $k \rightarrow \infty$, so the given series converges absolutely by the Ratio Test.

10.7.10 The absolute value of the ratio between successive terms is $\left| \frac{a_{k+1}}{a_k} \right| = \frac{2^{k+1}}{(k+1)!} \cdot \frac{(k)!}{2^k} = \frac{2}{k+1}$; the limit of this ratio is zero, so the given series converges absolutely by the Ratio Test.

10.7.11 The k th root of the absolute value of the k th term is $\frac{10k^3 + k}{9k^3 + k + 1}$. The limit of this as $k \rightarrow \infty$ is $\frac{10}{9} > 1$, so the given series diverges by the Root Test.

10.7.12 The k th root of the absolute value of the k th term is $\frac{2k}{k+1}$. The limit of this as $k \rightarrow \infty$ is $2 > 1$, so the given series diverges by the Root Test.

10.7.13 The absolute value of the ratio between successive terms is $\left| \frac{a_{k+1}}{a_k} \right| = \frac{(k+1)^2}{4(k+1)} \cdot \frac{4^k}{(k)^2} = \frac{1}{4} \left(\frac{k+1}{k} \right)^2$. The limit is $1/4$ as $k \rightarrow \infty$, so the given series converges absolutely by the Ratio Test.

10.7.14 The absolute value of the ratio between successive terms is

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{2^{(k+1)}}{(k+1)^{(k+1)}} \cdot \frac{(k)^k}{2^k} = \frac{2}{k+1} \left(\frac{k}{k+1} \right)^k.$$

Note that $\lim_{k \rightarrow \infty} \left(\frac{k}{k+1} \right)^k = \lim_{k \rightarrow \infty} \left(1 + \frac{-1}{k+1} \right)^k = \frac{1}{e}$, so the limit of the ratio is $0 \cdot \frac{1}{e} = 0$, so the given series converges absolutely by the Ratio Test.

10.7.15 The absolute value of the ratio between successive terms is $\left| \frac{a_{k+1}}{a_k} \right| = \frac{(k+1)e^{-(k+1)}}{(k)e^{-(k)}} = \frac{k+1}{(k)e}$. The limit of this ratio as $k \rightarrow \infty$ is $1/e < 1$, so the given series converges absolutely by the Ratio Test.

10.7.16 The absolute value of the ratio between successive terms is $\left| \frac{a_{k+1}}{a_k} \right| = \frac{(k+1)!}{(k+1)^{(k+1)}} \cdot \frac{(k)^k}{(k)!} = \left(\frac{k}{k+1} \right)^k$. This is the reciprocal of $\left(\frac{k+1}{k} \right)^k$ which has limit e as $k \rightarrow \infty$, so the limit of the ratio of successive terms is $1/e < 1$, so the given series converges absolutely by the Ratio Test.

10.7.17 The absolute value of the ratio between successive terms is $\left| \frac{a_{k+1}}{a_k} \right| = \frac{k^2 7^{k+1}}{(k+1)^{27^k}} = \frac{7k^2}{(k+1)^2}$ which has limit $7 > 1$ as $k \rightarrow \infty$. Therefore the given series diverges by the Ratio Test.

10.7.18 The k th root of the absolute value of the k th term is $\left(1 + \frac{3}{k} \right)^k$. The limit of this as $k \rightarrow \infty$ is $= e^3 > 1$, so the given series diverges by the Root Test.

10.7.19 The absolute value of the ratio between successive terms is

$$\frac{2^{k+1}}{(k+1)^{99}} \cdot \frac{(k)^{99}}{2^k} = 2 \left(\frac{k}{k+1} \right)^{99};$$

the limit as $k \rightarrow \infty$ is 2 , so the given series diverges by the Ratio Test.

10.7.20 The absolute value of the ratio between successive terms is

$$\frac{(k+1)!}{(k+1)^6} \cdot \frac{k^6}{k!} = \frac{k+1}{1} \left(\frac{k}{k+1} \right)^6;$$

the limit as $k \rightarrow \infty$ is ∞ , so the given series diverges by the Ratio Test.

10.7.21 The absolute value of the ratio between successive terms is

$$\frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{((k)!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)};$$

the limit as $k \rightarrow \infty$ is $1/4$, so the given series converges absolutely by the Ratio Test.

10.7.22 The absolute value of the ratio between successive terms is

$$\frac{(k+1)^4 2^{-(k+1)}}{(k)^4 2^{-k}} = \frac{1}{2} \left(\frac{k+1}{k} \right)^4;$$

the limit as $k \rightarrow \infty$ is $\frac{1}{2}$, so the given series converges absolutely by the Ratio Test.

10.7.23 The k th root of the absolute value of the k th term is $\left(\frac{k}{k+1}\right)^{2k}$. The limit of this as $k \rightarrow \infty$ is $e^{-2} < 1$, so the given series converges absolutely by the Root Test.

10.7.24 The k th root of the absolute value of the k th term is $\frac{3^k}{k}$, which has limit ∞ as $k \rightarrow \infty$, so the given series diverges by the Root Test.

10.7.25 The k th root of the absolute value of the k th term is $\frac{3k^2 + 4k}{2k^2 + 1}$ which has limit $\frac{3}{2} > 1$ as $k \rightarrow \infty$, so the given series diverges by the Root Test.

10.7.26 The k th root of the absolute value of the k th term is $\frac{1}{\ln(k+1)}$. The limit of this as $k \rightarrow \infty$ is 0, so the given series converges absolutely by the Root Test.

10.7.27 The k th root of the absolute value of the k th term is $\frac{1}{k}$. The limit of this as $k \rightarrow \infty$ is 0, so the given series converges absolutely by the Root Test.

10.7.28 The k th root of the absolute value of the k th term is $\frac{k^{1/k}}{e}$. The limit of this as $k \rightarrow \infty$ is $\frac{1}{e} < 1$, so the given series converges absolutely by the Root Test.

10.7.29 We use the Ratio Test. The limit of the absolute values of the ratio of the terms is

$$L = \lim_{k \rightarrow \infty} \frac{k!k!((k+1)^{2k+2})}{(k+1)!(k+1)!k^{2k}} = \lim_{k \rightarrow \infty} \frac{(k+1)^{2k+2}}{(k+1)(k+1)k^{2k}} = \lim_{k \rightarrow \infty} \frac{(k+1)^{2k}}{k^{2k}} = \lim_{k \rightarrow \infty} \left(\left(1 + \frac{1}{k}\right)^k \right)^2 = e^2 > 1.$$

So the given series diverges by the Ratio Test.

10.7.30 The k th root of the absolute value of the k th term is $\left| \frac{k}{1-5k} \right|$, which has limit $\frac{1}{5}$ as $k \rightarrow \infty$, so the given series converges absolutely by the Root Test.

10.7.31

a. False. For example, when $n = 1$ we have $1!1! = 1$ but $(2 \cdot 1)! = 2! = 2$.

b. True. $\frac{(2n)!}{(2n-1)!} = \frac{2n(2n-1)!}{(2n-1)!} = 2n$.

c. True. By the Root Test, $\sum a_k$ converges, so therefore $\sum 10a_k$ converges.

d. True. In this case, the limit of the ratio of terms is 1.

10.7.32 The k th root of the k th term is $\frac{4k^3 + 7}{9k^2 - 1}$ which has limit ∞ as $k \rightarrow \infty$. Therefore the given series diverges by the Root Test.

10.7.33 Note that $\sum_{k=1}^{\infty} \left((-1)^k \frac{2^{k+1}}{9^{k-1}} \right) = \sum_{k=1}^{\infty} 18 \left(-\frac{2}{9} \right)^k$ is a geometric series with $r = -\frac{2}{9}$. Because $|r| < 1$, the given series converges absolutely.

10.7.34 We use the Ratio Test. We have

$$\lim_{k \rightarrow \infty} \frac{\frac{(k+1)!}{(2k+3)!}}{\frac{k!}{(2k+1)!}} = \lim_{k \rightarrow \infty} \frac{(2k+1)!(k+1)!}{(2k+3)!k!} = \lim_{k \rightarrow \infty} \frac{k+1}{(2k+3)(2k+2)} = 0.$$

Therefore the given series converges absolutely by the Ratio Test.

10.7.35 We use the Ratio Test. We have

$$\lim_{k \rightarrow \infty} \frac{\frac{(2k+3)!}{((k+1)!)^2}}{\frac{(2k+1)!}{(k!)^2}} = \lim_{k \rightarrow \infty} \frac{(2k+3)!(k!)^2}{(2k+1)!((k+1)!)^2} = \lim_{k \rightarrow \infty} \frac{(2k+3)(2k+2)}{(k+1)^2} = 4 > 1.$$

Therefore the given series diverges by the Ratio Test.

10.7.36 Use the Root Test: The k th root of the absolute value of the k th term is $\frac{k^2}{2k^2+1}$. The limit of this as $k \rightarrow \infty$ is $\frac{1}{2} < 1$, so the given series converges absolutely by the Root Test.

10.7.37 Use the Ratio Test: the ratio of successive terms is $\frac{(k+1)^{100}}{(k+2)!} \cdot \frac{(k+1)!}{k^{100}} = \left(\frac{k+1}{k}\right)^{100} \cdot \frac{1}{k+2}$. This has limit $1^{100} \cdot 0 = 0$ as $k \rightarrow \infty$, so the given series converges absolutely by the Ratio Test.

10.7.38 $\lim_{k \rightarrow \infty} k^3 \sin(1/k^3) = \lim_{k \rightarrow \infty} \frac{\sin(1/k^3)}{1/k^3} = 1 \neq 0$, so the given series diverges by the Divergence Test.

10.7.39 We use the Limit Comparison Test with the divergent series $\sum_{k=1}^{\infty} \frac{1}{k}$ on the corresponding series with all positive terms to obtain

$$L = \lim_{k \rightarrow \infty} \frac{\frac{k^3}{\sqrt{k^8+1}}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{k^4}{\sqrt{k^8+1}} = \lim_{k \rightarrow \infty} \frac{1}{\sqrt{1+1/k^8}} = 1.$$

Therefore the given series is not absolutely convergent. To see that it is convergent (and therefore conditionally convergent), note that $\lim_{k \rightarrow \infty} \frac{k^3}{\sqrt{k^8+1}} = 0$, and that the sequence is decreasing, so by the Alternating Series Test, the series is convergent. To see that the sequence decreases, note that if $f(x) = \frac{x^3}{\sqrt{x^8+1}}$, that $f'(x) = \frac{x^2(3-x^8)}{(x^8+1)^{3/2}}$ which is less than 0 for $x \geq 2$. This shows that the the sequence $|a_k|$ is decreasing, at least for $k \geq 2$.

10.7.40 Use the Root Test. $\lim_{k \rightarrow \infty} \frac{1}{1+p} = \frac{1}{1+p} < 1$ because $p > 0$, so the given series converges absolutely.

10.7.41 Use the Root Test. The k th root of the absolute value of the k th term is $(k^{1/k} - 1)^2$, which has limit 0 as $k \rightarrow \infty$, so the given series converges absolutely by the Root Test.

10.7.42 Use the Ratio Test: $\left| \frac{a_{k+1}}{a_k} \right| = \frac{((k+1)!)^3}{((3k+3)!) \cdot (k!)^3} = \frac{(k+1)^3}{(3k+1)(3k+2)(3k+3)}$, which has limit $1/27$ as $k \rightarrow \infty$. Thus the given series converges absolutely.

10.7.43 Use the Ratio Test. $\left| \frac{a_{k+1}}{a_k} \right| = \frac{2^{k+1}(k+1)!}{(k+1)^{k+1}} \cdot \frac{(k)^k}{2^k(k)!} = 2 \left(\frac{k}{k+1} \right)^k$, which has limit $\frac{2}{e}$ as $k \rightarrow \infty$, so the given series converges absolutely.

10.7.44 Use the Root Test. $\lim_{k \rightarrow \infty} \left(1 - \frac{1}{k}\right)^k = e^{-1} < 1$, so the given series converges absolutely.

10.7.45 We have $|a_k| = \frac{1}{k^{0.99}}$, and $\sum_{k=1}^{\infty} \frac{1}{k^{0.99}}$ is a p -series with $p = 0.99$. Because $p < 1$, the series $\sum |a_k|$ diverges. However, because the sequence $\left\{ \frac{1}{k^{0.99}} \right\}$ decreases to 0, the given series converges by the Alternating Series Test. Therefore, the given series is conditionally convergent.

10.7.46 Use the Root Test. $\lim_{k \rightarrow \infty} \sqrt[k]{a_k} = \lim_{k \rightarrow \infty} \sqrt[k]{100} \cdot \frac{1}{k} = 0$, so the given series converges absolutely by the Root Test.

10.7.47 This series is $\sum_{k=1}^{\infty} \frac{k^2}{k!}$. By the Ratio Test, $\left| \frac{a_{k+1}}{a_k} \right| = \frac{(k+1)^2}{(k+1)!} \cdot \frac{k!}{k^2} = \frac{1}{k+1} \left(\frac{k+1}{k} \right)^2$, which has limit 0 as $k \rightarrow \infty$, so the given series converges absolutely.

10.7.48 Note that $\lim_{k \rightarrow \infty} \frac{1}{\tan^{-1} k} = \frac{2}{\pi} \neq 0$, so $\lim_{k \rightarrow \infty} \frac{(-1)^k}{\tan^{-1} k} \neq 0$, and thus the given series diverges by the Divergence Test.

10.7.49 For the corresponding series of all positive terms, we use the Limit Comparison Test with the divergent series $\sum_{k=1}^{\infty} \frac{1}{k^{3/4}}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{1}{\sqrt{k^{3/2} + k}}}{\frac{1}{k^{3/4}}} = \lim_{k \rightarrow \infty} \sqrt{\frac{k^{3/2}}{k^{3/2} + k}} = 1.$$

Therefore the given series does not converge absolutely. However, because $\left\{ \frac{1}{\sqrt{k^{3/2} + k}} \right\}$ is decreasing to 0, the given series is convergent by the Alternating Series Test. Thus, the given series is conditionally convergent.

10.7.50 For any p , if k is sufficiently large then $k^{1/p} > \ln k$ because powers grow faster than logs, so that $k > (\ln k)^p$ and thus $1/k < 1/(\ln k)^p$. Because $\sum_{k=1}^{\infty} 1/k$ diverges, we see that the original series diverges for all p .

10.7.51 For $p \leq 1$ and $k > e$, $\frac{\ln k}{k^p} > \frac{1}{k^p}$. The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges, so the given series diverges. For $p > 1$, let $q < p - 1$; then for sufficiently large k , $\ln k < k^q$, so that by the Comparison Test, $\frac{\ln k}{k^p} < \frac{k^q}{k^p} = \frac{1}{k^{p-q}}$. But $p - q > 1$, so that $\sum_{k=1}^{\infty} \frac{1}{k^{p-q}}$ is a convergent p -series. Thus the original series is convergent precisely when $p > 1$.

10.7.52 $\int_2^{\infty} \frac{dx}{x \ln x (\ln \ln x)^p} = \lim_{b \rightarrow \infty} \left(\frac{(\ln \ln x)^{1-p}}{1-p} \right) \Big|_2^b$. This improper integral converges if and only if $p > 1$. Thus the original series converges for $p > 1$.

10.7.53 For $p \leq 1$, $\frac{(\ln k)^p}{k^p} > \frac{1}{k^p}$, and $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges for $p \leq 1$, so the original series diverges. For $p > 1$, let $q < p - 1$; then for sufficiently large k , $(\ln k)^p < k^q$. Note that $\frac{(\ln k)^p}{k^p} < \frac{k^q}{k^p} = \frac{1}{k^{p-q}}$. But $p - q > 1$, so $\sum_{k=1}^{\infty} \frac{1}{k^{p-q}}$ converges, so the given series converges. Thus, the given series converges exactly for $p > 1$.

10.7.54 Using the Ratio Test, $\left| \frac{a_{k+1}}{a_k} \right| = \frac{(k+1)!p^{k+1}}{(k+2)^{k+1}} \cdot \frac{(k+1)^k}{(k)!p^k} = \frac{(k+1)p(k+1)^k}{(k+2)^{k+1}} = p \left(\frac{k+1}{k+2} \right)^{k+1} = p \cdot \left(\frac{1}{1 + \frac{1}{k+1}} \right)^{k+1}$, which has limit pe^{-1} . The series converges if the ratio limit is less than 1, so if $p < e$.

If $p > e$, the given series diverges by the Ratio Test. If $p = e$, the given series diverges by the Divergence Test.

10.7.55 Use the Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)p^{k+1}}{k+2} \cdot \frac{k+1}{kp^k} = p \cdot \frac{k^2 + 2k + 1}{k^2 + 2k},$$

and this expression has limit p as $k \rightarrow \infty$. Thus the series converges for $p > 1$.

10.7.56 $\ln \left(\frac{k}{k+1} \right)^p = p(\ln(k) - \ln(k+1))$, so

$$\sum_{k=1}^{\infty} \ln \left(\frac{k}{k+1} \right)^p = p \sum_{k=1}^{\infty} (\ln(k) - \ln(k+1))$$

which telescopes, and the n^{th} partial sum is $-p \ln(n+1)$, and $\lim_{n \rightarrow \infty} -p \ln(n+1)$ is not a finite number for any value of p other than 0. The given series diverges for all values of p other than $p = 0$.

10.7.57 $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} \left(1 - \frac{p}{k} \right)^k = e^{-p} \neq 0$, so this sequence diverges for all p by the Divergence Test.

10.7.58 $\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right| = \frac{|x|}{k+1}$. This has limit 0 as $k \rightarrow \infty$ for any value of x , so the series converges absolutely for all x .

10.7.59 $\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{x^{k+1}}{x^k} \right| = |x|$. This has limit $|x|$ as $k \rightarrow \infty$, so the series converges absolutely for $|x| < 1$. It clearly does not converge for $x = \pm 1$.

10.7.60 $\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{x^{k+1}}{k+1} \cdot \frac{k}{x^k} \right| = |x| \cdot \frac{k}{k+1}$, which has limit $|x|$ as $k \rightarrow \infty$. Thus this series converges for $|x| < 1$; additionally, for $x = 1$ (where the Ratio Test is inconclusive), the series is the harmonic series which diverges and for $x = -1$ the series is the alternating harmonic series which converges. So the series converges for $-1 \leq x < 1$.

10.7.61 $\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{x^{k+1}}{(k+1)^2} \cdot \frac{k^2}{x^k} \right| = |x| \left(\frac{k}{k+1} \right)^2$, which has limit $|x|$ as $k \rightarrow \infty$. Thus the series converges for $|x| < 1$. When $x = 1$, the series is $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which converges, and for $x = -1$, the series is $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ which converges. Thus the original series converges for $|x| \leq 1$.

10.7.62 $\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{x^{2k+2}}{(k+1)^2} \cdot \frac{k^2}{x^{2k}} \right| = x^2 \left(\frac{k}{k+1} \right)^2$, which has limit x^2 as $k \rightarrow \infty$, so the series converges for $|x| < 1$. When $x = \pm 1$, the series is $\frac{1}{k^2}$, which converges. Thus this series converges for $-1 \leq x \leq 1$.

10.7.63 $\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{x^{k+1}}{2^{k+1}} \cdot \frac{2^k}{x^k} \right| = \frac{|x|}{2}$, which has limit $\frac{|x|}{2}$ as $k \rightarrow \infty$. Thus the series converges for $|x| < 2$. For $x = \pm 2$, it is obviously divergent. So the series converges for $-2 < x < 2$.

10.7.64

a. Use the Ratio Test:

$$\frac{a_{k+1}}{a_k} = \frac{1 \cdot 3 \cdot 5 \cdots (2k+1)}{p^{k+1}(k+1)!} \cdot \frac{p^k(k)!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} = \frac{(2k+1)}{(k+1)p}$$

and this expression has limit $\frac{2}{p}$ as $k \rightarrow \infty$. Thus the series converges for $p > 2$.

b. Following the hint, when $p = 2$ we have $\sum_{k=1}^{\infty} \frac{(2k)!}{2^k k! (2 \cdot 4 \cdot 6 \cdots 2k)} = \sum_{k=1}^{\infty} \frac{(2k)!}{(2^k)^2 (k!)^2}$. Using Stirling's formula, the numerator is asymptotic to $2\sqrt{\pi}\sqrt{k}(2k)^{2k}e^{-2k} = 2\sqrt{\pi}\sqrt{k}(2^k)^2(k^k)^2e^{-2k}$ while the denominator is asymptotic to $(2^k)^2 2\pi k (k^k)^2 e^{-2k}$, so the quotient is asymptotic to $\frac{1}{\sqrt{\pi}\sqrt{k}}$. Thus the original series diverges for $p = 2$ by the Limit Comparison Test with the divergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$.

10.8 Choosing a Convergence Test

10.8.1 The presence of the power k makes the Root Test a good candidate for determining convergence.

10.8.2 The Limit Comparison Test is likely the easiest way to do determine convergence. However, we could factor the denominator as $(k+1)(k-2)$ and then obtain the partial fraction decomposition of the fraction in the sum, and then recognize that we have a telescoping series.

10.8.3 The Divergence Test is appropriate because $\lim_{k \rightarrow \infty} \frac{2k^2}{k^2 - k - 2} \neq 0$.

10.8.4 Let $f(x) = \frac{1}{x \ln^7 x}$ and apply the Integral Test.

10.8.5 Reindex the sum $\sum_{k=10}^{\infty} \frac{1}{(k-9)^5}$ to obtain $\sum_{k=1}^{\infty} \frac{1}{k^5}$ and apply the p -series Test. Another approach is to use the Limit Comparison Test with the comparison series $\sum_{k=10}^{\infty} \frac{1}{k^5}$.

10.8.6 The Ratio Test is an appropriate choice due to the presence of the factorial term.

10.8.7 Use the Comparison Test or Limit Comparison Test with the comparison series $\sum_{k=10}^{\infty} \frac{1}{k^2}$.

10.8.8 By writing out the first few terms of the series or by rewriting the series $\sum_{k=1}^{\infty} \frac{(-3)^k}{4^{k+1}}$ as $\sum_{k=1}^{\infty} \frac{1}{4} \left(-\frac{3}{4}\right)^k$, we can see that this is a geometric series. The value of r will determine whether the series converges or diverges.

10.8.9 The series is an alternating series and the function $f(x) = \frac{1}{\sqrt{2^x + \ln x}}$ is nonincreasing for $x \geq 1$, so the Alternating Series Test is an appropriate test to apply to this series.

10.8.10 Use a telescoping series argument to determine whether this series converges.

10.8.11 Taking the limit of the k th-term of the series, we have $\lim_{k \rightarrow \infty} \frac{2k^4 + k}{4k^4 - 8k} = \frac{1}{2}$. The terms of the series $\sum_{k=1}^{\infty} \frac{2k^4 + k}{4k^4 - 8k}$ do not approach 0, so the series diverges by the Divergence Test.

10.8.12 By rewriting the series, we see that it is a geometric series with $r = \frac{1}{49}$:

$$\sum_{k=1}^{\infty} 7^{-2k} = \sum_{k=1}^{\infty} \frac{1}{7^{2k}} = \sum_{k=1}^{\infty} \frac{1}{49^k}.$$

Because $|r| < 1$, the series converges.

10.8.13 We apply the Limit Comparison Test using the series $\sum_{k=3}^{\infty} \frac{1}{k}$ as the comparison series. This comparison series diverges because it is the tail of the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Calculating L , with the help of l'Hôpital's Rule, we have

$$L = \lim_{k \rightarrow \infty} \frac{5/(2 + \ln k)}{1/k} = \lim_{k \rightarrow \infty} \frac{5k}{2 + \ln k} = \lim_{k \rightarrow \infty} \frac{5}{1/k} = \lim_{k \rightarrow \infty} 5k = \infty.$$

By case (3) of the Limit Comparison Test, the series $\sum_{k=3}^{\infty} \frac{5}{2 + \ln k}$ diverges because the comparison series $\sum_{k=3}^{\infty} \frac{1}{k}$ diverges.

10.8.14 We use the Limit Comparison Test using the comparison series $\sum_{k=1}^{\infty} \frac{k^2}{k^4} = \sum_{k=1}^{\infty} \frac{1}{k^2}$. Note that

$$L = \lim_{k \rightarrow \infty} \frac{\frac{7k^2 - k - 2}{4k^4 - 3k + 1}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{7k^4 - k^3 - 2k^2}{4k^4 - 3k + 1} = \frac{7}{4},$$

and therefore $0 < L < \infty$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series, it follows that $\sum_{k=1}^{\infty} \frac{7k^2 - k - 2}{4k^4 - 3k + 1}$ converges by the Limit Comparison Test.

10.8.15 The presence of $k!$ in the sum suggests that the Ratio Test might be an appropriate approach here.

$$r = \lim_{k \rightarrow \infty} \left| \frac{(-7)^{k+1}/(k+1)!}{(-7)^k/k!} \right| = \lim_{k \rightarrow \infty} \frac{7^{k+1}}{(k+1)!} \cdot \frac{k!}{7^k} = \lim_{k \rightarrow \infty} \frac{7}{k+1} = 0$$

Because $r < 1$, the series $\sum_{k=1}^{\infty} \frac{(-7)^k}{k!}$ converges by the Ratio Test.

10.8.16 Because $\sum_{k=1}^{\infty} \frac{7^k}{k!}$ converges (previous solution) and $\frac{7^k}{k! + 10} < \frac{7^k}{k!}$, it follows from the Comparison Test that $\sum_{k=1}^{\infty} \frac{7^k}{k! + 10}$ converges.

10.8.17 The limit of the k th-term of the series is $\lim_{k \rightarrow \infty} \frac{(-k)^3}{3k^3 + 2} = -\frac{1}{3}$. Therefore $\sum_{k=1}^{\infty} \frac{(-k)^3}{3k^3 + 2}$ diverges by the Divergence Test because $\lim_{k \rightarrow \infty} \frac{(-k)^3}{3k^3 + 2} \neq 0$.

10.8.18 We apply the Divergence Test by showing that the limit $\lim_{k \rightarrow \infty} (1 + a/k)^k$ is nonzero. Noting that $\lim_{k \rightarrow \infty} (1 + a/k)^k = \lim_{k \rightarrow \infty} e^{\ln(1+a/k)^k} = e^{\lim_{k \rightarrow \infty} k \ln(1+a/k)}$, the first step is to evaluate the limit

$$L = \lim_{k \rightarrow \infty} k \ln \left(1 + \frac{a}{k} \right).$$

The limit has the form $\infty \cdot 0$, which we first put in the form $0/0$ so that l'Hôpital's Rule may be applied.

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} k \ln \left(1 + \frac{a}{k} \right) = \lim_{k \rightarrow \infty} \frac{\ln(1 + a/k)}{1/k} && k = \frac{1}{1/k} \\ &= \lim_{k \rightarrow \infty} \frac{\frac{1}{1+a/k} \cdot \left(-\frac{a}{k^2} \right)}{\left(-\frac{1}{k^2} \right)} && \text{L'Hôpital's Rule} \\ &= \lim_{k \rightarrow \infty} \frac{a}{1 + a/k} = a && \text{Simplify and evaluate.} \end{aligned}$$

Therefore

$$\lim_{k \rightarrow \infty} \left(1 + \frac{a}{k} \right)^k = e^L = e^a.$$

This limit does not equal 0 because $e^a \neq 0$ for any constant a . It follows from the Divergence Test that the series $\sum_{k=1}^{\infty} \left(1 + \frac{a}{k} \right)^k$ diverges for any constant a .

10.8.19 Writing out the first few terms of the series, we have

$$\sum_{k=0}^{\infty} \frac{3^{k+4}}{5^{k-2}} = \frac{3^4}{5^{-2}} + \frac{3^5}{5^{-1}} + \frac{3^6}{5^0} + \frac{3^7}{5^1} + \cdots.$$

This is a geometric series with $r = 3/5$. Because $r < 1$, the series converges.

10.8.20 We use the Limit Comparison Test using the comparison series $\sum_{j=1}^{\infty} \frac{j^{10}}{j^{11}} = \sum_{j=1}^{\infty} \frac{1}{j}$.

$$L = \lim_{j \rightarrow \infty} \frac{\frac{4j^{10}}{j^{11}+1}}{\frac{1}{j}} = \lim_{j \rightarrow \infty} \frac{4j^{11}}{j^{11}+1} = 4$$

Therefore $0 < L < \infty$. The series $\sum_{j=1}^{\infty} \frac{4j^{10}}{j^{11}+1}$ diverges by the Limit Comparison Test because the comparison series $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges.

10.8.21 The Alternating Series can be used here to establish convergence of the series. Another approach is to first establish that the positive-term series $\sum_{k=1}^{\infty} \frac{k}{k^3+1}$ converges. Notice that

$$\frac{k}{k^3+1} < \frac{k}{k^3} = \frac{1}{k^2}.$$

It follows that $\sum_{k=1}^{\infty} \frac{k}{k^3+1}$ converges by the Comparison Test because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series. This implies that the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^3+1}$ is absolutely convergent. Therefore $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^3+1}$ converges because every absolutely convergent series converges.

10.8.22 The series $\sum_{k=1}^{\infty} \left(\frac{e+1}{\pi}\right)^k$ is a geometric series with $r = \frac{e+1}{\pi}$. Because $r > 1$, the series diverges.

10.8.23 Apply the Ratio Test.

$$r = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^5/5^{k+1}}{k^5/5^k} \right| = \lim_{k \rightarrow \infty} \frac{(k+1)^5}{5^{k+1}} \cdot \frac{5^k}{k^5} = \lim_{k \rightarrow \infty} \frac{1}{5} \left(\frac{k+1}{k} \right)^5 = \frac{1}{5}$$

Because $r < 1$, the series $\sum_{k=1}^{\infty} \frac{k^5}{5^k}$ converges by the Ratio Test.

10.8.24 Reindex the series:

$$\sum_{k=1}^{\infty} \frac{4}{(k+3)^3} = \frac{4}{4^3} + \frac{4}{5^3} + \frac{4}{6^3} + \frac{4}{7^3} + \cdots = \sum_{k=4}^{\infty} \frac{4}{k^3}.$$

The series $\sum_{k=4}^{\infty} \frac{1}{k^3}$ converges because it is the tail of the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^3}$. It follows that $\sum_{k=1}^{\infty} \frac{4}{(k+3)^3}$ converges.

10.8.25 The terms of the series are positive and decreasing. We apply the Integral Test, with the help of the substitution $u = \sqrt{x}$.

$$\int_1^{\infty} \frac{dx}{\sqrt{x}e^{\sqrt{x}}} = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{\sqrt{x}e^{\sqrt{x}}} = \lim_{b \rightarrow \infty} \int_1^{\sqrt{b}} \frac{2}{e^u} du = \lim_{b \rightarrow \infty} \left(-\frac{2}{e^u} \right) \Big|_1^{\sqrt{b}} = \lim_{b \rightarrow \infty} \left(\frac{2}{e} - \frac{2}{e^{\sqrt{b}}} \right) = \frac{2}{e}$$

Because the integral converges, it follows from the Integral Test that the series converges.

10.8.26 Adding 5 to the inequality $-1 \leq \sin k$, and then dividing by $\sqrt{k}$, it follows that for $k \geq 1$,

$$\frac{4}{\sqrt{k}} \leq \frac{5 + \sin k}{\sqrt{k}}.$$

Because $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is a divergent p -series, $\sum_{k=1}^{\infty} \frac{4}{\sqrt{k}}$ diverges. So by the Comparison Test, $\sum_{k=1}^{\infty} \frac{5 + \sin k}{\sqrt{k}}$ diverges.

10.8.27 Adding 3 to the inequality $\cos 5k \leq 1$, and then dividing by k^3 , we obtain

$$\frac{3 + \cos 5k}{k^3} \leq \frac{4}{k^3}.$$

Because $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p -series, $\sum_{k=1}^{\infty} \frac{4}{k^3}$ converges. So by the Comparison Test, $\sum_{k=1}^{\infty} \frac{3 + \cos 5k}{k^3}$ converges.

10.8.28 For $k \geq 3$, the expression $\frac{\ln k}{k^{1/3}}$ is positive, which implies that $\sum_{k=3}^{\infty} \frac{(-1)^k \ln k}{k^{1/3}}$ is an alternating series.

We apply the Alternating Series Test to show that this series converges. It is not obvious that the terms of the series are decreasing in magnitude, so we use calculus to show that the derivative of function $f(x) = \frac{\ln x}{x^{1/3}}$ is negative after a fixed number of terms of the series. By the Quotient Rule,

$$f'(x) = \frac{x^{1/3} \cdot \frac{1}{x} - (\ln x) \frac{1}{3} x^{-2/3}}{x^{2/3}} = \frac{3 - \ln x}{3x^{4/3}}.$$

For $x \geq 27$, $\ln x \geq 3$ because $\ln x \geq \ln 3^3 > \ln e^3 = 3$. So $f'(x)$ is negative for $x \geq 27$ which implies that the terms of the series decrease for $x \geq 27$. By l'Hôpital's Rule,

$$\lim_{k \rightarrow \infty} \frac{\ln k}{k^{1/3}} = \lim_{k \rightarrow \infty} \frac{\frac{1}{k}}{\frac{1}{3} k^{-2/3}} = \lim_{k \rightarrow \infty} \frac{3}{k^{1/3}} = 0.$$

The conditions of the Alternating Series Test are met which implies that the series converges.

10.8.29 We first examine the series $\sum_{k=1}^{\infty} \frac{10^k}{k^{10}}$ using the Ratio Test.

$$r = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \frac{10^{k+1}/(k+1)^{10}}{10^k/k^{10}} = \lim_{k \rightarrow \infty} \frac{10^{k+1}}{(k+1)^{10}} \cdot \frac{k^{10}}{10^k} = 10 \lim_{k \rightarrow \infty} \left(\frac{k}{k+1} \right)^{10} = 10.$$

Because $r > 1$, the series $\sum_{k=1}^{\infty} \frac{10^k}{k^{10}}$ diverges by the Ratio Test. Because $\frac{10^k + 1}{k^{10}} > \frac{10^k}{k^{10}}$ and $\sum_{k=1}^{\infty} \frac{10^k}{k^{10}}$ diverges, the series $\sum_{k=1}^{\infty} \frac{10^k + 1}{k^{10}}$ diverges by the Comparison Test.

10.8.30 For $k \geq 3$, $\ln k > 1$. Multiplying both sides of this inequality by k^3 , we have $k^3 \ln k > k^3$ for $k \geq 3$. Inverting both sides we have $\frac{1}{k^3 \ln k} < \frac{1}{k^3}$ for $k \geq 3$. Because $\sum_{k=3}^{\infty} \frac{1}{k^3}$ is a convergent p -series, it follows that

$\sum_{k=3}^{\infty} \frac{1}{k^3 \ln k}$ converges by the Comparison Test.

10.8.31 Note that $\frac{1}{j^2 + 4} < \frac{1}{j^2}$. Because $\sum_{j=1}^{\infty} \frac{1}{j^2}$ is a convergent p -series, it follows that $\sum_{j=1}^{\infty} \frac{1}{j^2 + 4}$ converges

by the Comparison Test. Therefore $\sum_{j=1}^{\infty} \frac{5}{j^2 + 4}$ converges.

10.8.32 In the solution to Exercise 18, we showed that $\lim_{k \rightarrow \infty} (1 + a/k)^k = e^a$. It follows that

$$\lim_{k \rightarrow \infty} \frac{k^k}{(k+2)^k} = \lim_{k \rightarrow \infty} \left(\frac{k}{k+2} \right)^k = \frac{1}{\lim_{k \rightarrow \infty} \left(\frac{k+2}{k} \right)^k} = \frac{1}{\lim_{k \rightarrow \infty} \left(1 + \frac{2}{k} \right)^k} = \frac{1}{e^2}.$$

Because the terms of the series $\sum_{k=1}^{\infty} \frac{k^k}{(k+2)^k}$ do not approach 0 as $k \rightarrow \infty$, the series diverges by the Divergence Test.

10.8.33 Using the comparison series $\sum_{k=3}^{\infty} \frac{1}{k}$, we calculate the limit L using L'Hôpital's Rule.

$$L = \lim_{k \rightarrow \infty} \frac{1/(k^{1/3} \ln k)}{1/k} = \lim_{k \rightarrow \infty} \frac{1}{k^{1/3} \ln k} \cdot \frac{k}{1} = \lim_{k \rightarrow \infty} \frac{k^{2/3}}{\ln k} = \lim_{k \rightarrow \infty} \frac{\frac{2}{3}k^{-1/3}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{2}{3}k^{2/3} = \infty$$

The series $\sum_{k=3}^{\infty} \frac{1}{k}$ diverges because it is the tail of the divergent harmonic series. Because $L = \infty$, it

follows from case (3) of the Limit Comparison Test that $\sum_{k=3}^{\infty} \frac{1}{k^{1/3} \ln k}$ diverges.

10.8.34 We apply the Alternating Series Test. The first step is to show that the terms of the series decrease in magnitude after a fixed number of terms of the series. Let $f(x) = \frac{5x^2}{\sqrt{3x^5 + 1}}$. Then

$$f'(x) = \frac{10x\sqrt{3x^5 + 1} - 5x^2 \frac{15x^4}{2\sqrt{3x^5 + 1}}}{3x^5 + 1} = \frac{20x(3x^5 + 1) - 75x^6}{2(3x^5 + 1)^{3/2}} = \frac{20x - 15x^6}{2(3x^5 + 1)^{3/2}} = \frac{5x(4 - 3x^5)}{2(3x^5 + 1)^{3/2}}.$$

Because $f'(x) < 0$ for $x \geq 2$, it follows that the terms of the series are decreasing in magnitude for $k \geq 2$. Furthermore,

$$\lim_{k \rightarrow \infty} \frac{5k^2}{\sqrt{3k^5 + 1}} = \lim_{k \rightarrow \infty} \frac{5k^2/k^2}{(\sqrt{3k^5 + 1})/k^2} = \lim_{k \rightarrow \infty} \frac{5}{\sqrt{3k + 1/k^4}} = 0.$$

The conditions of the Alternating Series Test are met which implies that the series $\sum_{k=1}^{\infty} \frac{(-1)^k 5k^2}{\sqrt{3k^5 + 1}}$ converges.

10.8.35 We apply the Root Test.

$$\rho = \lim_{k \rightarrow \infty} \left| \frac{2^k 3^k}{k^k} \right|^{1/k} = \lim_{k \rightarrow \infty} \frac{6}{k} = 0$$

Because $\rho < 1$, the series $\sum_{k=1}^{\infty} \frac{2^k 3^k}{k^k}$ converges by the Root Test.

10.8.36 Applying the Ratio Test, we have

$$\begin{aligned} r &= \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+2} (2(k+1) + 1) / (k+1)!}{(-1)^{k+1} (2k + 1) / (k!)} \right| = \lim_{k \rightarrow \infty} \frac{(2k+3)k!}{(k+1)!(2k+1)} \\ &= \lim_{k \rightarrow \infty} \frac{(2k+3)}{(k+1)(2k+1)} = \lim_{k \rightarrow \infty} \frac{2k+3}{2k^2 + 3k + 1} = 0. \end{aligned}$$

Because $r < 1$, the series $\sum_{k=1}^{\infty} \frac{2k+1}{k!}$ converges by the Ratio Test.

10.8.37 We apply the Root Test.

$$\rho = \lim_{k \rightarrow \infty} \left| (-1)^k \left(\frac{5k}{3k+7} \right)^k \right|^{1/k} = \lim_{k \rightarrow \infty} \frac{5k}{3k+7} = \frac{5}{3}$$

Because $\rho > 1$, the series $\sum_{k=1}^{\infty} \left(\frac{5k}{3k+7} \right)^k$ diverges by the Root Test.

10.8.38 Simplifying the terms of the series, we have $\sum_{k=1}^{\infty} \frac{2^k(k-1)!}{k!} = \sum_{k=1}^{\infty} \frac{2^k(k-1)!}{k(k-1)!} = \sum_{k=1}^{\infty} \frac{2^k}{k}$. Applying l'Hôpital's Rule, we have

$$\lim_{k \rightarrow \infty} \frac{2^k}{k} = \lim_{k \rightarrow \infty} (2^k \ln 2) = \infty.$$

Because the terms of this series do not approach 0, the series diverges by the Divergence Test.

10.8.39 Applying the Ratio Test, we have

$$r = \lim_{k \rightarrow \infty} \left| \frac{\frac{5^{k+1}((k+1)!)^2}{(2k+2)!}}{\frac{5^k(k!)^2}{(2k)!}} \right| = \lim_{k \rightarrow \infty} \frac{5^k 5(k+1)!(k+1)!}{(2k+2)(2k+1)(2k)!} \cdot \frac{(2k)!}{5^k(k!)(k!)} = \lim_{k \rightarrow \infty} \frac{5(k+1)}{2(2k+1)} = \frac{5}{4}.$$

Because $r > 1$, the series $\sum_{k=1}^{\infty} \frac{5^k(k!)^2}{(2k)!}$ diverges by the Ratio Test.

10.8.40 We simplify the series first by observing that $\cos((2j+1)\pi) = -1$ for any integer j . Therefore

$$\sum_{j=1}^{\infty} \frac{\cos((2j+1)\pi)}{j^2+1} = \sum_{j=1}^{\infty} \frac{-1}{j^2+1}.$$

Because $\sum_{j=1}^{\infty} \frac{1}{j^2}$ is a convergent p -series and $\frac{1}{j^2+1} \leq \frac{1}{j^2}$, it follows that $\sum_{j=1}^{\infty} \frac{1}{j^2+1}$ converges by the Comparison Test. Therefore $\sum_{j=1}^{\infty} \frac{\cos((2j+1)\pi)}{j^2+1} = \sum_{j=1}^{\infty} \frac{-1}{j^2+1}$ converges.

10.8.41 We apply the Ratio Test.

$$r = \lim_{k \rightarrow \infty} \left| \frac{2^{k+1}/(3^{k+1}-2^{k+1})}{2^k/(3^k-2^k)} \right| = \lim_{k \rightarrow \infty} \frac{2(3^k-2^k)}{3^{k+1}-2^{k+1}} = \lim_{k \rightarrow \infty} \frac{2(3^k/3^k-2^k/3^k)}{3^{k+1}/3^k-2^{k+1}/3^k} = \lim_{k \rightarrow \infty} \frac{2(1-(2/3)^k)}{3-2(2/3)^k} = \frac{2}{3}$$

Because $r < 1$, it follows that $\sum_{k=1}^{\infty} \frac{2^k}{3^k-2^k}$ converges.

10.8.42 Apply the Comparison test with the comparison series $\sum_{k=1}^{\infty} \frac{3}{2^k}$. The inequality $2+(-1)^k \leq 3$ implies that

$$\frac{2+(-1)^k}{2^k} \leq \frac{3}{2^k}.$$

Because $\sum_{k=1}^{\infty} \frac{3}{2^k}$ is a convergent geometric series, it follows from the Comparison Test $\sum_{k=1}^{\infty} \frac{2+(-1)^k}{2^k}$ converges.

10.8.43 The values of $\cos \frac{(3k-1)\pi}{3}$ oscillate between $-1/2$ and $1/2$, which implies that $\lim_{k \rightarrow \infty} \cos \frac{(3k-1)\pi}{3} \neq 0$. So by the Divergence Test, the series $\sum_{k=1}^{\infty} \cos \frac{(3k-1)\pi}{3}$ diverges.

10.8.44 Apply the Limit Comparison test using $\sum_{k=10}^{\infty} \frac{1}{k}$, which is a tail of the divergent harmonic series.

$$L = \lim_{k \rightarrow \infty} \frac{\frac{1}{\sqrt{k(k-9)}}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \sqrt{\frac{k^2}{k^2 - 9k}} = \sqrt{\lim_{k \rightarrow \infty} \frac{1}{1 - 9/k}} = 1$$

We see that $0 < L < \infty$. This implies that $\sum_{k=10}^{\infty} \frac{1}{\sqrt{k(k-9)}}$ diverges by the Limit Comparison Test.

10.8.45 We will use the Integral Test, and we first show that the terms of the series $\sum_{k=1}^{\infty} \frac{k^4}{e^{k^5}}$ are decreasing

by demonstrating that the positive and continuous function $f(x) = \frac{x^4}{e^{x^5}}$ is decreasing. In this case,

$$f'(x) = \frac{e^{x^5} 4x^3 - x^4 e^{x^5} 5x^4}{e^{2x^5}} = \frac{x^3 e^{x^5} (4 - 5x^5)}{e^{2x^5}} = \frac{x^3 (4 - 5x^5)}{e^{x^5}}.$$

Therefore $f'(x) < 0$ for $x \geq 1$ which implies that the terms of the series $\sum_{k=1}^{\infty} \frac{k^4}{e^{k^5}}$ are decreasing. Next, we integrate with the help of the substitution $u = x^5$, which implies that $du = 5x^4 dx$.

$$\begin{aligned} \int_1^{\infty} \frac{x^4}{e^{x^5}} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{x^4}{e^{x^5}} dx \\ &= \lim_{b \rightarrow \infty} \frac{1}{5} \int_1^{b^5} \frac{1}{e^u} du \\ &= \lim_{b \rightarrow \infty} \frac{1}{5} \left(-\frac{1}{e^u} \right) \Big|_1^{b^5} \\ &= \lim_{b \rightarrow \infty} \frac{1}{5} \left(\frac{1}{e} - \frac{1}{e^{b^5}} \right) = \frac{1}{5e} \end{aligned}$$

Because the integral converges, it follows by the Integral Test that the series $\sum_{k=1}^{\infty} \frac{k^4}{e^{k^5}}$ converges.

10.8.46 The denominator $(k+1)! - k! = (k+1)k! - k! = k!(k+1-1) = k! \cdot k$. Therefore

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)! - k!} = \sum_{k=1}^{\infty} \frac{1}{k! \cdot k}.$$

Apply the Ratio Test by taking the limit of successive terms of the series.

$$r = \lim_{k \rightarrow \infty} \left| \frac{\frac{1}{(k+1)!(k+1)}}{\frac{1}{k! \cdot k}} \right| = \lim_{k \rightarrow \infty} \frac{k! \cdot k}{(k+1)!(k+1)} = \lim_{k \rightarrow \infty} \frac{k! \cdot k}{(k+1)k!(k+1)} = \lim_{k \rightarrow \infty} \frac{k}{k^2 + 2k + 1} = 0$$

Because $r < 1$, it follows by the Ratio Test that the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)! - k!}$ converges.

10.8.47 We apply the Ratio Test.

$$\begin{aligned}
r &= \lim_{k \rightarrow \infty} \left| \frac{\frac{(4k+4)!}{((k+1)!)^4}}{\frac{(4k)!}{(k!)^4}} \right| \\
&= \lim_{k \rightarrow \infty} \frac{(4k+4)!}{((k+1)!)^4} \cdot \frac{(k!)^4}{(4k)!} \\
&= \lim_{k \rightarrow \infty} \frac{(4k+4)(4k+3)(4k+2)(4k+1)(4k)!}{((k+1)k!)^4} \cdot \frac{(k!)^4}{(4k)!} \\
&= \lim_{k \rightarrow \infty} \frac{(4k+4)(4k+3)(4k+2)(4k+1)}{(k+1)^4} = 4^4
\end{aligned}$$

Because $r > 1$, it follows from the Ratio Test that the series $\sum_{k=1}^{\infty} \frac{(4k)!}{(k!)^4}$ diverges.

10.8.48 The series $\frac{2}{3} + \frac{3}{8} + \frac{4}{15} + \frac{5}{24} + \frac{6}{35} + \cdots$ is represented in summation notation as $\sum_{k=1}^{\infty} \frac{k+1}{k(k+2)}$. We apply the Limit Comparison Test using the divergent comparison series $\sum_{k=1}^{\infty} \frac{1}{k}$.

$$L = \lim_{k \rightarrow \infty} \frac{\frac{k+1}{k(k+2)}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{k+1}{k+2} = 1$$

Because $0 < L < \infty$, the series $\sum_{k=1}^{\infty} \frac{k+1}{k(k+2)}$ diverges by the Limit Comparison Test.

10.8.49 We apply the Comparison Test using the comparison series $\sum_{k=1}^{\infty} \frac{1}{k^{6/5}}$. Note that

$$\frac{\sqrt[5]{k}}{\sqrt[5]{k^7+1}} < \frac{\sqrt[5]{k}}{\sqrt[5]{k^7}} = \frac{1}{k^{6/5}}.$$

Because $\sum_{k=1}^{\infty} \frac{1}{k^{6/5}}$ is a convergent p -series, $\sum_{k=1}^{\infty} \frac{\sqrt[5]{k}}{\sqrt[5]{k^7+1}}$ converges by the Comparison Test.

10.8.50 Note that $e^{k^3} > k^3$ for $k > 1$, so that $\frac{1}{e^{k^3}} < \frac{1}{k^3}$. Because $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p -series, it follows by the Comparison Test that $\sum_{k=1}^{\infty} \frac{1}{e^{k^3}}$ is a convergent series.

10.8.51 The limit of the k th-term of the series is

$$\lim_{k \rightarrow \infty} \frac{7^k + 11^k}{11^k} = \lim_{k \rightarrow \infty} \left(\frac{7^k}{11^k} + \frac{11^k}{11^k} \right) = \lim_{k \rightarrow \infty} \left(\left(\frac{7}{11} \right)^k + 1 \right) = 1$$

Because $\lim_{k \rightarrow \infty} \frac{7^k + 11^k}{11^k} \neq 0$, it follows that $\sum_{k=1}^{\infty} \frac{7^k + 11^k}{11^k}$ diverges by the Divergence Test.

10.8.52 The series $\sum_{k=1}^{\infty} \frac{7^k + 11^k}{13^k}$ can be written as the sum of two geometric series:

$$\sum_{k=1}^{\infty} \frac{7^k + 11^k}{13^k} = \sum_{k=1}^{\infty} \frac{7^k}{13^k} + \sum_{k=1}^{\infty} \frac{11^k}{13^k} = \sum_{k=1}^{\infty} \left(\frac{7}{13}\right)^k + \sum_{k=1}^{\infty} \left(\frac{11}{13}\right)^k.$$

Because both $\frac{7}{13}$ and $\frac{11}{13}$ are less than one, the series $\sum_{k=1}^{\infty} \frac{7^k + 11^k}{13^k}$ converges because it is the sum of two convergent geometric series.

10.8.53 We apply the Limit Comparison Test using the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^9}$ as the comparison series.

Observe that

$$L = \lim_{k \rightarrow \infty} \frac{\sin \frac{1}{k^9}}{\frac{1}{k^9}} = 1.$$

Because $0 < L < \infty$, we conclude that the series $\sum_{k=1}^{\infty} \sin \frac{1}{k^9}$ converges by the Limit Comparison Test.

10.8.54 Observe that

$$L = \lim_{j \rightarrow \infty} j^9 \sin \frac{1}{j^9} = \lim_{j \rightarrow \infty} \frac{\sin \frac{1}{j^9}}{\frac{1}{j^9}} = 1.$$

Because $\lim_{j \rightarrow \infty} j^9 \sin \frac{1}{j^9} \neq 0$, the series $\sum_{j=1}^{\infty} j^9 \sin \frac{1}{j^9}$ diverges by the Divergent Test.

10.8.55 Because $\lim_{k \rightarrow \infty} \cos \frac{1}{k^9} = 1 \neq 0$, $\sum_{k=1}^{\infty} \cos \frac{1}{k^9}$ diverges by the Divergence Test.

10.8.56 We apply the Root Test with the help of the solution to Exercise 18 where we showed that $\lim_{k \rightarrow \infty} (1 + a/k)^k = e^a$.

$$\rho = \lim_{k \rightarrow \infty} \sqrt[k]{\left| \left(\frac{k}{k+10} \right)^{-k^2} \right|} = \lim_{k \rightarrow \infty} \left(\frac{k+10}{k} \right)^k = \lim_{k \rightarrow \infty} \left(1 + \frac{10}{k} \right)^k = e^{10}$$

Because $\rho > 1$, the series $\sum_{k=1}^{\infty} \left(\frac{k}{k+10} \right)^{-k^2}$ diverges.

10.8.57 We note that

$$\sum_{k=1}^{\infty} 5^{1-2k} = \sum_{k=1}^{\infty} 5 \cdot 5^{-2k} = \sum_{k=1}^{\infty} \frac{5}{25^k}.$$

So $\sum_{k=1}^{\infty} 5^{1-2k}$ is a geometric series with $r = \frac{1}{25}$. Because $|r| < 1$, the series converges.

10.8.58 Observe that $\frac{1}{\sqrt{\ln(k+2)}}$ decreases as k increases and $\lim_{k \rightarrow \infty} \frac{1}{\sqrt{\ln(k+2)}} = 0$. So by the Alternating

Series Test, $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{\ln(k+2)}}$ converges.

10.8.59 We first establish that $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ is a convergent series using the Ratio Test. Taking the limit of successive terms of $\sum_{k=1}^{\infty} \frac{k!}{k^k}$, we have

$$r = \lim_{k \rightarrow \infty} \left| \frac{\frac{(k+1)!}{(k+1)^{k+1}}}{\frac{k!}{k^k}} \right| = \lim_{k \rightarrow \infty} \frac{(k+1)k!}{(k+1)^k(k+1)} \cdot \frac{k^k}{k!} = \lim_{k \rightarrow \infty} \frac{k^k}{(k+1)^k} = \lim_{k \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{k}\right)^k} = \frac{1}{e},$$

where the last equality follows from the fact that $\lim_{k \rightarrow \infty} \left(1 + \frac{a}{k}\right)^k = e^a$ for any real number a (see the solution to Exercise 18). Because $r < 1$, the series $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converges by the Ratio Test. It is also the case that

$\frac{k!}{k^k + 3} < \frac{k!}{k^k}$ which implies that $\sum_{k=1}^{\infty} \frac{k!}{k^k + 3}$ converges by the Comparison Test.

10.8.60 The series equals the sum of two convergent p -series

$$\sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{1}{k^5} \right) = \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{k^5}.$$

Therefore $\sum_{k=1}^{\infty} \left(\frac{1}{k^2} + \frac{1}{k^5} \right)$ converges.

10.8.61 We apply the Limit Comparison Test by using the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ as our comparison series. With the help of l'Hôpital's Rule, we calculate L .

$$L = \lim_{k \rightarrow \infty} \frac{\frac{1}{\ln(e^k + 1)}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{k}{\ln(e^k + 1)} = \lim_{k \rightarrow \infty} \frac{1}{\frac{e^k}{e^k + 1}} = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{e^k} \right) = 1$$

Because $0 < L < \infty$, $\sum_{k=1}^{\infty} \frac{1}{\ln(e^k + 1)}$ diverges by the Limit Comparison Test.

10.8.62 Apply the Root Test.

$$\rho = \lim_{k \rightarrow \infty} \sqrt[k]{\left| \left(\frac{\tan^{-1} k}{\pi} \right)^k \right|} = \frac{1}{\pi} \lim_{k \rightarrow \infty} \tan^{-1} k = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$$

Because $\rho < 1$, $\sum_{k=0}^{\infty} \left(\frac{\tan^{-1} k}{\pi} \right)^k$ converges.

10.8.63 In the solution to Exercise 18, we showed that $\lim_{k \rightarrow \infty} \left(1 + \frac{a}{k} \right)^k = e^a$. We use this result in the following limit calculation. Applying the Root Test, we have

$$\rho = \lim_{k \rightarrow \infty} \sqrt[k]{\left| \left(\frac{k+a}{k} \right)^{k^2} \right|} = \lim_{k \rightarrow \infty} \left(1 + \frac{a}{k} \right)^k = e^a.$$

Because $a > 0$, $\rho = e^a > 1$. So the series diverges by the Root Test.

10.8.64 Rewrite the series using summation notation.

$$\frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 7} + \frac{1}{3 \cdot 10} + \frac{1}{4 \cdot 13} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k(3k+1)}$$

Using the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ as a comparison series, we apply the Limit Comparison Test:

$$L = \lim_{k \rightarrow \infty} \frac{\frac{1}{k(3k+1)}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{k^2}{k(3k+1)} = \frac{1}{3}.$$

Because $0 < L < \infty$, the series $\sum_{k=1}^{\infty} \frac{1}{k(3k+1)}$ converges.

10.8.65 The n th-partial sum of $\sum_{k=1}^{\infty} \left(\cos \frac{1}{k} - \cos \frac{1}{k+1} \right)$ is

$$S_n = \left(\cos 1 - \cos \frac{1}{2} \right) + \left(\cos \frac{1}{2} - \cos \frac{1}{3} \right) + \cdots + \left(\cos \frac{1}{n} - \cos \frac{1}{n+1} \right) = \cos 1 - \cos \frac{1}{n+1}.$$

It follows that $\lim_{n \rightarrow \infty} S_n = \cos 1 - \cos 0 = \cos 1 - 1$. Because the sequence of partial sums converges, the series

$$\sum_{k=1}^{\infty} \left(\cos \frac{1}{k} - \cos \frac{1}{k+1} \right) \text{ converges.}$$

10.8.66 Use the Ratio Test with the help of l'Hôpital's Rule.

$$r = \lim_{k \rightarrow \infty} \left| \frac{\frac{4^{(k+1)^2}}{(k+1)!}}{\frac{4^{k^2}}{k!}} \right| = \lim_{k \rightarrow \infty} \frac{4^{(k+1)^2 - k^2} k!}{(k+1)!} = \lim_{k \rightarrow \infty} \frac{4^{2k+1}}{(k+1)} = \lim_{k \rightarrow \infty} \frac{4^{2k+1} \cdot 2 \ln 4}{1} = \infty$$

Because $r > 1$, the series $\sum_{k=1}^{\infty} \frac{4^{k^2}}{k!}$ diverges by the Ratio Test.

10.8.67 Apply the Limit Comparison Test using the convergent geometric series $\sum_{j=1}^{\infty} \frac{1}{2^j}$ as the comparison series.

$$L = \lim_{j \rightarrow \infty} \frac{\frac{\cot^{-1} \frac{1}{j}}{2^j}}{\frac{1}{2^j}} = \lim_{j \rightarrow \infty} \cot^{-1} \frac{1}{j} = \frac{\pi}{2}$$

Because $0 < L < \infty$, the series $\sum_{j=1}^{\infty} \frac{\cot^{-1} \frac{1}{j}}{2^j}$ converges by the Limit Comparison Test.

10.8.68 Multiplying numerator and denominator by $\sqrt{k+1} + \sqrt{k}$, we have

$$\sum_{k=2}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2 - k}} \cdot \frac{\sqrt{k+1} + \sqrt{k}}{\sqrt{k+1} + \sqrt{k}} = \sum_{k=2}^{\infty} \frac{1}{\sqrt{k^2 - k}(\sqrt{k+1} + \sqrt{k})}$$

Using the convergent p -series $\sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$ as the comparison series, we use the Limit Comparison Test.

$$\begin{aligned}
L &= \lim_{k \rightarrow \infty} \frac{\left(\frac{1}{\sqrt{k^2 - k}(\sqrt{k+1} + \sqrt{k})} \right)}{\frac{1}{k^{3/2}}} \\
&= \lim_{k \rightarrow \infty} \frac{k^{3/2}}{\sqrt{k^2 - k}(\sqrt{k+1} + \sqrt{k})} \\
&= \lim_{k \rightarrow \infty} \frac{k^{3/2}/k^{3/2}}{\frac{\sqrt{k^2 - k}}{k} \frac{\sqrt{k+1} + \sqrt{k}}{k^{1/2}}} \\
&= \lim_{k \rightarrow \infty} \frac{1}{\sqrt{1 - \frac{1}{k}} \left(\sqrt{1 + \frac{1}{k}} + 1 \right)} = \frac{1}{2}
\end{aligned}$$

Because $0 < L < \infty$, $\sum_{k=2}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2 - k}}$ converges by the Limit Comparison Test.

10.8.69 Note that

$$\lim_{k \rightarrow \infty} \left(1 + \frac{1}{2k} \right)^k = e^{1/2}.$$

Because the limit does not equal 0, the series $\sum_{k=1}^{\infty} \left(1 + \frac{1}{2k} \right)^k$ diverges by the Divergence Test.

10.8.70 We have

$$\lim_{k \rightarrow \infty} e^{k/100} = \infty.$$

Because the limit does not equal 0, the series $\sum_{k=1}^{\infty} e^{k/100}$ diverges by the Divergence Test.

10.8.71 We apply the Limit Comparison Test with the comparison series $\sum_{k=1}^{\infty} \frac{1}{k^{5/4}}$.

$$L = \lim_{k \rightarrow \infty} \frac{\ln^2 k / k^{3/2}}{1/k^{5/4}} = \lim_{k \rightarrow \infty} \frac{\ln^2 k}{k^{1/4}} = \lim_{k \rightarrow \infty} \frac{2(\ln k)/k}{\frac{1}{4}k^{-3/4}} = 8 \lim_{k \rightarrow \infty} \frac{\ln k/k}{k^{-3/4}} = 8 \lim_{k \rightarrow \infty} \frac{\ln k}{k^{1/4}} = 32 \lim_{k \rightarrow \infty} \frac{1}{k^{1/4}} = 0.$$

Because the series $\sum_{k=1}^{\infty} \frac{1}{k^{5/4}}$ converges, it follows from case (2) of the Limit Comparison Test that $\sum_{k=1}^{\infty} \frac{\ln^2 k}{k^{3/2}}$ converges.

10.8.72 Calculate the following limit.

$$\lim_{k \rightarrow \infty} \cos \frac{1}{k} = \cos 0 = 1$$

It follows that $\lim_{k \rightarrow \infty} (-1)^k \cos \frac{1}{k} \neq 0$. So by the Divergence Test, $\sum_{k=1}^{\infty} (-1)^k \cos \frac{1}{k}$ diverges.

10.8.73 Use the Ratio Test.

$$r = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^2 \cdot 1.0001^{-k-1}}{k^2 \cdot 1.0001^{-k}} \right| = \frac{1}{1.001}$$

Because $r < 1$, the series $\sum_{k=0}^{\infty} k^2 \cdot 1.001^{-k}$ converges by the Ratio Test.

10.8.74 Using the Ratio Test, we have

$$r = \lim_{k \rightarrow \infty} \left| \frac{(k+1) \cdot 0.999^{-k-1}}{k \cdot 0.999^{-k}} \right| = \frac{1}{0.999}$$

Because $r > 1$, the series $\sum_{k=0}^{\infty} k \cdot 0.999^{-k}$ diverges by the Ratio Test.

10.8.75 The terms of the series decrease in magnitude, for $k \geq 1$. Furthermore $\lim_{k \rightarrow \infty} \frac{1}{k^k} = 0$. Therefore, the series converges by the Alternating Series Test.

10.8.76 We first establish that the related positive-term series $\sum_{k=3}^{\infty} \frac{1}{k(k-2)}$ converges using the Limit Comparison Test with the help of the convergent comparison series $\sum_{k=3}^{\infty} \frac{1}{k^2}$.

$$L = \lim_{k \rightarrow \infty} \frac{1/(k(k-2))}{1/k^2} = \lim_{k \rightarrow \infty} \frac{k^2}{k^2 - 2k} = 1$$

The value of L is positive and finite which implies that $\sum_{k=3}^{\infty} \frac{1}{k(k-2)}$ converges. It follows that the series $\sum_{k=3}^{\infty} \frac{2}{k(2-k)}$ converges because it equals $\sum_{k=3}^{\infty} \frac{-2}{k(k-2)}$.

10.8.77 The series diverges by the Divergence Test because

$$\lim_{k \rightarrow \infty} \frac{3k}{\sqrt[4]{k^4 + 3}} = \lim_{k \rightarrow \infty} \frac{3k/k}{(\sqrt[4]{k^4 + 3})/k} = \lim_{k \rightarrow \infty} \frac{3}{\sqrt[4]{k^4/k^4 + 3/k^4}} = \lim_{k \rightarrow \infty} \frac{3}{\sqrt[4]{1 + 3/k^4}} \neq 0.$$

10.8.78 We use the Root Test.

$$\rho = \lim_{k \rightarrow \infty} \sqrt[k]{|k \cdot 3^{-k^2}|} = \lim_{k \rightarrow \infty} \frac{k^{1/k}}{3^k} = \frac{\lim_{k \rightarrow \infty} k^{1/k}}{\lim_{k \rightarrow \infty} 3^k} = \frac{1}{\lim_{k \rightarrow \infty} 3^k} = 0$$

Because $\rho < 1$ it follows that $\sum_{k=0}^{\infty} k \cdot 3^{-k^2}$ converges.

10.8.79 Apply the Limit Comparison Test using the divergent p -series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ as the comparison series.

$$L = \lim_{k \rightarrow \infty} \frac{\tan^{-1} \frac{1}{\sqrt{k}}}{\frac{1}{\sqrt{k}}} = \lim_{k \rightarrow \infty} \frac{\frac{1}{1+(1/\sqrt{k})^2} \left(-\frac{1}{2}k^{-3/2}\right)}{-\frac{1}{2}k^{-3/2}} = \lim_{k \rightarrow \infty} \frac{1}{1 + (1/\sqrt{k})^2} = 1$$

Because $0 < L < \infty$, $\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{\sqrt{k}}$ diverges by the Limit Comparison Test.

10.8.80 Apply the Root Test.

$$\rho = \lim_{k \rightarrow \infty} \sqrt[k]{\left| \left(\frac{3k^8 - 2}{3k^9 + 2} \right)^k \right|} = \lim_{k \rightarrow \infty} \frac{3k^8 - 2}{3k^9 + 2} = 0$$

Because $\rho < 1$, the series $\sum_{k=1}^{\infty} \left(\frac{3k^8 - 2}{3k^9 + 2} \right)^k$ converges.

10.8.81 The n th-partial sum of the series $\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k+2}} - \frac{1}{\sqrt{k}} \right)$ is

$$S_n = \left(\frac{1}{\sqrt{3}} - 1 \right) + \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{3}} \right) + \left(\frac{1}{\sqrt{6}} - \frac{1}{2} \right) + \cdots + \left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n-1}} \right) + \left(\frac{1}{\sqrt{n+2}} - \frac{1}{\sqrt{n}} \right).$$

For $n \geq 4$, S_n simplifies to $S_n = -1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}}$. So the limit of the sequence of partial sums is $\lim_{n \rightarrow \infty} S_n = -1 - \frac{1}{\sqrt{2}}$. Therefore the series $\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k+2}} - \frac{1}{\sqrt{k}} \right)$ converges (to this limit).

10.8.82 The series can be rewritten as

$$\sum_{k=0}^{\infty} \left(\left(\frac{2}{3} \right)^{k+1} - \left(\frac{3}{2} \right)^{-k+1} \right) = \sum_{k=0}^{\infty} \left(\left(\frac{2}{3} \right)^{k+1} - \left(\frac{2}{3} \right)^{k-1} \right).$$

Because $\sum_{k=0}^{\infty} \left(\frac{2}{3} \right)^{k+1}$ and $\sum_{k=0}^{\infty} \left(\frac{2}{3} \right)^{k-1}$ are geometric series, both with $r = 2/3$, they both converge. Therefore the series converges.

10.8.83 We apply the Integral Test using the function $f(x) = \frac{1}{x \ln^{10} x}$. This function is continuous, positive, and

decreasing for $x \geq 2$. Integrating and using the substitution $u = \ln x$, we have

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x \ln^{10} x} &= \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln^{10} x} \\ &= \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u^{10}} \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{9u^9} \right) \Big|_{\ln 2}^{\ln b} \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{9 \ln^9 b} + \frac{1}{9 \ln^9 2} \right) = \frac{1}{9 \ln^9 2} \end{aligned}$$

The integral converges which implies that the series $\sum_{j=2}^{\infty} \frac{1}{j \ln^{10} j}$ converges by the Integral Test.

10.8.84 We apply the Limit Comparison Test using the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ as a comparison series.

$$L = \lim_{k \rightarrow \infty} \frac{\tan^{-1} \frac{1}{k^3}}{\frac{1}{k^3}} = \lim_{k \rightarrow \infty} \frac{\frac{1}{1+1/k^6} \cdot \left(-\frac{3}{k^4} \right)}{-\frac{3}{k^4}} = \lim_{k \rightarrow \infty} \frac{1}{1 + 1/k^6} = 1$$

Because $0 < L < \infty$, the series $\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{k^3}$ converges by Limit Comparison Test.

10.8.85 The series $\frac{1}{2 \cdot 3} + \frac{1}{4 \cdot 5} + \frac{1}{6 \cdot 7} + \frac{1}{8 \cdot 9} + \cdots$ is expressed in summation notation as $\sum_{k=1}^{\infty} \frac{1}{2k(2k+1)}$.

We apply the Limit Comparison Test using the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

$$L = \lim_{k \rightarrow \infty} \frac{1/(2k(2k+1))}{1/k^2} = \lim_{k \rightarrow \infty} \frac{k^2}{4k^2 + 2k} = \frac{1}{4}$$

So we have $0 < L < \infty$ and it follows that $\sum_{k=1}^{\infty} \frac{1}{2k(2k+1)}$ converges by the Limit Comparison Test.

10.8.86 Applying the Ratio Test, we have

$$r = \lim_{k \rightarrow \infty} \frac{|a|^{k+1}/(k+1)!}{|a|^k/k!} = \lim_{k \rightarrow \infty} \frac{|a|}{k+1} = 0.$$

Because $r < 1$, the series $\sum_{k=1}^{\infty} \frac{|a|^k}{k!}$ converges and it follows that the series $\sum_{k=1}^{\infty} \frac{a^k}{k!}$ converges.

10.8.87

- a. False. For example, the Limit Comparison Series can be used to show that $\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$ converges using the comparison series $\sum_{k=2}^{\infty} \frac{1}{k^2}$, but this comparison series will not work with the Comparison Test.
- b. True.
- c. True.
- d. False. The Alternating Series Test can only be used to show that a series converges.

10.8.88 Multiply the numerator and denominator by $(\sqrt{k^4 + 1} + k^2)$:

$$\sum_{k=1}^{\infty} (\sqrt{k^4 + 1} - k^2) \frac{(\sqrt{k^4 + 1} + k^2)}{(\sqrt{k^4 + 1} + k^2)} = \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k^4 + 1} + k^2} \right).$$

The inequalities $\sqrt{k^4 + 1} + k^2 > \sqrt{k^4 + 1} > \sqrt{k^4} = k^2$ imply that

$$\sqrt{k^4 + 1} - k^2 = \frac{1}{\sqrt{k^4 + 1} + k^2} < \frac{1}{k^2}.$$

Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p -series, it follows from the Comparison Test that $\sum_{k=1}^{\infty} (\sqrt{k^4 + 1} - k^2)$ converges.

10.8.89 Using the sum $1 + 2 + \cdots + k = k(k+1)/2$ from Chapter 5, we have

$$\sum_{k=1}^{\infty} \frac{k}{1 + 2 + \cdots + k} = \sum_{k=1}^{\infty} \frac{2k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{2}{k+1}.$$

We apply the Limit Comparison Test using the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ as the comparison series.

$$L = \lim_{k \rightarrow \infty} \frac{2/(k+1)}{1/k} = \lim_{k \rightarrow \infty} \frac{2k}{k+1} = 2$$

Because $0 < L < \infty$, $\sum_{k=1}^{\infty} \frac{k}{1 + 2 + \cdots + k}$ diverges by the Limit Comparison Test.

10.8.90 A Chapter 5 result states that

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

This implies that $\sum_{k=1}^{\infty} \left(\frac{1^3 + 2^3 + 3^3 + \cdots + k^3}{k^5} \right)$ equals $\sum_{k=1}^{\infty} \left(\frac{k^2(k+1)^2/4}{k^5} \right) = \sum_{k=1}^{\infty} \frac{(k+1)^2}{4k^3}$. Using the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ as a comparison series, we have

$$L = \lim_{k \rightarrow \infty} \frac{(k+1)^2/(4k^3)}{1/k} = \lim_{k \rightarrow \infty} \frac{k^3 + 2k^2 + k}{4k^3} = \frac{1}{4}.$$

Because $0 < L < \infty$, the series $\sum_{k=1}^{\infty} \left(\frac{1^3 + 2^3 + 3^3 + \cdots + k^3}{k^5} \right)$ diverges by the Limit Comparison Test.

10.8.91 The series $\sum_{k=0}^{\infty} \ln \left(\frac{2k+1}{2k+4} \right)$ consists of negative terms because $(2k+1)/(2k+4) < 1$. Notice that $\sum_{k=0}^{\infty} \ln \left(\frac{2k+1}{2k+4} \right) = -\sum_{k=0}^{\infty} \ln \left(\frac{2k+4}{2k+1} \right)$ where $\sum_{k=0}^{\infty} \ln \left(\frac{2k+4}{2k+1} \right)$ consists of positive terms. So we show that the series $\sum_{k=0}^{\infty} \ln \left(\frac{2k+4}{2k+1} \right)$ diverges by showing that $\sum_{k=0}^{\infty} \ln \left(\frac{2k+4}{2k+1} \right)$ diverges. To do this, we apply the Limit Comparison test using the divergent comparison series $\sum_{k=0}^{\infty} \frac{1}{k}$.

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} \frac{\ln((2k+4)/(2k+1))}{1/k} && \text{Limit Comparison Test} \\ &= \lim_{k \rightarrow \infty} \frac{\frac{2}{2k+4} - \frac{2}{2k+1}}{-1/k^2} && \text{L'Hôpital's Rule} \\ &= \lim_{k \rightarrow \infty} \frac{2(2k+1) - 2(2k+4)}{(2k+1)(2k+4)} \cdot \frac{-k^2}{1} && \text{Common denominators.} \\ &= \lim_{k \rightarrow \infty} \frac{6k^2}{4k^2 + 10k + 4} = \frac{3}{2} && \text{Evaluate.} \end{aligned}$$

Because $0 < L < \infty$, the series $\sum_{k=0}^{\infty} \ln \left(\frac{2k+4}{2k+1} \right)$ diverges and so $\sum_{k=0}^{\infty} \ln \left(\frac{2k+1}{2k+4} \right)$ diverges.

10.8.92 Observe that $10^{\ln k} = e^{\ln 10^{\ln k}} = e^{(\ln k)(\ln 10)} = (e^{\ln k})^{\ln 10} = k^{\ln 10}$. So $\sum_{k=1}^{\infty} \frac{1}{10^{\ln k}} = \sum_{k=1}^{\infty} \frac{1}{k^{\ln 10}}$ and because $\sum_{k=1}^{\infty} \frac{1}{k^{\ln 10}}$ is a p -series with $p = \ln 10 > 1$, $\sum_{k=1}^{\infty} \frac{1}{10^{\ln k}}$ converges by the p -series test.

(Alternate solution): We apply the Limit Comparison Test with the comparison series $\sum_{k=1}^{\infty} \frac{1}{k^2}$:

$$L = \lim_{k \rightarrow \infty} \frac{1/10^{\ln k}}{1/k^2} = \lim_{k \rightarrow \infty} \frac{e^{2 \ln k}}{10^{\ln k}} = \lim_{k \rightarrow \infty} \left(\frac{e^2}{10} \right)^{\ln k} = 0.$$

Because $L = 0$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (because it is a convergent p -series), it follows by part 2 of the Limit Comparison Test that $\sum_{k=1}^{\infty} \frac{1}{10^{\ln k}}$ converges.

10.8.93 We apply the Limit Comparison Test with the comparison series $\sum_{k=1}^{\infty} \frac{1}{k}$:

$$\lim_{k \rightarrow \infty} \frac{1/(2^{\ln k} + 2)}{1/k} = \lim_{k \rightarrow \infty} \frac{e^{\ln k}}{2^{\ln k} + 2} = \lim_{k \rightarrow \infty} \frac{(e/2)^{\ln k}}{1 + 2^{(1-\ln k)}} = \frac{\lim_{k \rightarrow \infty} \left(\frac{e}{2}\right)^{\ln k}}{1 - 0} = \infty.$$

Because $L = \infty$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (because it is the harmonic series), it follows by part 3 of the Limit Comparison Test that $\sum_{k=1}^{\infty} \frac{1}{2^{\ln k} + 2}$ diverges.

Chapter Ten Review

1

- False. Let $a_n = 1 - \frac{1}{n}$. This sequence has limit 1.
- False. The terms of a sequence tending to zero is necessary but not sufficient for convergence of the series.
- True. This is the definition of convergence of a series.
- False. If a series converges absolutely, the definition says that it does not converge conditionally.
- True. It has limit 1 as $n \rightarrow \infty$.
- False. The subsequence of the even terms has limit 1 and the subsequence of odd terms has limit -1 , so the sequence does not have a limit.
- False. It diverges by the Divergence Test because $\lim_{k \rightarrow \infty} \frac{k^2}{k^2 + 1} = 1 \neq 0$.
- True. The given series converges by the Limit Comparison Test with the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, and thus its sequence of partial sums converges.

2

- $\lim_{n \rightarrow \infty} \frac{12n^3 + 4n}{6n^3 + 5} = \lim_{n \rightarrow \infty} \frac{12 + 4/n^2}{6 + 5/n^3} = 2$.
- Because the sequence $\left\{ \frac{12n^3 + 4n}{6n^3 + 5} \right\}$ does not converge to 0, the sequence $\left\{ (-1)^n \frac{12n^3 + 4n}{6n^3 + 5} \right\}$ oscillates and diverges.

3 $\sum_{k=0}^9 (0.2)^k = \frac{1 - (0.2)^{10}}{1 - 0.2} \approx 1.25$, and $\sum_{k=2}^9 (0.2)^k = (0.2)^2 \frac{1 - (0.2)^8}{1 - 0.2} \approx 0.0499999$.

4

- a. $B_n = 1.0025B_{n-1} + 100$ for $n \geq 1$, and $B_0 = 100$.
- b. $B_1 = 100 + 1.0025(100)$.
 $B_2 = 100 + 1.0025(B_1) = 100 + 1.0025(100 + 1.0025(100)) = 100 + 1.0025(100) + 1.0025(100^2)$.
 $B_n = 100 + 100(1.0025) + \cdots + 100(1.0025)^n = 100 \left(\frac{1 - 1.0025^{n+1}}{1 - 1.0025} \right)$.
- c. $B_{30} = 100 \left(\frac{1 - 1.0025^{31}}{1 - 1.0025} \right) \approx 3219.11$ dollars.

5 Because the series converges, we must have $\lim_{k \rightarrow \infty} a_k = 0$. Because it converges to 8, the partial sums converge to 8, so that $\lim_{k \rightarrow \infty} S_k = 8$.

6 $\sum_{k=0}^{\infty} 3r^k = \frac{3}{1-r} = 6$, so $\frac{1}{2} = 1-r$, so $r = \frac{1}{2}$.

7 Consider the constant sequence with $a_k = 1$ for all k . The sequence $\{a_k\}$ converges to 1, but the corresponding series $\sum a_k$ diverges by the divergence test.

8 This is not possible. If the series $\sum_{k=1}^{\infty} a_k$ converges, then we must have $\lim_{k \rightarrow \infty} a_k = 0$.

9

- a. For $|x| < 1$, $\lim_{k \rightarrow \infty} x^k = 0$, so this limit is zero.
- b. This is a geometric series with ratio $-4/5$, so the sum is $\frac{1}{1 + 4/5} = \frac{5}{9}$.

10

- a. $\lim_{k \rightarrow \infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \lim_{k \rightarrow \infty} \frac{1}{k(k+1)} = 0$.
- b. This series telescopes, and $S_n = 1 - \frac{1}{n+1}$, so $\lim_{n \rightarrow \infty} S_n = 1$, which is the sum of the series.

11

- a. This sequence converges because $\lim_{k \rightarrow \infty} \frac{k}{k+1} = \lim_{k \rightarrow \infty} \frac{1}{1 + \frac{1}{k}} = \frac{1}{1+0} = 1$.
- b. Because the sequence of terms has limit 1 (which means its limit isn't zero) this series diverges by the Divergence Test.

12 $\lim_{n \rightarrow \infty} \frac{3n^3 + 4n}{6n^3 + 5} = \lim_{n \rightarrow \infty} \frac{3 + 4/n^2}{6 + 5/n^3} = \frac{3}{6} = \frac{1}{2}$.

13 Because $\lim_{n \rightarrow \infty} \frac{3n^3 + 4n}{6n^3 + 5} = \frac{1}{2} \neq 0$, the sequence $\left\{ (-1)^n \frac{3n^3 + 4n}{6n^3 + 5} \right\}$ oscillates and doesn't converge.

14 $\lim_{n \rightarrow \infty} \frac{(2n+2)!}{(2n)!n^2} = \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)(2n)!}{(2n)!n^2} = \lim_{n \rightarrow \infty} \frac{4n^2 + 6n + 2}{n^2} = 4$.

15 $\lim_{n \rightarrow \infty} \frac{2^n + 5^{n+1}}{5^n} = \lim_{n \rightarrow \infty} \left(\frac{2}{5} \right)^n + 5 = 0 + 5 = 5$.

$$16 \quad \lim_{n \rightarrow \infty} \frac{n^2 + 4}{\sqrt{4n^4 + 1}} = \lim_{n \rightarrow \infty} \frac{1 + 4n^{-2}}{\sqrt{4 + n^{-4}}} = \frac{1}{2}.$$

$$17 \quad \lim_{n \rightarrow \infty} \frac{8^n}{n!} = 0 \text{ because exponentials grow more slowly than factorials.}$$

18 After taking logs, we want to compute

$$\lim_{n \rightarrow \infty} 2n \ln(1 + 3/n) = \lim_{n \rightarrow \infty} \frac{\ln(1 + 3/n)}{1/(2n)}.$$

By L'Hôpital's rule, this is $\lim_{n \rightarrow \infty} \frac{6n}{n+3}$ (after some algebraic manipulations), which is 6. Thus the original limit is e^6 .

19 Because $-1 \leq \sin n \leq 1$, we have

$$\frac{-(3n^2 + 2n + 1)}{4n^3 + n} \leq \frac{(3n^2 + 2n + 1) \sin n}{4n^3 + n} \leq \frac{(3n^2 + 2n + 1)}{4n^3 + n}.$$

Because $\lim_{n \rightarrow \infty} \frac{-(3n^2 + 2n + 1)}{4n^3 + n} = 0 = \lim_{n \rightarrow \infty} \frac{(3n^2 + 2n + 1)}{4n^3 + n}$, we must have the the original sequence has limit 0.

$$20 \quad \lim_{n \rightarrow \infty} (n - \sqrt{n^2 - 1}) = \lim_{n \rightarrow \infty} \frac{n - \sqrt{n^2 - 1}}{1} \cdot \frac{n + \sqrt{n^2 - 1}}{n + \sqrt{n^2 - 1}} = \lim_{n \rightarrow \infty} \frac{1}{n + \sqrt{n^2 + 1}} = 0.$$

$$21 \quad \text{Take logs, and then evaluate } \lim_{n \rightarrow \infty} \frac{1}{\ln n} \ln(1/n) = \lim_{n \rightarrow \infty} (-1) = -1, \text{ so the original limit is } e^{-1}.$$

22 This series oscillates among the values $\pm 1/2, \pm \sqrt{3}/2, \pm 1$, and 0, so it has no limit.

23 $a_n = (-1/0.9)^n = (-10/9)^n$. The terms grow without bound so the sequence does not converge.

$$24 \quad \lim_{n \rightarrow \infty} \left(\frac{\pi}{2} - \tan^{-1} n \right) = \frac{\pi}{2} - \frac{\pi}{2} = 0.$$

25

a. 80, 48, 32, 24, 20

b. Assume that $L = \lim_{n \rightarrow \infty} a_n$. Then $L = \frac{L}{2} + 8$, so $2L = L + 16$, so $L = 16$.

26

a. 81, 7.81025, 6.05008, 6.00139, 6.00004

b. Assume that $L = \lim_{n \rightarrow \infty} a_n$. Then $L = \sqrt{L/3 + 34}$, so $L^2 = L/3 + 34$, so $3L^2 - L - 102 = 0$, so $(L - 6)(3L + 17) = 0$. So we either have $L = 6$ or $L = -17/3$, but because all the terms of a_n are positive, we must have $L = 6$.

$$27 \quad \sum_{k=1}^{\infty} 3(1.001)^k = 3 \sum_{k=1}^{\infty} (1.001)^k. \text{ This is a geometric series with ratio greater than 1, so it diverges.}$$

$$28 \quad \text{This is a geometric series with ratio } 9/10, \text{ so the sum is } \frac{9/10}{1 - 9/10} = 9.$$

$$29 \quad \text{Because } \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \text{ converges, but } \sum_{k=0}^{\infty} \left(\frac{4}{3}\right)^k \text{ diverges, the given series diverges.}$$

$$30 \quad \sum_{k=1}^{\infty} \frac{3^k - 5^k}{10^k} = \sum_{k=1}^{\infty} \left(\left(\frac{3}{10} \right)^k - \left(\frac{5}{10} \right)^k \right) = \frac{3/10}{1 - 3/10} - \frac{5/10}{1 - 5/10} = \frac{3}{7} - 1 = -\frac{4}{7}.$$

31 Because $\ln \left(\frac{2k+1}{2k-1} \right) = \ln(2k+1) - \ln(2k-1)$, this series telescopes. The formula for S_n is $S_n = (\ln 3 - \ln 1) + (\ln 5 - \ln 3) + \cdots + (\ln(2n+1) - \ln(2n-1)) = \ln(2n+1)$. The given series therefore diverges because $\lim_{n \rightarrow \infty} S_n = \infty$.

32 This is a geometric series with ratio $-\frac{1}{5}$, so it converges and the sum is $\frac{1}{1 + 1/5} = \frac{5}{6}$.

33 This series telescopes, and the formula for S_n is

$$\begin{aligned} S_n &= (\tan^{-1} 2 - \tan^{-1} 0) + (\tan^{-1} 3 - \tan^{-1} 1) + (\tan^{-1} 4 - \tan^{-1} 2) + \cdots + (\tan^{-1}(n+2) - \tan^{-1} n) \\ &= \tan^{-1}(n+2) + \tan^{-1}(n+1) - \tan^{-1} 1 - \tan^{-1} 0. \end{aligned}$$

So

$$\lim_{n \rightarrow \infty} S_n = \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} = \frac{3\pi}{4}.$$

34 This series clearly telescopes, and

$$S_n = \left(\frac{1}{\sqrt{2}} - 1 \right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} \right) + \cdots + \left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}} \right) = \frac{1}{\sqrt{n+1}} - 1,$$

so $\lim_{n \rightarrow \infty} S_n = -1$.

35 Note that $\frac{9}{(3k-2)(3k+1)} = \frac{3}{3k-2} - \frac{3}{3k+1}$. This series telescopes.

$$S_n = \left(3 - \frac{3}{4} \right) + \left(\frac{3}{4} - \frac{3}{7} \right) + \cdots + \left(\frac{3}{3n-2} - \frac{3}{3n+1} \right) = 3 - \frac{3}{3n+1},$$

so that $\lim_{n \rightarrow \infty} S_n = 3$, which is the value of the series.

36 $\sum_{k=1}^{\infty} 4^{-3k} = \sum_{k=1}^{\infty} (1/64)^k$. This is a geometric series with ratio $1/64$, so its sum is $\frac{1/64}{1 - 1/64} = \frac{1}{63}$.

$$37 \quad \sum_{k=1}^{\infty} \frac{2^k}{3^{k+2}} = \frac{1}{9} \sum_{k=1}^{\infty} \left(\frac{2}{3} \right)^k = \frac{1}{9} \cdot \frac{2/3}{1 - 2/3} = \frac{2}{9}.$$

$$38 \quad 0.29292929 \dots = 0.29 + 0.0029 + 0.000029 + \cdots = \frac{0.29}{1 - 0.01} = \frac{0.29}{0.99} = \frac{29}{99}.$$

$$39 \quad 0.314141414 \dots = 0.3 + (0.014 + 0.00014 + 0.00000014 + \cdots) = \frac{3}{10} + \frac{0.014}{1 - 0.01} = \frac{3}{10} + \frac{14}{990} = \frac{311}{990}.$$

40

$$\text{a. } d_{n+1} = 100 + \frac{1}{2}d_n; d_1 = 100.$$

b. Assume $\lim_{n \rightarrow \infty} d_n = L$. Then $L = 100 + \frac{1}{2}L$, so $L = 200$. So the steady-state amount of medication in the blood is 200 mg.

41 Note that $d_2 = 100 + \frac{1}{2}(100)$, $d_3 = 100 + \frac{1}{2}\left(100 + \frac{1}{2}(100)\right) = 100 + \frac{1}{2}(100) + \frac{1}{2^2}(100)$, and

$$d_n = 100 + \frac{1}{2}(100) + \cdots + \frac{1}{2^{n-1}}(100).$$

The steady state is $\lim_{n \rightarrow \infty} d_n$, which is the sum of the geometric series which is given by $\frac{100}{1 - (1/2)} = 200$ mg. So the steady-state amount of medication in the blood is 200 mg.

42 We have $\frac{k}{k^{5/2} + 1} < \frac{k}{k^{5/2}} = \frac{1}{k^{3/2}}$, and because $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a convergent p -series, the given series converges.

43 The series can be written $\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$, which is a p -series with $p = 2/3 < 1$, so this series diverges.

44 Apply the Limit Comparison Test with the convergent p -series $\sum_{j=1}^{\infty} \frac{1}{j^2}$. We have

$$\lim_{j \rightarrow \infty} \frac{\frac{2j^4 + j^3 + 1}{3j^6 + 4}}{\frac{1}{j^2}} = \lim_{j \rightarrow \infty} \frac{2j^6 + j^5 + j^2}{3j^6 + 4} = \frac{2}{3}.$$

So the given series converges by the Limit Comparison Test.

45 The k th term does not have limit 0, so the given series diverges by the Divergence Test. Note that

$$\lim_{k \rightarrow \infty} \frac{2k^4 + k^3 + 1}{3k^4 + 4} = \frac{2}{3} \neq 0.$$

46 We use the Ratio Test. The ratio of successive terms is

$$\frac{2(k+1)^{(k+1)/2}}{k^{k/2}\sqrt{k+1}} = 2 \frac{(k+1)^{k/2}}{k^{k/2}} = 2 \left(\left(1 + \frac{1}{k}\right)^k \right)^{1/2},$$

which as limit $2\sqrt{e}$ as $k \rightarrow \infty$. Because $2\sqrt{e} > 1$, the given series diverges by the Ratio Test.

47 We have $\frac{7 + \sin k}{k^2} \leq \frac{8}{k^2}$ and $\sum_{k=1}^{\infty} \frac{8}{k^2}$ converges (it is a p -series with $p > 1$), so by the Comparison Test, the given series converges.

48 $\lim_{k \rightarrow \infty} \frac{2k^4 + 1}{\sqrt{9k^8 + 2}} = \lim_{k \rightarrow \infty} \frac{2 + 1/k^4}{\sqrt{9 + 2/k^8}} = \frac{2}{\sqrt{9}} = \frac{2}{3} \neq 0$, so the given series diverges by the Divergence Test.

49 Note that $\sqrt{9k^{12} + 2} \geq \sqrt{9k^{12}} \geq \sqrt{k^{12}} = k^6$ for $k \geq 1$. It follows that

$$\frac{k^4}{\sqrt{9k^{12} + 2}} \leq \frac{k^4}{k^6} = \frac{1}{k^2}.$$

Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series converges by the Comparison Test.

50 We use the Ratio Test. The ratio of successive terms is

$$\sqrt{\frac{(k+1)!k^k}{(k+1)^{k+1}k!}} = \sqrt{\frac{k^k}{(k+1)^k}} = \frac{1}{\sqrt{\left(1+\frac{1}{k}\right)^k}},$$

which has limit $\sqrt{\frac{1}{e}} < 1$ as $k \rightarrow \infty$. The given series therefore converges by the Ratio Test.

51 This is a geometric series with ratio $2/e < 1$, so the given series converges.

52 Note that $\frac{1}{a_k} = \left(1 + \frac{3}{k}\right)^k$, so $\lim_{k \rightarrow \infty} \frac{1}{a_k} = \lim_{k \rightarrow \infty} \left(1 + \frac{3}{k}\right)^k = (e^3)^2$, so $\lim_{k \rightarrow \infty} a_k = \frac{1}{e^6} \neq 0$, so the given series diverges by the Divergence Test.

53 We use the Root Test. The k th root of the absolute value of the k th term is $\frac{k}{2k+3}$ which has limit $\frac{1}{2}$ as $k \rightarrow \infty$, so the given series converges.

54 We use the Limit Comparison Test with the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^4}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\sin \frac{1}{k^4}}{\frac{1}{k^4}} = 1,$$

so the given series converges by the Limit Comparison Test.

55 Applying the Ratio Test:

$$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} \frac{\frac{(k+1)!}{e^{k+1}(k+1)^{k+1}}}{\frac{k!}{e^k k^k}} = \lim_{k \rightarrow \infty} \frac{(k+1)k^k}{e(k+1)^{k+1}} = \lim_{k \rightarrow \infty} \frac{1}{e} \cdot \left(\frac{k}{k+1}\right)^k = \frac{1}{e^2},$$

so the given series converges.

56 Use the Limit Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{k}$:

$$\frac{1}{\sqrt{k^2+k}} \bigg/ \frac{1}{k} = \frac{k}{\sqrt{k^2+k}} = \sqrt{\frac{k^2}{k^2+k}},$$

which has limit 1 as $k \rightarrow \infty$. Because $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the original series does as well.

57 The series can be written as $2 \sum_{k=1}^{\infty} \frac{5^k}{4^k}$, and because $\sum_{k=1}^{\infty} \left(\frac{5}{4}\right)^k$ is a divergent geometric series, the given series is divergent.

58 By the Ratio Test: $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} \frac{k+1}{e^{k+1}} \cdot \frac{e^k}{k} = \lim_{k \rightarrow \infty} \frac{1}{e} \cdot \frac{k+1}{k} = \frac{1}{e} < 1$. Thus the given series converges.

59 Use the Ratio Test. The ratio of successive terms is $\frac{2 \cdot 4^{j+1}}{(2j+3)!} \cdot \frac{(2j+1)!}{2 \cdot 4^j} = \frac{4}{(2j+3)(2j+2)}$. This has limit 0 as $j \rightarrow \infty$, so the given series converges.

60 Use the Ratio Test. The ratio of successive term is $\frac{9^{k+1}}{(2k+2)!} \cdot \frac{(2k)!}{9^k} = \frac{9}{(2k+2)(2k+1)}$. This has limit 0 as $k \rightarrow \infty$, so the given series converges.

61 We use the Integral Test. Let $f(x) = \frac{\ln x}{x^{3/2}}$. Then $f'(x) = \frac{2-3\ln x}{2x^{5/2}}$, which is less than zero for $x \geq 2$. So f is decreasing, positive and continuous for $x \geq 3$. Using Integration by Parts, it can be shown that

$$\int f(x) dx = -\frac{4}{\sqrt{x}} - \frac{2\ln x}{\sqrt{x}},$$

so $\lim_{b \rightarrow \infty} \int_3^b f(x) dx = \frac{4}{\sqrt{3}} + \frac{2\ln 3}{\sqrt{3}}$. Because this integral converges, the given series converges by the Integral Test.

62 We use the Ratio Test. The absolute value of the ratio of successive terms is

$$\frac{(k+1)^2(2k-1)!}{k^2(2k+1)!} = \frac{(k+1)^2}{k^2(2k+1)(2k)},$$

which has limit zero as $k \rightarrow \infty$. The given series is therefore convergent by the Ratio Test.

63 Use the Comparison Test: $\frac{3}{2+e^k} < \frac{3}{e^k}$, but $\sum_{k=1}^{\infty} \frac{3}{e^k}$ converges because it is a geometric series with ratio $\frac{1}{e} < 1$. Thus the original series converges as well.

64 $\lim_{k \rightarrow \infty} a_k = \lim_{k \rightarrow \infty} k \sin(1/k) = \lim_{k \rightarrow \infty} \frac{\sin(1/k)}{1/k} = 1$, so the given series diverges by the Divergence Test.

65 $a_k = \frac{k^{1/k}}{k^3} = \frac{1}{k^{3-1/k}}$. For $k \geq 2$, then, $a_k < \frac{1}{k^2}$. Because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the given series also converges, by the Comparison Test.

66 Use the Comparison Test: $\frac{1}{1+\ln k} > \frac{1}{k}$ for $k > 1$. Because $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the given series does as well.

67 Use the Ratio Test: $\frac{a_{k+1}}{a_k} = \frac{(k+1)^5}{e^{k+1}} \cdot \frac{e^k}{k^5} = \frac{1}{e} \cdot \left(\frac{k+1}{k}\right)^5$, which has limit $1/e < 1$ as $k \rightarrow \infty$. Thus the given series converges.

68 For $k > 5$, we have $k^2 - 10 > (k-1)^2$, so that $a_k = \frac{2}{k^2 - 10} < \frac{2}{(k-1)^2}$. Because $\sum_{k=4}^{\infty} \frac{2}{(k-1)^2}$ converges, the original series does as well.

69 We use the Limit Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{k^2}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{1 - \cos \frac{1}{k}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{\left(\sin \frac{1}{k}\right) \frac{1}{k^2}}{\frac{2}{k^3}} = \frac{1}{2} \lim_{k \rightarrow \infty} \frac{\sin \frac{1}{k}}{\frac{1}{k}} = \frac{1}{2}.$$

Because $0 < L < \infty$, and because $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, we have that the given series converges.

70 We use the Limit Comparison Test with $\sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$L = \lim_{k \rightarrow \infty} \frac{\frac{\pi}{2} - \tan^{-1} k}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{\frac{1}{1+k^2}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{k^2}{1+k^2} = 1.$$

Because $0 < L < \infty$, and because the Harmonic Series diverges, the given series diverges.

71 We use the Limit Comparison Test with the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. We have

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} \frac{\left(1 - \cos \frac{1}{k}\right)^2}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{2 \left(1 - \cos \frac{1}{k}\right) \left(\sin \frac{1}{k}\right) \left(-\frac{1}{k^2}\right)}{-\frac{2}{k^3}} \\ &= \lim_{k \rightarrow \infty} k \left(1 - \cos \frac{1}{k}\right) \left(\sin \frac{1}{k}\right) = \lim_{k \rightarrow \infty} \left(1 - \cos \frac{1}{k}\right) \lim_{k \rightarrow \infty} \frac{\sin(1/k)}{(1/k)} = 0 \cdot 1 = 0. \end{aligned}$$

Because $L = 0$ and the comparison series converges, by the Limit Comparison Test, the given series converges.

72 We use the Limit Comparison Test with the convergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. We have

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} \frac{\left(\frac{\pi}{2} - \tan^{-1} k\right)^2}{1/k} = \lim_{k \rightarrow \infty} \frac{2 \left(\frac{\pi}{2} - \tan^{-1} k\right) \frac{1}{1+k^2}}{2/k^3} = \lim_{k \rightarrow \infty} \left(\frac{\pi}{2} - \tan^{-1} k\right) \frac{k^3}{1+k^2} \\ &= \lim_{k \rightarrow \infty} \frac{\frac{\pi}{2} - \tan^{-1} k}{1/k} \cdot \frac{k^2}{1+k^2} = \lim_{k \rightarrow \infty} \frac{1/(1+k^2)}{1/k^2} = \lim_{k \rightarrow \infty} \frac{k^2}{1+k^2} = 1. \end{aligned}$$

Thus the given integral converges by the Limit Comparison Test.

73 Use the Limit Comparison Test with the harmonic series. Note that $\lim_{k \rightarrow \infty} \frac{\coth k}{k} \cdot \frac{k}{1} = \lim_{k \rightarrow \infty} \coth k = 1$. Because the harmonic series diverges, the given series does as well.

74 Use the Limit Comparison Test with the convergent geometric series whose k th term is $\frac{1}{e^k}$. We have

$$\lim_{k \rightarrow \infty} \frac{1}{\sinh k} \cdot \frac{e^k}{1} = \lim_{k \rightarrow \infty} \frac{2e^k}{e^k - e^{-k}} = 2 \lim_{k \rightarrow \infty} \frac{1}{1 - e^{-2k}} = 2.$$

The given series is therefore convergent.

75 Use the Divergence Test. $\lim_{k \rightarrow \infty} \tanh k = \lim_{k \rightarrow \infty} \frac{e^k + e^{-k}}{e^k - e^{-k}} = 1 \neq 0$, so the given series diverges.

76 Use the Limit Comparison Test with the convergent geometric series whose k th term is $\frac{1}{e^k}$. We have

$$\lim_{k \rightarrow \infty} \frac{1}{\cosh k} \cdot \frac{e^k}{1} = \lim_{k \rightarrow \infty} \frac{2e^k}{e^k + e^{-k}} = 2 \lim_{k \rightarrow \infty} \frac{1}{1 + e^{-2k}} = 2.$$

The given series is therefore convergent.

77 The corresponding series of all positive terms is the divergent p -series $\sum_{k=1}^{\infty} \frac{1}{k^{3/7}}$. However, the sequence $\left\{ \frac{1}{k^{3/7}} \right\}$ is nonincreasing and has limit 0, so the given alternating series is convergent by the Alternating Series Test. Therefore, the given series is conditionally convergent.

78 The corresponding series of all positive terms is convergent, because it is $\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$ which is a convergent p -series. The given series is therefore absolutely convergent.

79 $|a_k| = \frac{1}{k^2 - 1}$. Use the Limit Comparison Test with the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Because

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{k^2 - 1}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{k^2}{k^2 - 1} = 1,$$

the given series converges absolutely.

80 We have $\ln(x+3) < x+3$, or $\frac{1}{\ln(x+3)} > \frac{1}{x+3}$ for $x \geq 1$. The series $\sum_{j=1}^{\infty} \frac{1}{j+3} = \sum_{j=4}^{\infty} \frac{1}{j}$ diverges, so the series $\sum_{j=1}^{\infty} \frac{1}{\ln(j+3)}$ diverges. However, the sequence $\left\{ \frac{1}{\ln(j+3)} \right\}$ is decreasing and has limit 0, so the given series is convergent by the Alternating Series Test. The given series is therefore conditionally convergent.

81 The sequence $\left\{ \frac{k^2 + 4}{2k^2 + 1} \right\}$ has limit $\frac{1}{2}$, which implies that the sequence $\left\{ (-1)^{k+1} \frac{k^2 + 4}{2k^2 + 1} \right\}$ oscillates and diverges. The given series therefore diverges, because by the Divergence Test, the k th term does not have limit 0.

82 Using the Limit Comparison Test with the harmonic series, we consider

$$\lim_{k \rightarrow \infty} \left| \frac{a_k}{\frac{1}{k}} \right| = \lim_{k \rightarrow \infty} \frac{k}{\sqrt{k^2 + 1}} = \lim_{k \rightarrow \infty} \sqrt{\frac{k^2}{k^2 + 1}} = 1;$$

because the comparison series diverges, so does the original series. Thus the series is not absolutely convergent. However, the terms are clearly decreasing to zero, so it is conditionally convergent.

83 Use the Root Test on the absolute values of the sequence of terms:

$$\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \sqrt[k]{ke^{-k}} = \lim_{k \rightarrow \infty} \frac{k^{1/k}}{e} = \frac{1}{e}.$$

Thus, the original series is absolutely convergent by the Root Test.

84 The series of absolute values converges by the Comparison Test. Note that $e^k + e^{-k} > e^k$, so $\frac{1}{e^k + e^{-k}} < \frac{1}{e^k}$ for $k \geq 1$. Because the series $\sum_{k=0}^{\infty} \frac{1}{e^k}$ is a convergent geometric series, the given series converges absolutely.

85 Use the Ratio Test on the absolute values of the sequence of terms: $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \frac{10}{k+1} = 0$, so the series converges absolutely.

86 $\sum \frac{1}{k \ln k}$ does not converge because $\int_2^\infty \frac{1}{x \ln x} dx = \lim_{b \rightarrow \infty} \ln(\ln x) \Big|_2^\infty = \infty$, so the improper integral diverges. Thus the given series does not converge absolutely. However, it does converge conditionally because the terms are decreasing and approach zero.

87 Because $k^2 \ll 2^k$, $\lim_{k \rightarrow \infty} \frac{-2 \cdot (-2)^k}{k^2} \neq 0$. The given series thus diverges by the Divergence Test. Alternatively, this result can be derived from either the Ratio Test or the Root Test.

88 By the Integral Test, the series converges if and only if the following integral converges:

$$\int_2^\infty \frac{1}{x \ln^p(x)} dx = \lim_{b \rightarrow \infty} \left(\frac{1}{1-p} \ln^{(1-p)}(x) \Big|_2^b \right) = \lim_{b \rightarrow \infty} \frac{1}{1-p} \ln^{(1-p)}(b) - \left(\frac{1}{1-p} \right) \cdot \ln^{(1-p)}(2).$$

This limit exists only if $1-p < 0$, i.e. $p > 1$. Note that the above calculation is for the case $p \neq 1$. In the case $p = 1$, the integral also diverges.

89

a. $\sum_{k=1}^5 \frac{1}{k^5} \approx 1.03666.$

b. $R_5 < \int_5^\infty \frac{dx}{x^5} = \lim_{b \rightarrow \infty} \int_5^b \frac{dx}{x^5} = \lim_{b \rightarrow \infty} -\frac{1}{4} \left(\frac{1}{b^4} - \frac{1}{5^4} \right) = \frac{1}{4 \cdot 5^4} = 0.0004.$

c. Note that $\int_6^\infty \frac{dx}{x^5} = \frac{1}{4 \cdot 6^4} \approx 0.00019$. So $L_5 = S_5 + \int_6^\infty \frac{dx}{x^5} \approx 1.03685$, and $U_5 = S_5 + \int_5^\infty \frac{dx}{x^5} \approx 1.03706.$

90

a. $\sum_{k=1}^{20} \frac{1}{k^2 + 1} \approx 1.0279418.$

b. $R_{20} < \int_{20}^\infty \frac{dx}{x^2 + 1} = \lim_{b \rightarrow \infty} \int_{20}^b \frac{dx}{x^2 + 1} = \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 20) = \frac{\pi}{2} - \tan^{-1} 20 \approx 0.0499584.$

c. Note that $\int_{21}^\infty \frac{dx}{x^2 + 1} = \frac{\pi}{2} - \tan^{-1} 21 \approx 0.0475831$. So $L_{20} = S_{20} + \int_{21}^\infty \frac{dx}{x^2 + 1} \approx 1.0755249$, and $U_{20} = S_{20} + \int_{20}^\infty \frac{dx}{x^2 + 1} \approx 1.0779002.$

91 We have $R_n = \int_n^\infty \frac{dx}{(2x+5)^3} = \frac{1}{4(2n+5)^2}$. We want $\frac{1}{4(2n+5)^2} < \frac{1}{10^4}$, which leads to $n \geq 23$.

Therefore $\sum_{k=1}^\infty \frac{1}{(2k+5)^3} \approx \sum_{k=1}^{23} \frac{1}{(2k+5)^3} \approx 0.0067.$

92 $\sum_{k=1}^\infty \frac{(-1)^{k+1}}{(2k+5)^3} \approx \sum_{k=1}^5 \frac{(-1)^{k+1}}{(2k+5)^3} \approx 0.00213615$. We have $|S - S_5| \leq \frac{1}{(2 \cdot 6 + 5)^3} = \frac{1}{17^3} = 0.000203542.$

93 The maximum error is a_{n+1} , so we want $a_{n+1} = \frac{1}{(k+1)^4} < 10^{-8}$, or $(k+1)^4 > 10^8$, so $k = 100$.

94

a. $\sum_{k=0}^\infty e^{kx} = \sum_{k=0}^\infty (e^x)^k = \frac{1}{1 - e^x} = 2$, so $1 - e^x = 1/2$. Thus $e^x = 1/2$ and $x = -\ln(2).$

b. $\sum_{k=0}^{\infty} (3x)^k = \frac{1}{1-3x} = 4$, so that $1-3x = \frac{1}{4}$, $x = \frac{1}{4}$.

95

- a. Let T_n be the amount of additional tunnel dug during week n . Then $T_0 = 100$ and $T_n = .95 \cdot T_{n-1} = (.95)^n T_0 = 100(0.95)^n$, so the total distance dug in N weeks is

$$S_N = 100 \sum_{k=0}^{N-1} (0.95)^k = 100 \left(\frac{1 - (0.95)^N}{1 - 0.95} \right) = 2000(1 - 0.95^N).$$

Then $S_{10} \approx 802.5$ meters and $S_{20} \approx 1283.03$ meters.

- b. The longest possible tunnel is $S_{\infty} = 100 \sum_{k=0}^{\infty} (0.95)^k = \frac{100}{1 - 0.95} = 2000$ meters.

96 Let t_n be the time required to dig meters $(n-1) \cdot 100$ through $n \cdot 100$, so that $t_1 = 1$ week. Then $t_n = 1.1 \cdot t_{n-1} = (1.1)^{n-1} t_1 = (1.1)^{n-1}$ weeks. The time required to dig 1500 meters is then

$$\sum_{k=1}^{15} t_k = \sum_{k=1}^{15} (1.1)^{k-1} \approx 31.77 \text{ weeks.}$$

So it is not possible.

97

- a. The area of a circle of radius r is πr^2 . For $r = 2^{1-n}$, this is $2^{2-2n}\pi$. There are 2^{n-1} circles on the n^{th} page, so the total area of circles on the n^{th} page is $2^{n-1} \cdot \pi 2^{2-2n} = 2^{1-n}\pi$.
- b. The sum of the areas on all pages is $\sum_{k=1}^{\infty} 2^{1-k}\pi = 2\pi \sum_{k=1}^{\infty} 2^{-k} = 2\pi \cdot \frac{1/2}{1/2} = 2\pi$.

98 Because the given sequence is non-decreasing and bounded above by 1, it must have a limit. A reasonable conjecture is that the limit is 1.

99

- a. $T_1 = \frac{\sqrt{3}}{16}$ and $T_2 = \frac{7\sqrt{3}}{64}$.
- b. At stage n , 3^{n-1} triangles of side length $1/2^n$ are removed. Each of those triangles has an area of $\frac{\sqrt{3}}{4 \cdot 4^n} = \frac{\sqrt{3}}{4^{n+1}}$, so a total of

$$3^{n-1} \cdot \frac{\sqrt{3}}{4^{n+1}} = \frac{\sqrt{3}}{16} \cdot \left(\frac{3}{4}\right)^{n-1}$$

is removed at each stage. Thus

$$T_n = \frac{\sqrt{3}}{16} \sum_{k=1}^n \left(\frac{3}{4}\right)^{k-1} = \frac{\sqrt{3}}{16} \sum_{k=0}^{n-1} \left(\frac{3}{4}\right)^k = \frac{\sqrt{3}}{4} \left(1 - \left(\frac{3}{4}\right)^n\right).$$

- c. $\lim_{n \rightarrow \infty} T_n = \frac{\sqrt{3}}{4}$ because $\left(\frac{3}{4}\right)^n \rightarrow 0$ as $n \rightarrow \infty$.
- d. The area of the triangle was originally $\frac{\sqrt{3}}{4}$, so none of the original area is left.

100

- a. The fraction of available wealth spent each month is $1 - p$, so the amount spent in the n^{th} month is $W(1 - p)^n$ (so that all $\$W$ is spent during the first month). The total amount spent is then

$$\sum_{n=1}^{\infty} W(1 - p)^n = \frac{W(1 - p)}{1 - (1 - p)} = W \left(\frac{1 - p}{p} \right) \text{ dollars.}$$

- b. As $p \rightarrow 1$, the total amount spent approaches 0. This makes sense, because in the limit, if everyone saves all of the money, none will be spent. As $p \rightarrow 0$, the total amount spent gets larger and larger. This also makes sense, because almost all of the available money is being respent each month.

101 Ignoring the initial drop for the moment, the height after the n^{th} bounce is $10p^n$, so the total time spent in that bounce is $2 \cdot \sqrt{2 \cdot 10p^n/g}$ seconds. The total time before the ball comes to rest (now including the time

for the initial drop) is then $\sqrt{20/g} + \sum_{i=1}^{\infty} 2 \cdot \sqrt{2 \cdot 10p^i/g} = \sqrt{\frac{20}{g}} + 2\sqrt{\frac{20}{g}} \sum_{i=1}^{\infty} (\sqrt{p})^i = \sqrt{\frac{20}{g}} + 2\sqrt{\frac{20}{g}} \frac{\sqrt{p}}{1 - \sqrt{p}} =$

$$\sqrt{\frac{20}{g}} \left(1 + \frac{2\sqrt{p}}{1 - \sqrt{p}} \right) = \sqrt{\frac{20}{g}} \left(\frac{1 + \sqrt{p}}{1 - \sqrt{p}} \right) \text{ seconds.}$$